ORF522 – Linear and Nonlinear Optimization

17. Operator theory I

Bartolomeo Stellato — Fall 2022

Today's lecture

[Chapter 4, FMO][PA][PMO][LSMO]

Operator theory I

- Proximal gradient method
- Operators
- Monotone and cocoercive operators
- Fixed-point Iterations

Proximal gradient method

Remember: gradient descent interpretation

Problem

minimize $\quad f(x)$

Iterations

$$x^{k+1} = x^k - t\nabla f(x^k)$$

Quadratic approximation, replacing Hessian $\nabla^2 f(x^k)$ with $\dfrac{1}{t}I$

$$x^{k+1} = \underset{z}{\text{argmin}}\ f(x^k) + \nabla f(x^k)^T(z - x^k) + \frac{1}{2t}\|z - x^k\|_2^2$$

Let’s exploit the smooth part

minimize $\quad f(x) + g(x)$

$f(x)$ convex and smooth

$g(x)$ convex (may be not differentiable)

Quadratic approximation of f while keeping g

$$x^{k+1} = \underset{z}{\text{argmin}}\; g(z) + f(x^k) + \nabla f(x^k)^T(z - x^k) + \frac{1}{2t}\|z - x^k\|_2^2 \longleftarrow$$

same as gradient descent

Equivalent to

Proximal operator

$$x^{k+1} = \underset{z}{\text{argmin}}\; \underbrace{tg(z)}_{\uparrow} + \underbrace{\frac{1}{2}\left\|z - (x^k - t\nabla f(x^k))\right\|_2^2}_{\uparrow} = \mathbf{prox}_{tg}\left(x^k - t\nabla f(x^k)\right)$$

make g small

stay close to gradient update

Proximal gradient method

minimize $\quad f(x) + g(x)$

$f(x)$ convex and smooth
$g(x)$ convex (may be not differentiable)

Iterations

$$x^{k+1} = \mathbf{prox}_{tg}\left(x^k - t\nabla f(x^k)\right)$$

Properties

- Alternates between gradient updates of f and proximal updates on g
- Useful if $\mathbf{prox}_{tg}$ is inespensive
- Can handle nonsmooth and constrained problems

Special cases

Generalized gradient descent

Problem

minimize $f(x) + g(x)$

Iterations

$$x^{k+1} = \mathbf{prox}_{tg}\left(x^k - t\nabla f(x^k)\right)$$

Smooth

$g(x) = 0 \quad \Longrightarrow \quad \mathbf{prox}_{tg}(x) = x$

Gradient descent

$\Longrightarrow \quad x^{k+1} = x^k - t\nabla f(x^k)$

Constraints

$g(x) = \mathcal{I}_C(x) \quad \Longrightarrow \quad \mathbf{prox}_{tg}(x) = \Pi_C(x)$

Projected gradient descent

$\Longrightarrow \quad x^{k+1} = \Pi_C(x^k - t\nabla f(x^k))$

Non smooth

$f(x) = 0$

Proximal minimization

$\Longrightarrow \quad x^{k+1} = \mathbf{prox}_{tg}(x^k)$

Note: useful if $\mathbf{prox}_{tg}$ is cheap

What happens if we cannot evaluate the prox?

At every iteration, it can be very expensive to evaluate

$$\mathbf{prox}_g(x) = \underset{z}{\operatorname{argmin}} \left(g(z) + \frac{1}{2}\|z - x\|_2^2 \right)$$

Idea: solve it approximately!

If you precisely control the $\mathbf{prox}_g(x)$ evaluation errors
you can obtain the same convergence guarantees (and rates)
as the exact evaluations.

