ORF522 – Linear and Nonlinear Optimization

17. Operator theory II

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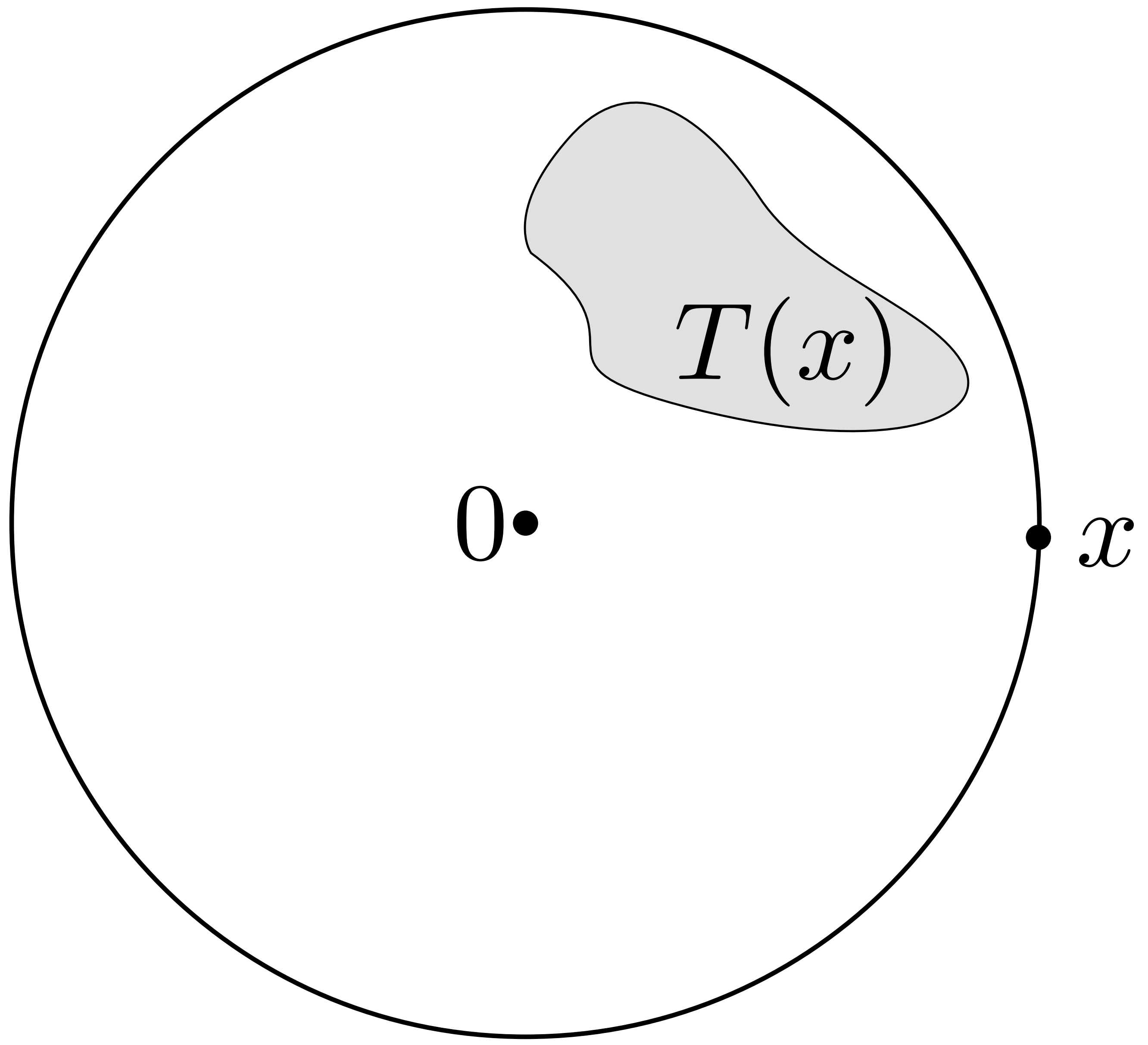
Recap

Operators

An operator T maps each point in $\mathbf{R}^n$ to a subset of $\mathbf{R}^n$

- **set valued** $T(x)$ returns a set
- **single-valued** $T(x)$ (function) returns a singleton

The **domain** of T is the set $\mathbf{dom}\, T = \{x \mid T(x) \neq \emptyset\}$



Example

- The subdifferential ∂f is a set-valued operator
- The gradient ∇f is a single-valued operator

Summary of monotone and cocoercive operators

Monotone

$$(T(x) - T(y))^T (x - y) \geq 0$$

$\mu = 0$

Strongly monotone

$$(T(x) - T(y))^T (x - y) \geq \mu \|x - y\|^2$$

$F = T^{-1}$

Lipschitz

$$\|F(x) - F(y)\| \leq L \|x - y\|$$

$L = 1/\mu$

Cocoercive

$$(F(x) - F(y))^T (x - y) \geq \mu \|F(x) - F(y)\|^2$$

$G = I - 2\mu F$

Nonexpansive

$$\|G(x) - G(y)\| \leq \|x - y\|$$

Zeros

Zero

x is a **zero** of T if $\quad 0 \in T(x)$

Zero set

The set of all the zeros $\quad T^{-1}(0) = \{x \mid 0 \in T(x)\}$

Example
If $T = \partial f$ and $f : \mathbf{R}^n \to \mathbf{R}$, then
$0 \in T(x)$ means that x minimizes f