[Schmidt et al. (2011), “Convergence rates of inexact proximal-gradient methods for convex optimization”]

Example: Lasso

Iterative Soft Thresholding Algorithm (ISTA)

$$\text{minimize} \quad \underset{f(x)}{(1/2)\|Ax - b\|_2^2} + \underset{g(x)}{\lambda\|x\|_1}$$

Proximal gradient descent

$$x^{k+1} = \mathbf{prox}_{tg}\left(x^k - t\nabla f(x^k)\right)$$

$$\nabla f(x) = A^T(Ax - b)$$

$$\mathbf{prox}_{tg}(x) = S_{\lambda t}(x)$$

(component wise soft-thresholding)

Closed-form iterations

$$x^{k+1} = S_{\lambda t}\left(x^k - tA^T(Ax^k - b)\right)$$

Example: Lasso

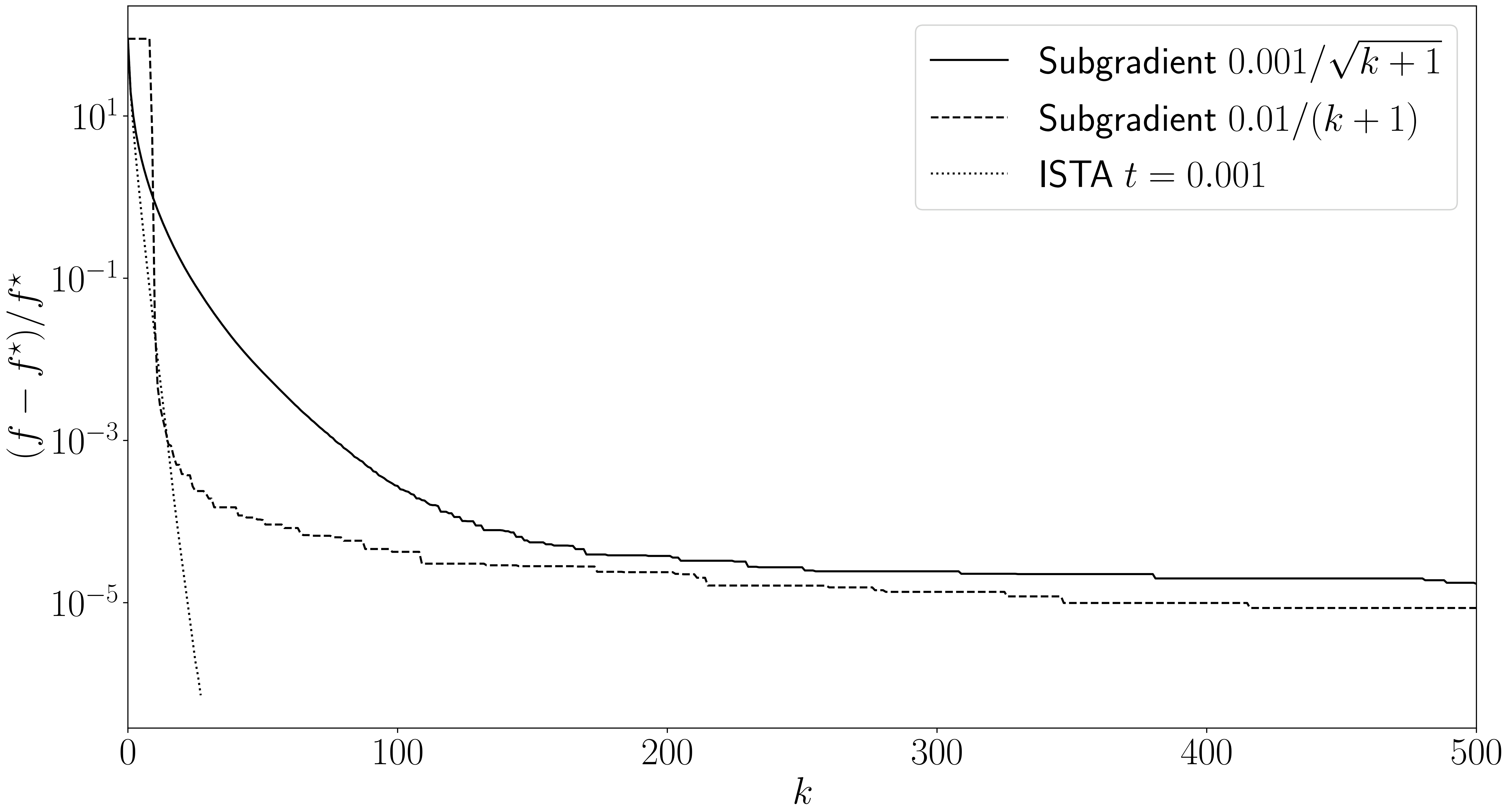
Iterative Soft Thresholding Algorithm (ISTA)

$A \in \mathbf{R}^{500 \times 100}$

$$\text{minimize} \quad (1/2)\|Ax - b\|_2^2 + \lambda\|x\|_1$$

Closed-form iterations

$$x^{k+1} = S_{\lambda t}\left(x^k - tA^T(Ax^k - b)\right)$$



Better convergence

Can we prove convergence generally?

Can we combine different operators?

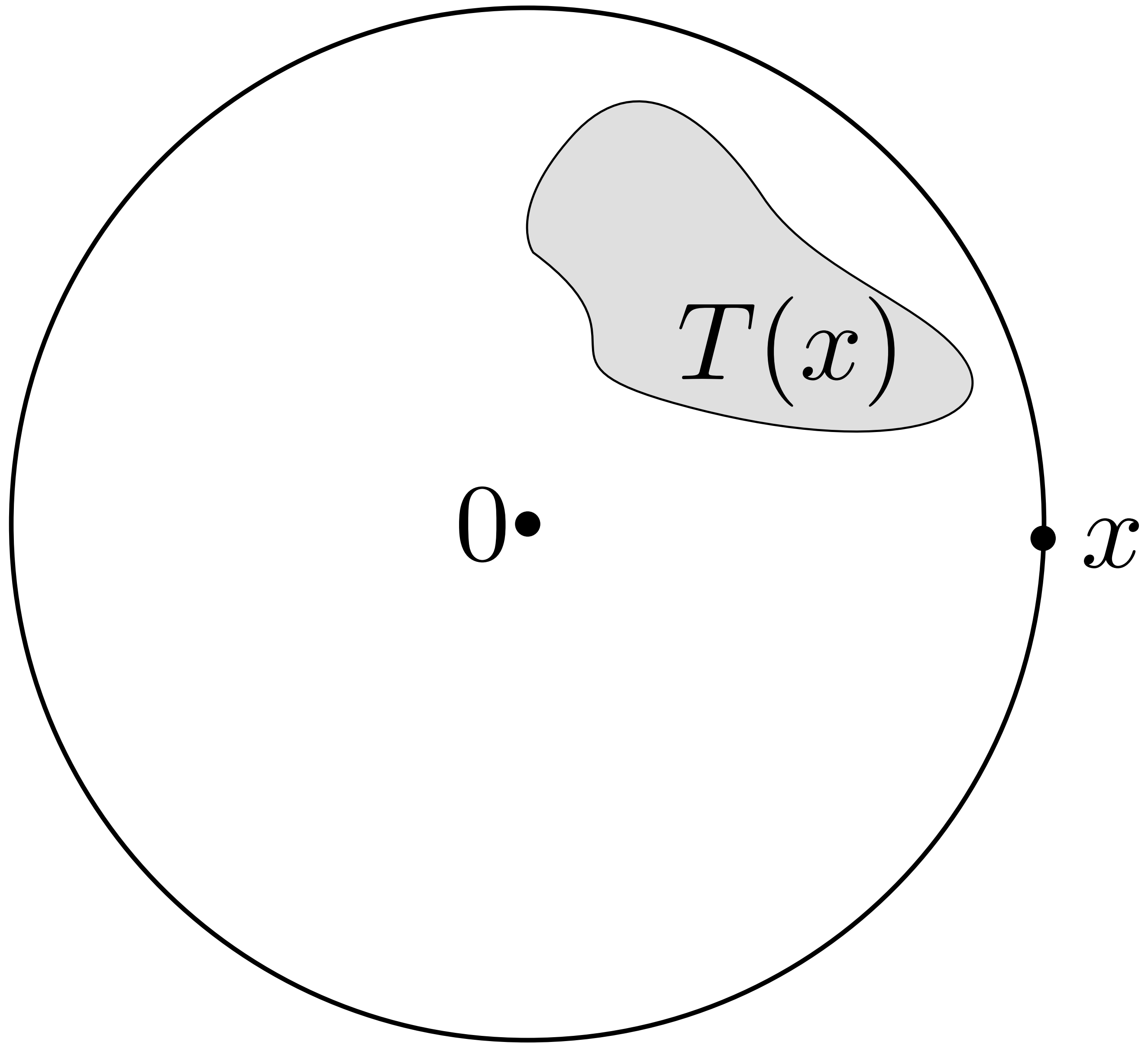
Operators

Operators

An operator T maps each point in $\mathbf{R}^n$ to a subset of $\mathbf{R}^n$

- **set valued** $T(x)$ returns a set
- **single-valued** $T(x)$ (function) returns a singleton