Many problems
can be posed as finding zeros
of an operator

Fixed points

$\bar{x}$ is a **fixed-point** of a single-valued operator T if

$$\bar{x} = T(\bar{x})$$

Set of fixed points $\quad \mathbf{fix}\, T = \{x \in \mathbf{dom}\, T \mid x = T(x)\} = (I - T)^{-1}(0)$

Examples

- **Identity** $T(x) = x$. Any point is a fixed point
- **Zero operator** $T(x) = 0$. Only 0 is a fixed point

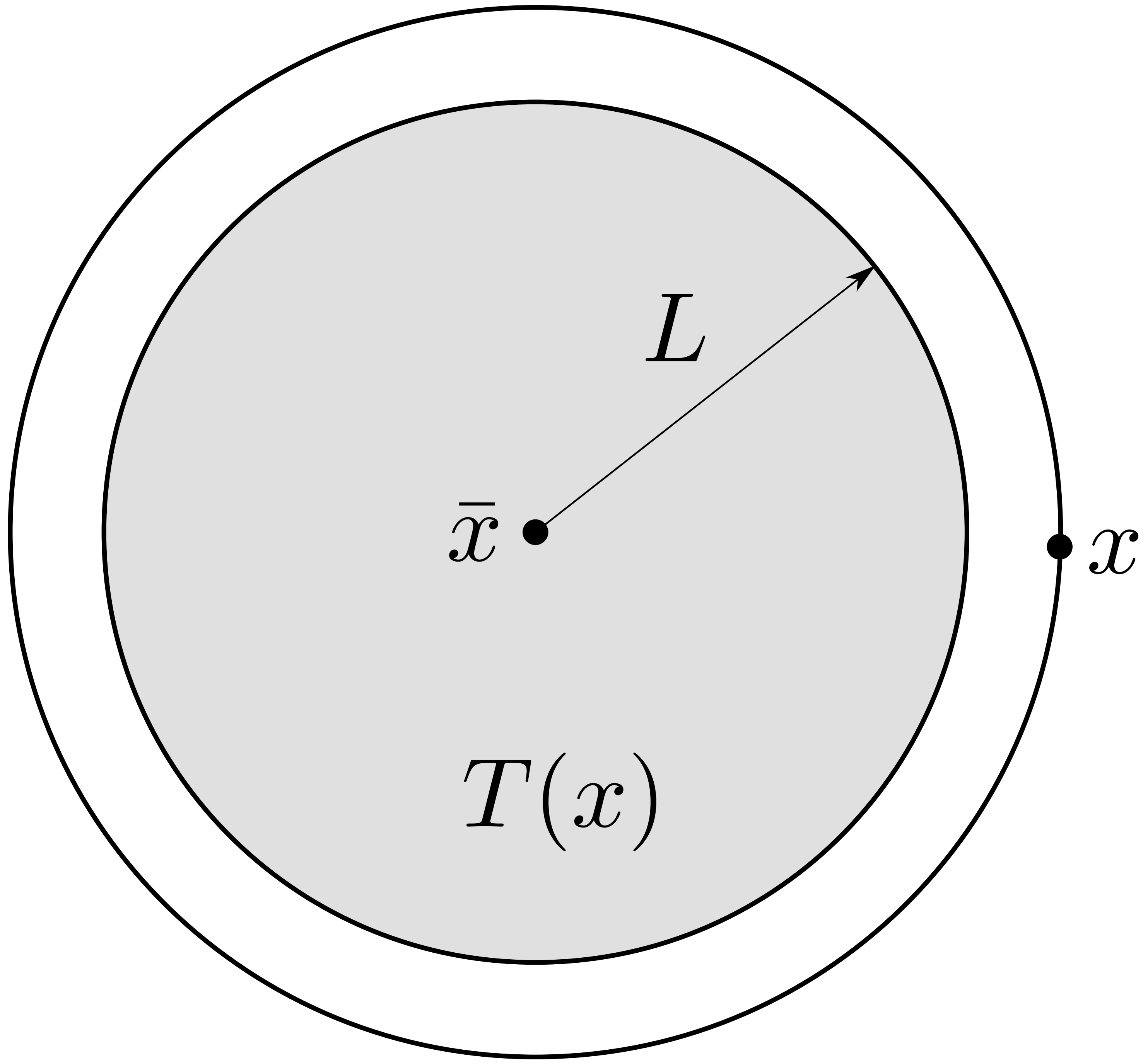
Lipschitz operators and fixed points

Given a L-Lipschitz operator T and a fixed point $\bar{x} = T\bar{x}$,

$$\|Tx - \bar{x}\| = \|Tx - T\bar{x}\| \leq L\|x - \bar{x}\|$$

A contractive operator ($L < 1$) can have at most one fixed point, i.e., $\mathbf{fix}\, T = \{\bar{x}\}$