The **domain** of T is the set $\mathbf{dom}\, T = \{x \mid T(x) \neq \emptyset\}$



Example

- The subdifferential ∂f is a set-valued operator
- The gradient ∇f is a single-valued operator

Graph and inverse operators

Graph

The graph of an operator T is defined as

$$\mathbf{gph}T = \{(x,y) \mid y \in T(x)\}$$

In other words, all the pairs of points (x,y) such that $y \in T(x)$.

Inverse

The graph of the inverse operator T^{-1} is defined as

$$\mathbf{gph}T^{-1} = \{(y,x) \mid (x,y) \in \mathbf{gph}T\}$$

Therefore, $y \in T(x)$ if and only if $x \in T^{-1}(y)$.

Zeros

Zero

x is a **zero** of T if $\quad 0 \in T(x)$

Zero set

The set of all the zeros $\quad T^{-1}(0) = \{x \mid 0 \in T(x)\}$

Example
If $T = \partial f$ and $f : \mathbf{R}^n \to \mathbf{R}$, then
$0 \in T(x)$ means that x minimizes f

Many problems
can be posed as finding zeros
of an operator

Fixed points

$\bar{x}$ is a **fixed-point** of a single-valued operator T if

$$\bar{x} = T(\bar{x})$$

Set of fixed points $\quad \mathbf{fix}\, T = \{x \in \mathbf{dom}\, T \mid x = T(x)\} = (I - T)^{-1}(0)$

Examples

- **Identity** $T(x) = x$. Any point is a fixed point
- **Zero operator** $T(x) = 0$. Only 0 is a fixed point

Lipschitz operators

An operator T is L-Lipschitz if

$$\|T(x) - T(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbf{dom}\, T$$

Fact If T is Lipschitz, then it is single-valued

Proof If $y = T(x), z = T(x)$, then $\|y - z\| \leq L\|x - x\| = 0 \Longrightarrow y = z$ ■

For $L = 1$ we say T is **nonexpansive**

For $L < 1$ we say T is **contractive** (with contraction factor L)

Lipschitz operators examples

Lipschitz affine functions

$T(x) = Ax + b$ $\longleftrightarrow$

maximum singular value

$L = \|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$

Lipschitz differentiable functions

T such that there exists derivative DT $\longleftrightarrow$

derivative is bounded

$\|DT\|_2 \leq L$

Lipschitz operators and fixed points

Given a L-Lipschitz operator T and a fixed point $\bar{x} = T\bar{x}$,

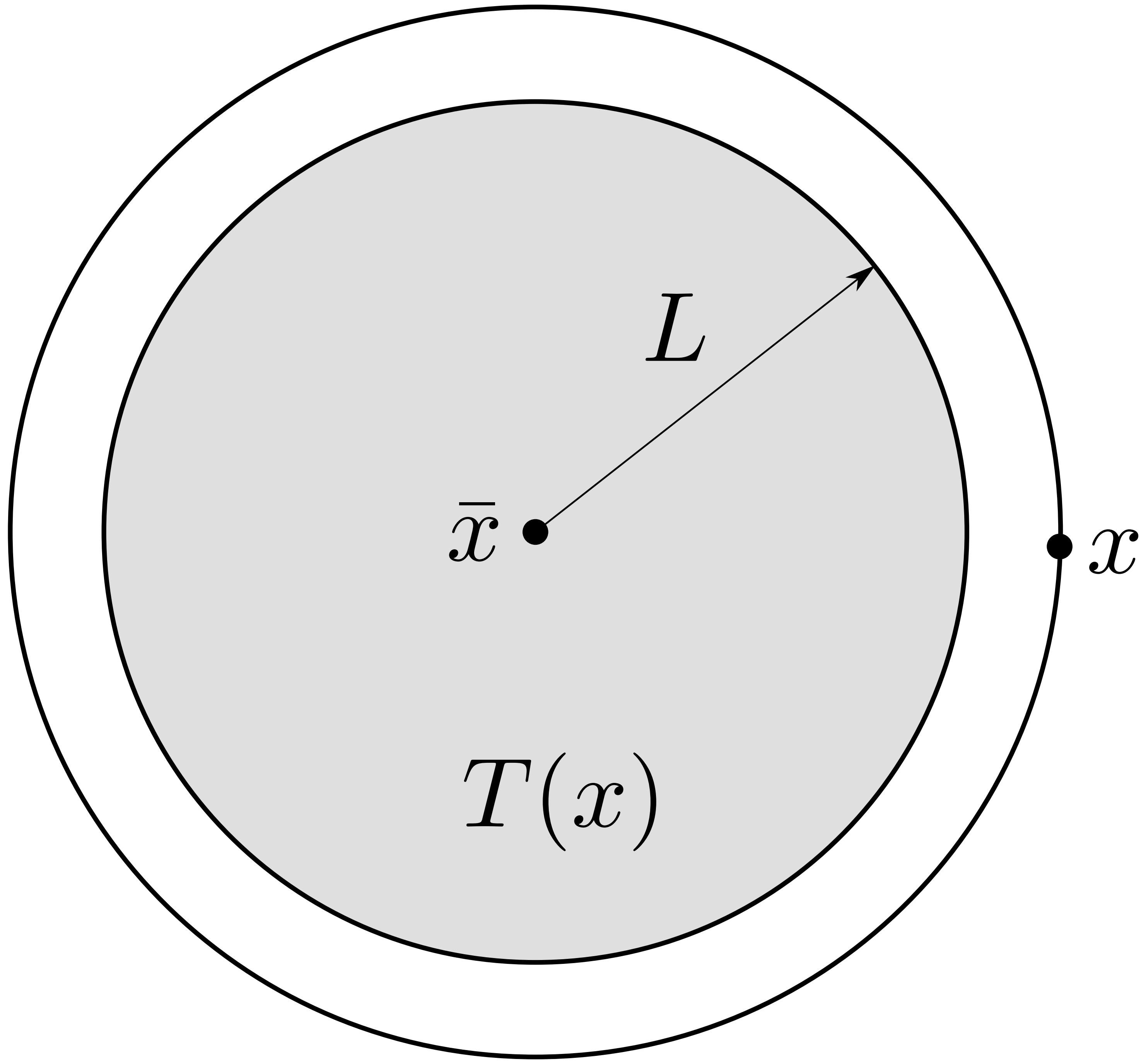
$$\|Tx - \bar{x}\| = \|Tx - T\bar{x}\| \leq L\|x - \bar{x}\|$$

A contractive operator ($L < 1$) can have at most one fixed point, i.e., $\mathbf{fix}\, T = \{\bar{x}\}$

Proof

If $\bar{x}, \bar{y} \in \mathbf{fix}\, T$ and $\bar{x} \neq \bar{y}$ then

$\|\bar{x} - \bar{y}\| = \|T(\bar{x}) - T(\bar{y})\| < \|\bar{x} - \bar{y}\|$ (contradiction) ■



A nonexpansive operator ($L = 1$) need not have a fixed point

Example $T(x) = x + 2$

Combining Lipschitz operators

T_1 is L_1-Lipschitz and T_2 is L_2-Lipschitz

The **composition** T_1T_2 is L_1L_2-Lipschitz

Proof $\|T_1T_2x - T_1T_2y\|_2 \leq L_1\|T_2x - T_2y\|_2 \leq L_1L_2\|x-y\|_2$ $\blacksquare$

- Composition of *nonexpansive* is nonexpansive
- Composition of *nonexpansive* and *contractive* is contractive

The **weighted average** $\theta T_1 + (1-\theta)T_2,\ \theta \in (0,1)$ is $(\theta L_1 + (1-\theta)L_2)$-Lipschitz

Proof (exercise)

- Weighted average of *nonexpansive* is nonexpansive
- Weighted average of *nonexpansive* and *contractive* is contractive