Proof
If $\bar{x}, \bar{y} \in \mathbf{fix}\, T$ and $\bar{x} \neq \bar{y}$ then
$\|\bar{x} - \bar{y}\| = \|T(\bar{x}) - T(\bar{y})\| < \|x - y\|$ (contradiction) ■



A nonexpansive operator ($L = 1$) need not have a fixed point

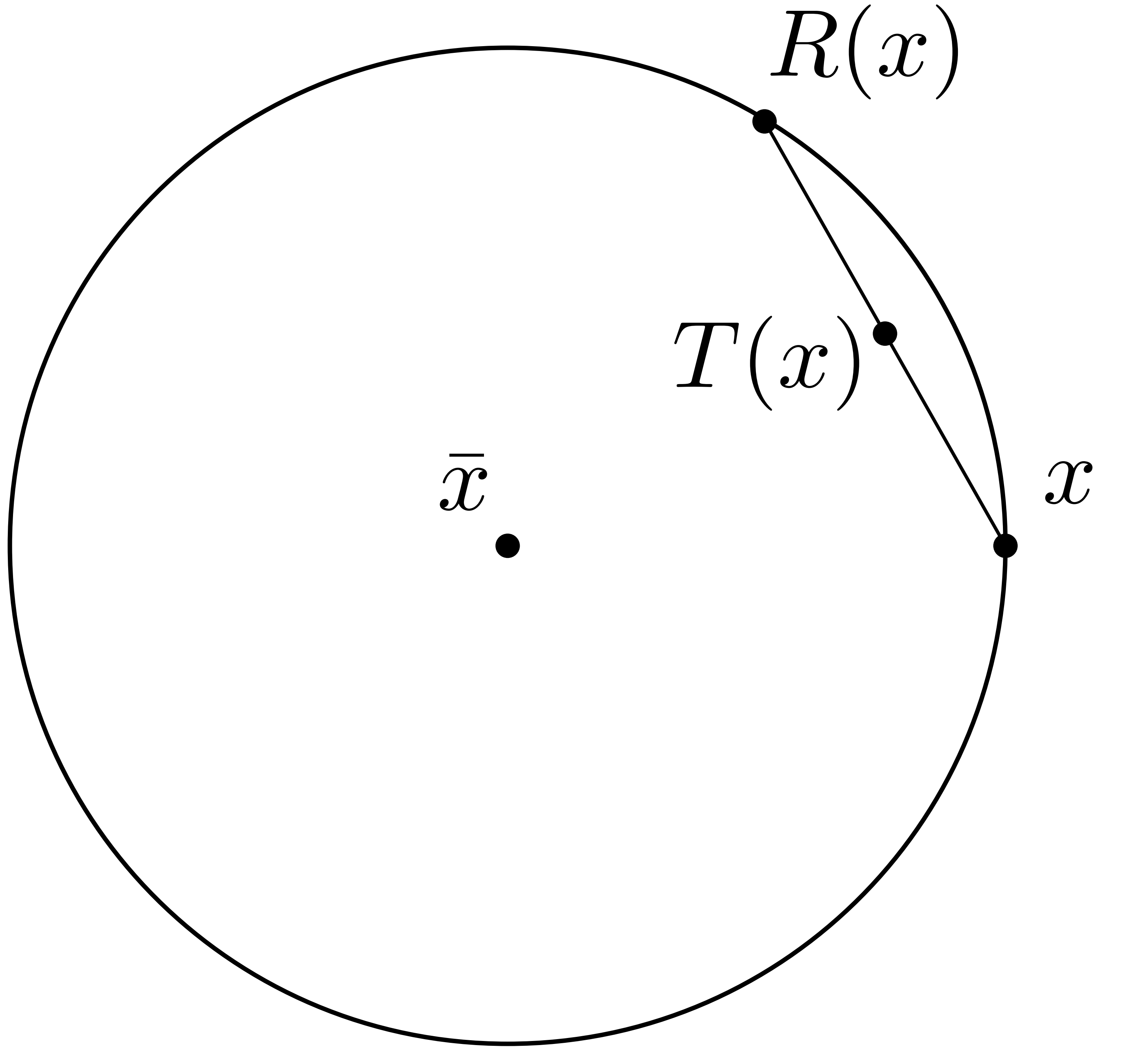
Example $T(x) = x + 2$

Averaged operators

We say that an operator T is $\alpha-$**averaged** with $\alpha \in (0,1)$ if

$$T = (1-\alpha)I + \alpha R$$

and R is nonexpansive.