Monotone cocoercive operators

Monotone operators

An operator T on $\mathbf{R}^n$ is **monotone** if

$$(u-v)^T(x-y) \geq 0, \quad \forall (x,u),(y,v) \in \mathbf{gph} T$$

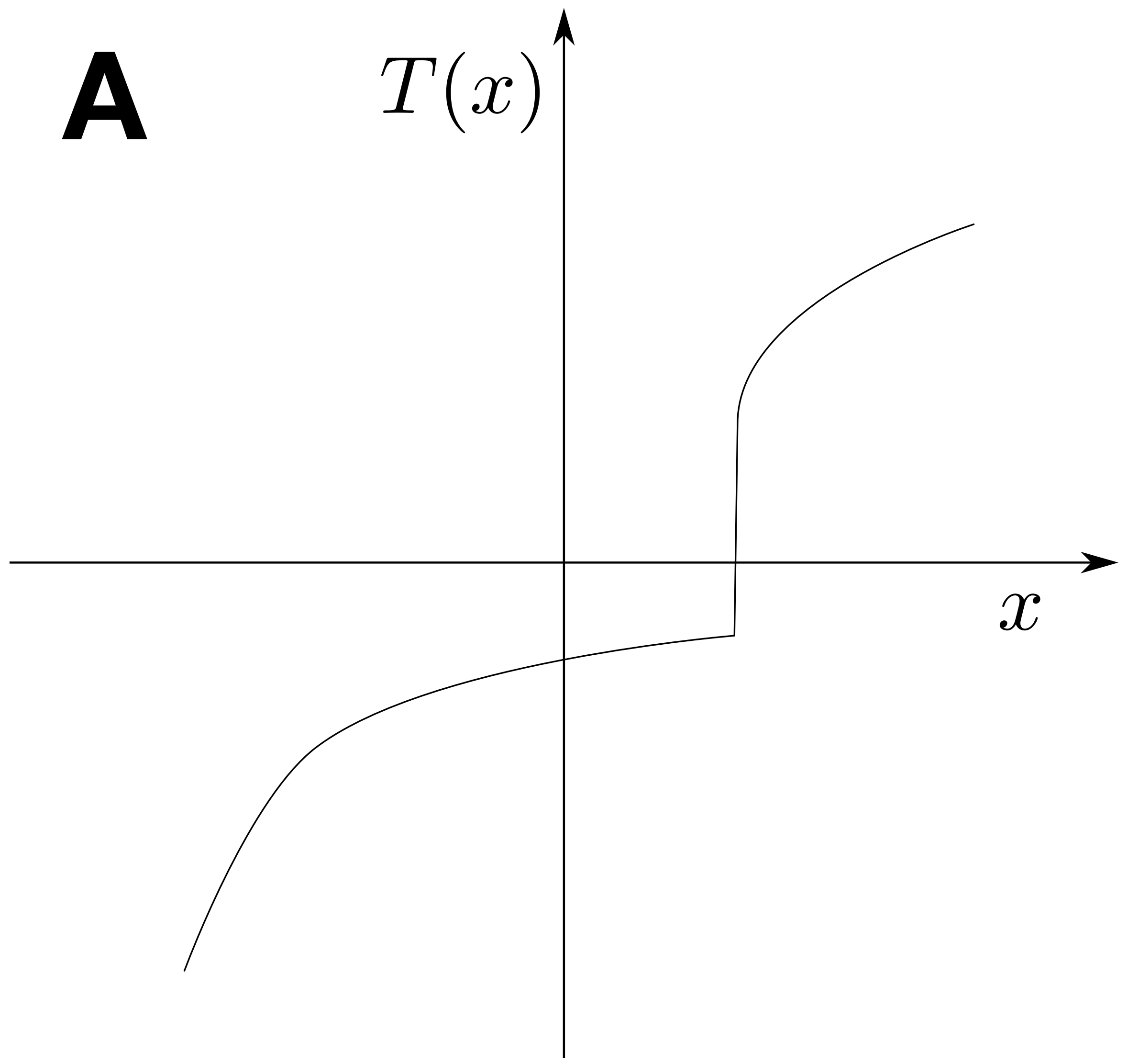
T is **maximal monotone** if
$\nexists (\bar{x},\bar{u}) \notin \mathbf{gph} T$ such that