How to design an algorithm

Problem

minimize $\quad f(x)$

Algorithm (operator) construction

1. Find a suitable T such that $\bar{x} \in \mathbf{fix}\, T$ solve your problem
2. Show that the fixed point iteration converges

If T is contractive $\quad \Longrightarrow \quad$ **linear convergence**
If T is averaged $\quad \Longrightarrow \quad$ **sublinear convergence**

Most first order algorithms can be constructed in this way

Today's lecture

[Chapter 4, FMO][PA][PMO][LSMO]

Operator theory

- Linking operators and functions
  - Conjugate functions and duality
  - Subdifferential operator
- Operators in optimization problems
- Operators in algorithms
- Building contractions

Conjugate functions and duality

Convex closed proper functions

A function $f : \mathbf{R}^n \to \mathbf{R}$ is called **CCP** if it is

closed $\quad \mathbf{epi}\, f$ is a closed set

convex $\quad f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y), \quad \alpha \in [0,1]$

proper $\quad \mathbf{dom}\, f$ is nonempty

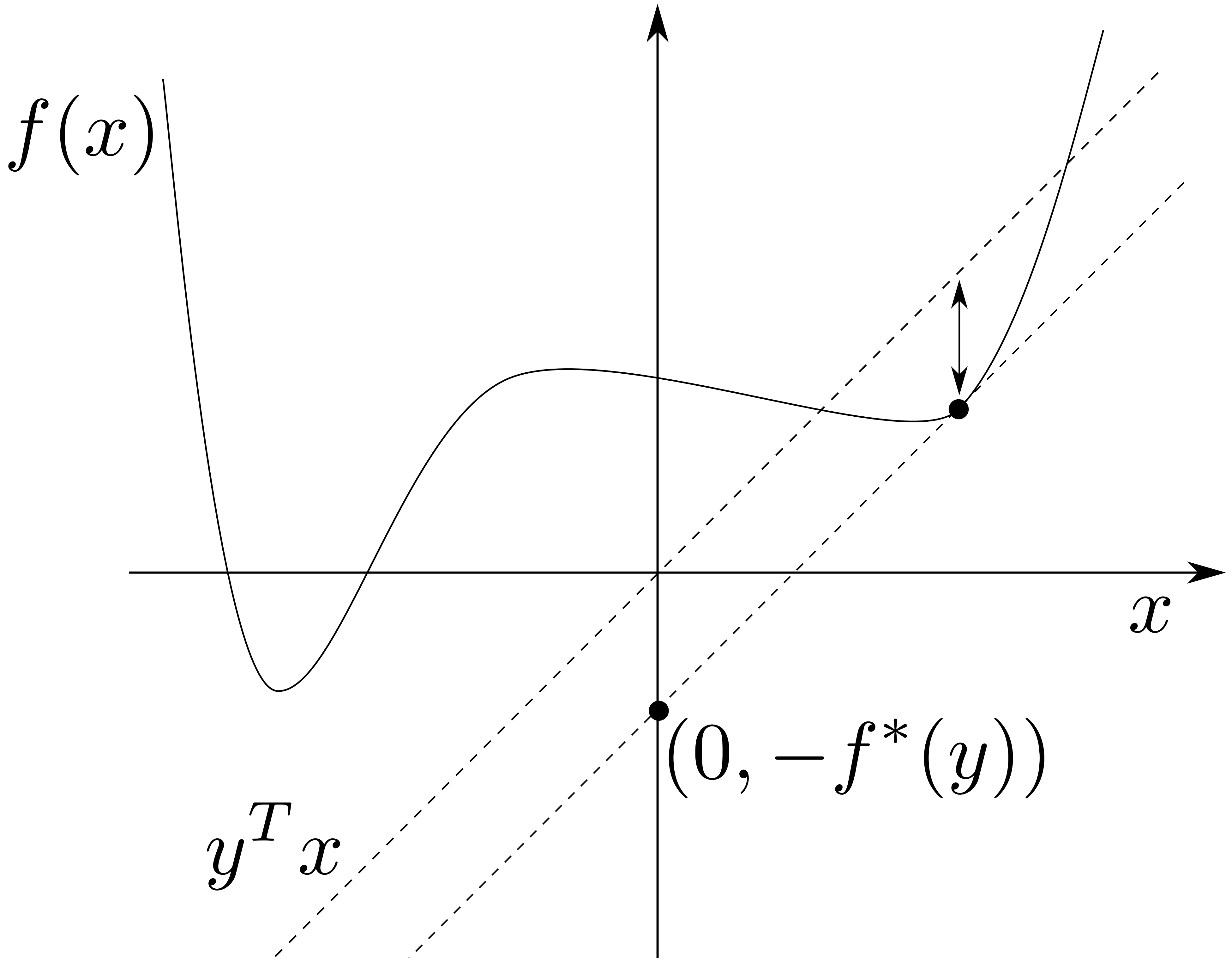
If not otherwise stated, we assume functions to be **CCP**

Conjugate function

Given a function $f : \mathbf{R}^n \to \mathbf{R}$ we define its **conjugate** $f^* : \mathbf{R}^n \to \mathbf{R}$ as

$$f^*(y) = \max_x \; y^T x - f(x)$$

Note f^* is always convex (pointwise maximum of affine functions in y)



f^* is the *maximum gap* between $y^T x$ and $f(x)$

Conjugate function properties and examples

Properties

Fenchel's inequality $f(x) + f^*(y) \geq y^T x$ (from $\max$ inside conjugate)

Biconjugate $f^{**}(x) = \max_y \; x^T y - f^*(y) \quad \Longrightarrow \quad f(x) \geq f^{**}(x)$

Biconjugate for CCP functions If f CCP, then $f^{**} = f$

Examples

Norm $f(x) = \|x\|$: $f^*(y) = \mathcal{I}_{\|y\|_* \leq 1}(y)$ **indicator function of dual norm set**

Indicator function $f(x) = \mathcal{I}_C(x)$: $f^*(y) = \mathcal{I}_C^*(y) = \max_{x \in C} \; y^T x = \sigma_C(y)$ **support function**

More examples of conjugate functions [Page 101, FMO]

Fenchel dual

Dual using conjugate functions

minimize $\quad f(x) + g(x)$ $\longrightarrow$

Equivalent form (variables split)

minimize $\quad f(x) + g(z)$

subject to $\quad x = z$

Lagrangian

$$L(x, z, y) = f(x) + g(z) + y^T(z - x) = -(y^T x - f(x)) - (-y^T z - g(z))$$

Dual function

$$\min_{x,z} L(x, z, y) = -f^*(y) - g^*(-y)$$

Dual problem

maximize $\; -f^*(y) - g^*(-y)$

Fenchel dual example

Constrained optimization

minimize $f(x) + \mathcal{I}_C(x)$ $\longrightarrow$

Dual problem

maximize $-f^*(y) - \sigma_C(-y)$

Norm penalization

minimize $f(x) + \|x\|$ $\longrightarrow$

Dual problem

maximize $-f^*(y)$

subject to $\|y\|_* \leq 1$

Remarks

- Fenchel duality can simplify derivations
- Useful when conjugates are known
- Very common in operator splitting algorithms

Subdifferential operator and monotonicity

Subdifferential operator monotonicity

$$\partial f(x) = \{g \mid f(y) \geq f(x) + g^T(y - x)\}$$

$\partial f(x)$ is **monotone** (also for nonconvex functions)

Proof Suppose $u \in \partial f(x)$ and $v \in \partial f(y)$ then

$$f(y) \geq f(x) + u^T(y - x), \qquad f(x) \geq f(y) + v^T(x - y)$$

By adding them, we can write $(u - v)^T(x - y) \geq 0$ ■

Maximal monotonicity

If f is convex, closed and proper (CCP), then $\partial f(x)$ is maximal monotone

Strongly monotone and cocoercive subdifferential

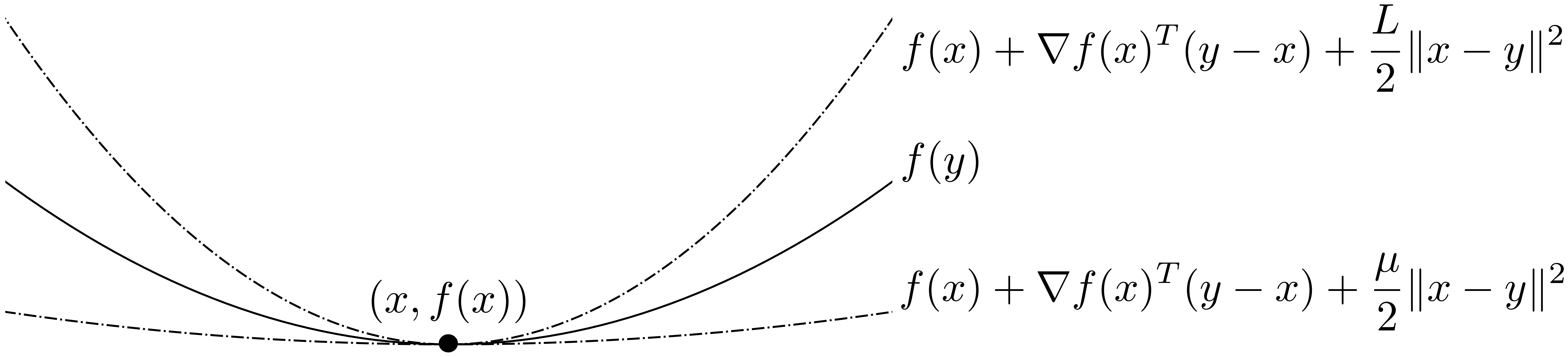
f is μ **-strongly convex** $\iff$ ∂f μ**-strongly monotone**

$$(\partial f(x) - \partial f(y))^T(x - y) \geq \mu \|x - y\|^2$$

f is L**-smooth**

$\iff$ ∂f L**-Lipschitz** and $\partial f = \nabla f$: $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$

$\iff$ ∂f $(1/L)$**-cocoercive**: $(\nabla f(x) - \nabla f(y))^T(x - y) \geq (1/L)\|\nabla f(x) - \nabla f(y)\|^2$



Inverse of subdifferential

If f is CCP, then, $\quad (\partial f)^{-1} = \partial f^*$

Proof

$$
\begin{aligned}
(u, v) \in \mathbf{gph}(\partial f)^{-1} &\iff (v, u) \in \mathbf{gph}\partial f \\
&\iff u \in \partial f(v) \\
&\iff 0 \in \partial f(v) - u \\
&\iff v \in \operatorname*{argmin}_x f(x) - u^T x \\
&\iff f^*(u) = u^T v - f(v)
\end{aligned}
$$

Therefore, $f(v) + f^*(u) = u^T v$. If f is CCP, then $f^{**} = f$ and we can write

$$f^{**}(v) + f^*(u) = u^T v \quad \iff \quad (u, v) \in \mathbf{gph}\partial f^* \qquad \blacksquare$$

Strong convexity is the dual of smoothness

f is μ-strongly convex $\quad \Longleftrightarrow \quad$ f^* is $(1/\mu)$-smooth

Proof

$$
\begin{aligned}
f \quad \mu\text{-strongly convex} \quad &\Longleftrightarrow \quad \partial f \quad \mu\text{-strongly monotone} \\
&\Longleftrightarrow \quad (\partial f)^{-1} = \partial f^* \quad \mu\text{-cocoercive} \\
&\Longleftrightarrow \quad f^* \quad (1/\mu)\text{-smooth} \qquad \blacksquare
\end{aligned}
$$

Remark: strong convexity and (strong) smoothness are **dual**

Operators in optimization problems

KKT operator

minimize $\quad f(x)$
subject to $\quad Ax = b$

→

Lagrangian

$$L(x, y) = f(x) + y^T(Ax - b)$$

KKT operator

$$T(x,y) = \begin{bmatrix} \partial_x L(x,y) \\ -\partial_y L(x,y) \end{bmatrix} = \begin{bmatrix} \partial f(x) + A^T y \\ b - Ax \end{bmatrix} = \begin{bmatrix} r^{\text{dual}} \\ -r^{\text{prim}} \end{bmatrix}$$

zero set $\{(x,y) \mid 0 \in T(x,y)\}$ is the set of **primal-dual optimal points**

Monotonicity

$$T(x,y) = \begin{bmatrix} \partial f(x) \\ b \end{bmatrix} + \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

skew-symmetric

sum of monotone operators

"multiplier to residual" mapping

minimize $f(x)$
subject to $Ax = b$

$\longrightarrow$

Lagrangian

$$L(x, y) = f(x) + y^T(Ax - b)$$

Dual problem

maximize $\quad g(y) = \min_x L(x, y) = -\max_x -L(x, y) = -(f^*(-A^T y) + y^T b)$

Operator

$T(y) = b - Ax$, where $x = \operatorname{argmin}_z L(z, y)$ $\longrightarrow$

Monotonicity

If f CCP, then T is monotone

Proof

$$0 \in \partial_x L(x, y) = \partial f(x) + A^T y \quad \Longleftrightarrow \quad x = (\partial f)^{-1}(-A^T y)$$

Therefore, $T(y) = b - A(\partial f)^{-1}(-A^T y) = \partial_y \left(b^T y + f^*(-A^T y)\right) = \partial(-g)$ ■

monotone

Operators in algorithms

Forward step operator

The **forward step operator** of T is defined as

$$I - \gamma T$$

In general **monotonicity of T** is not enough for convergence

Example

minimize $\quad x$

subject to $\quad x = 0$

KKT operator

$$T(x, y) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Monotone (skew-symmetric)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A + A^T = 0 \succeq 0$$

Forward step

$$(I - \gamma T) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -\gamma \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow$$

Expansive

$$\left\| \begin{bmatrix} 1 & -\gamma \\ \gamma & 1 \end{bmatrix} \right\|_2 > 1, \quad \forall \gamma$$

Gradient step: special case of forward step

f L-smooth $\iff$ ∇f $(1/L)$-cocoercive $\iff$ $I - (2/L)\nabla f$ nonexpansive

Construct averaged iterations

$$I - \gamma \nabla f = (1-\alpha)I + \alpha(I - (2/L)\nabla f)$$

where $\alpha = \gamma L/2 \in (0,1)$ $\iff$ $\gamma \in (0, 2/L)$

(to be averaged)

Remark

- Only smoothness assumption gives **sublinear convergence**
- Similar result we obtained in gradient descent lecture

Resolvent and Cayley operators

The **resolvent** of operator A is defined as

$$R_A = (I + A)^{-1}$$

The **Cayley (reflection) operator** of A is defined as

$$C_A = 2R_A - I = 2(I + A)^{-1} - I$$

Properties

- If A is maximal monotone, $\mathbf{dom}\, R_A = \mathbf{dom}\, C_A = \mathbf{R}^n$ (Minty's theorem)
- If A is **monotone**, R_A and C_A are **nonexpansive** (thus functions)
- **Zeros** of A are **fixed points** of R_A and C_A

Key result we can solve $0 \in A(x)$ by finding fixed points of C_A or R_A

Fixed points of R_A and C_A are zeros of A

Proof

$$R_A = (I + A)^{-1}$$

$x \in \mathbf{fix}\, R_A$

$$\begin{aligned} 0 \in A(x) &\iff x \in (I + A)(x) \\ &\iff (I + A)^{-1}(x) = x \\ &\iff x = R_A(x) \end{aligned}$$

$x \in \mathbf{fix}\, C_A$

$$C_A(x) = 2R_A(x) - I(x) = 2x - x = x \qquad \blacksquare$$

If A is monotone, then R_A is nonexpansive

Proof

If $(x, u) \in \mathbf{gph} R_A$ and $(y, v) \in \mathbf{gph} R_A$, then

$$u + A(u) \ni x, \qquad v + A(v) \ni y$$

Subtract to get $u - v + (A(u) - A(v)) \ni x - y$

Multiply by $(u-v)^T$ and use monotonicity of A (being also a function: $\in \to =$),

$$\|u - v\|^2 \leq (x-y)^T(u-v)$$

Apply Cauchy-Schwarz and divide by $\|u - v\|$ to get

$$\|u - v\| \leq \|x - y\| \qquad \blacksquare$$

If A is monotone, then C_A is nonexpansive

Proof

Given $u = R_A(x)$ and $v = R_A(y)$ (R_A is a function)

$$\begin{aligned}\|C(x) - C(y)\|^2 &= \|(2u - x) - (2v - y)\|^2 \\ &= \|2(u - v) - (x - y)\|^2 \\ &= 4\|u - v\|^2 - 4(u - v)^T(x - y) + \|x - y\|^2 \\ &\leq \|x - y\|^2\end{aligned}$$

Note R_A monotonicity (prev slide): $\|u - v\|^2 \leq (u - v)^T(x - y)$ ■

Remark

R_A is nonexpansive since it is the average of I and C_A:

$$R_A = (1/2)I + (1/2)C_A = (1/2)I + (1/2)(2R_A - 1)$$

Role of maximality

We mostly consider **maximal** operators A because of

Theory: R_A and C_A do not bring iterates outside their domains

Practice: hard to compute R_A and C_A for non-maximal monotone operators, e.g., when $A = \partial f(x)$ where f nonconvex.

Resolvent of subdifferential: proximal operator

$$\mathbf{prox}_f = R_{\partial f} = (I + \partial f)^{-1}$$

Proof

Let $z = \mathbf{prox}_f(x)$, then

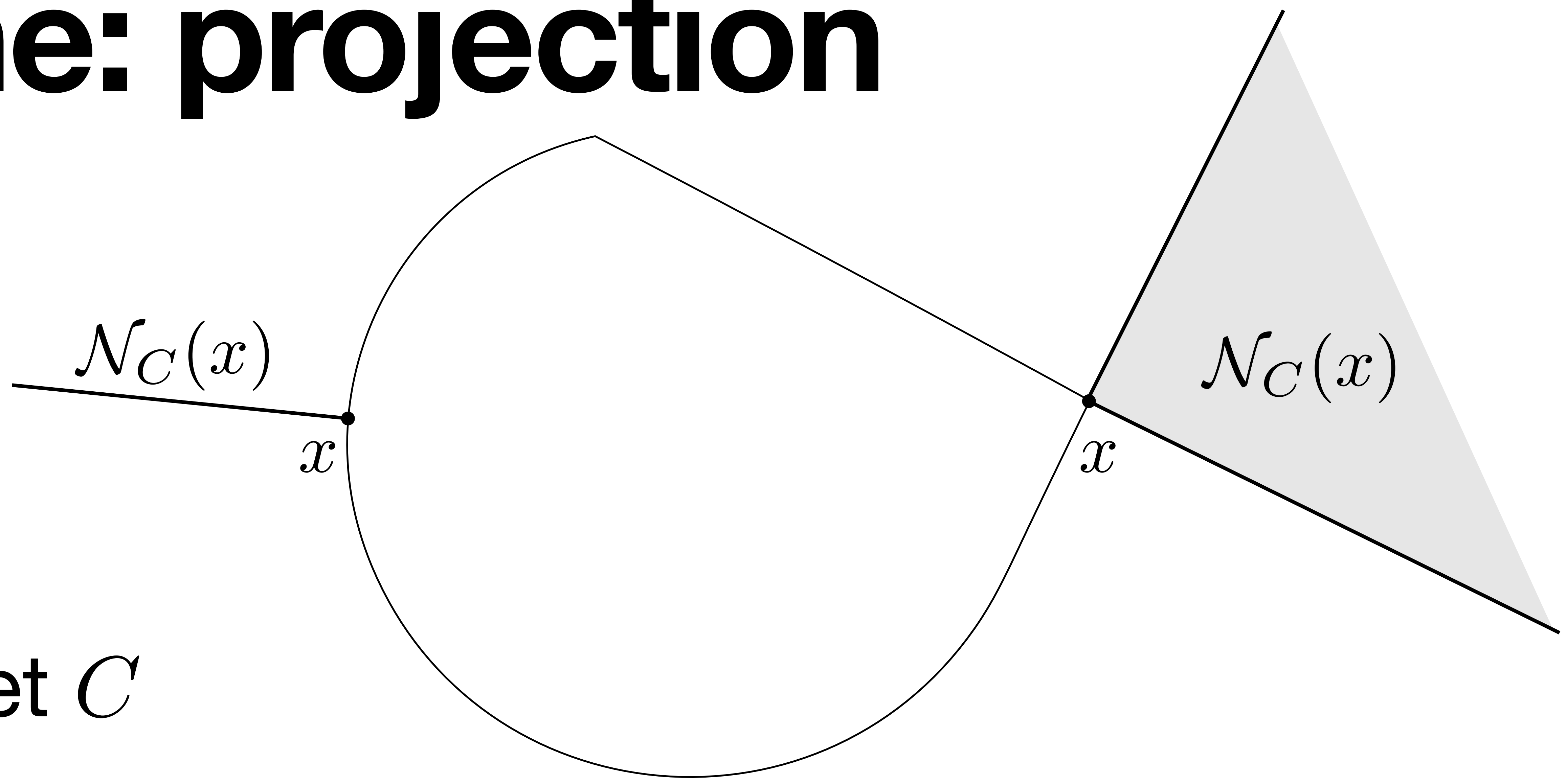
$$z = \underset{u}{\operatorname{argmin}}\, f(u) + \frac{1}{2}\|u - x\|^2$$

$$\iff 0 \in \partial f(z) + z - x \qquad \text{(optimality conditions)}$$

$$\iff x \in (I + \partial f)(z)$$

$$\iff z = (I + \partial f)^{-1}(x) \qquad \blacksquare$$

Resolvent of normal cone: projection



$$R_{\partial \mathcal{I}_C} = \Pi_C(x)$$

Proof

Let $f = \mathcal{I}_C$, the indicator function of a convex set C

Recall: $\partial \mathcal{I}_C(x) = \mathcal{N}_C(x)$ **normal cone operator**

$$u = (I + \partial \mathcal{I}_C)^{-1}(x) \quad \Longleftrightarrow \quad u = \underset{z}{\operatorname{argmin}} \ \mathcal{I}_C(u) + (1/2)\|z - x\|^2 = \Pi_C(x)$$ ■

$\mathcal{N}_C$ monotone $\implies$ Π_C nonexpansive

Proof of monotonicity

$$u \in \mathcal{N}_C(x) \ \Rightarrow \ u^T(z - x) \leq 0, \ \forall z \in C \ \Rightarrow \ u^T(y - x) \leq 0$$

$$v \in \mathcal{N}_C(y) \ \Rightarrow \ v^T(z - y) \leq 0, \ \forall z \in C \ \Rightarrow \ v^T(x - y) \leq 0$$

⟶ add to obtain monotonicity ■

Building contractions

Forward step contractions

Given T L-Lipschitz and μ-strongly monotone, then $I - \gamma T$ converges linearly at rate $\sqrt{1 - 2\gamma\mu + \gamma^2 L^2}$, with optimal step $\gamma = \mu / L^2$.

Proof

$$\begin{aligned}\|(I - \gamma T)(x) - (I - \gamma T)(y)\|^2 &= \|x - y + \gamma T(x) - \gamma T(y)\|^2 \\ &= \|x - y\|^2 - 2\gamma (T(x) - T(y))^T (x - y) + \gamma^2 \|T(x) - T(y)\|^2 \\ &\leq (1 - 2\gamma\mu + \gamma^2 L^2)\|x - y\|^2\end{aligned}$$

strongly monotone

Lipschitz

■

Remarks

- It applies to **gradient descent** with L-smooth and μ-strongly convex f
- Better rate in gradient descent lecture. We can get it by bounding derivative: $\|D(I - \gamma \nabla^2 f(x))\|_2 \leq \max\{|1 - \gamma L|, |1 - \gamma\mu|\}$. Optimal step $\gamma = 2/(\mu + L)$ and factor $(\mu/L - 1)(\mu/L + 1)$.

Resolvent contractions

If A is μ-strongly monotone, then

$$R_A = (I + A)^{-1}$$

is a contraction with Lipschitz parameter $1/(1+\mu)$

Proof

$$\begin{aligned} A\ \mu\text{-strongly monotone} \quad &\Longrightarrow \quad (I + A) \quad (1+\mu)\text{-strongly monotone} \\ &\Longrightarrow \quad R_A = (I+A)^{-1} \quad (1+\mu)\text{-cocoercive} \\ &\Longrightarrow \quad R_A \quad (1/(1+\mu))\text{-Lipschitz} \end{aligned}$$

■

Cayley contractions

If A is μ-strongly monotone and L-Lipschitz, then

$$C_{\gamma A} = 2R_{\gamma A} - I = 2(I + \gamma A)^{-1} - I$$

is a contraction with factor $\sqrt{1 - 4\gamma\mu/(1+\gamma L)^2}$

Remark need also Lipschitz condition

Proof [Page 20, PMO]

If, in addition, $A = \partial f$ where f is CCP, then $C_{\gamma A}$ converges with factor $(\sqrt{\mu/L} - 1)/(\sqrt{\mu/L} + 1)$ and optimal step $\gamma = 1/\sqrt{\mu L}$

Proof
[Linear Convergence and Metric Selection for Douglas-Rachford Splitting and ADMM, Giselsson and Boyd]

Requirements for contractions

	Operator A	**Function** f ($A = \partial f$)
Forward step $I - \gamma A$	μ-strongly monotone	μ-strongly convex L-smooth
Resolvent $R_A = (I + A)^{-1}$	μ-strongly monotone	μ-strongly convex L-smooth
Cayley $C_A = 2(I + A)^{-1} - I$	μ-strongly monotone L-Lipschitz	μ-strongly convex L-smooth

faster convergence

Key to contractions: strong monotonicity/convexity

Operator theory

Today, we learned to:

- **Use** conjugate functions to define duality
- **Relate** subdifferential operator and monotonicity
- **Recognize** monotone operators in optimization problems
- **Apply** operators in algorithms: forward step, resolvent, Cayley
- **Understand requirements** for building contractions

Next lecture

- Operator splitting algorithms