$$(\bar{u}-u)^T(\bar{x}-x) \geq 0$$

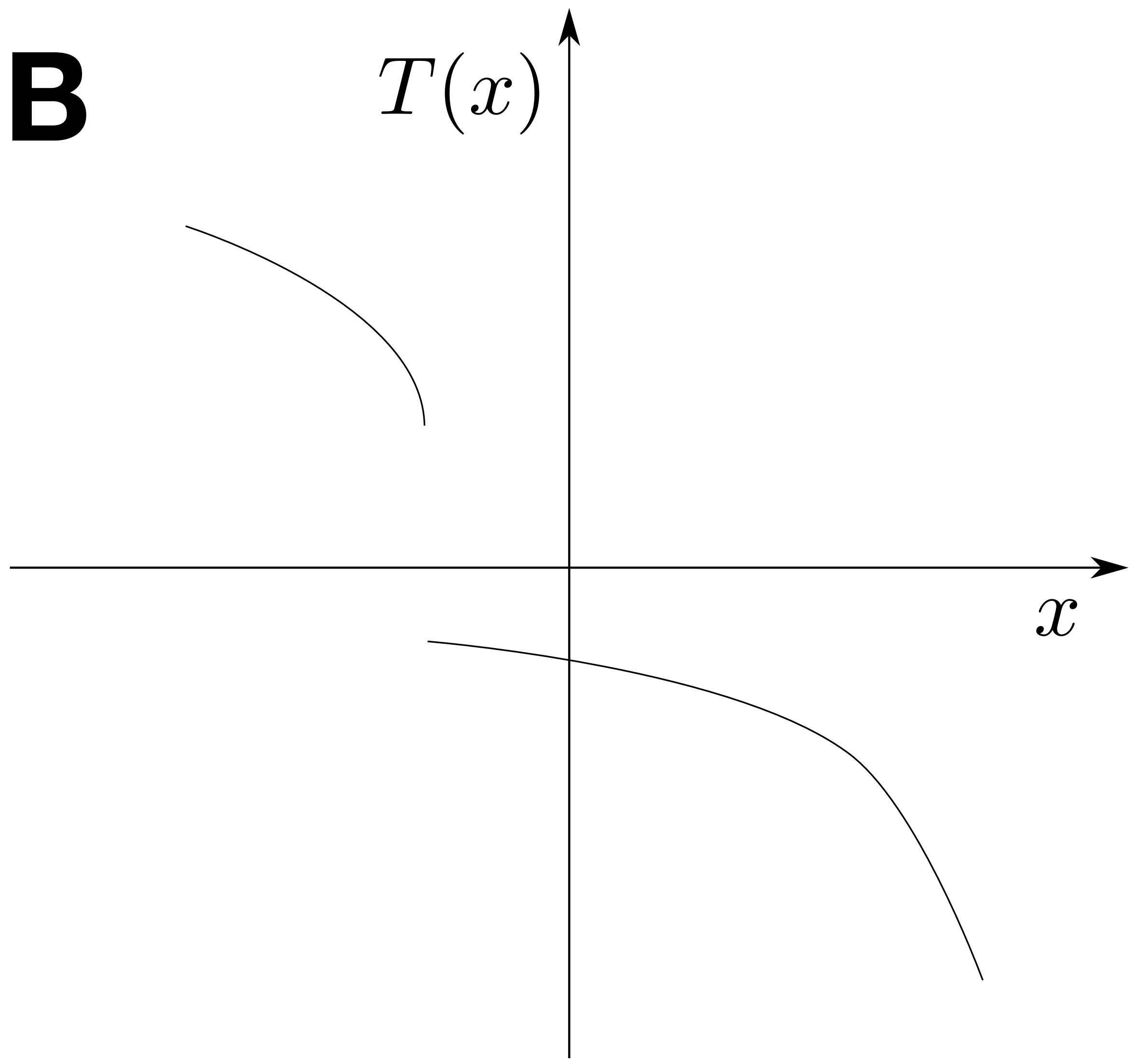
Equivalently: $\nexists$ monotone R
such that $\mathbf{gph} T \subset \mathbf{gph} R$

Monotone operators in 1D

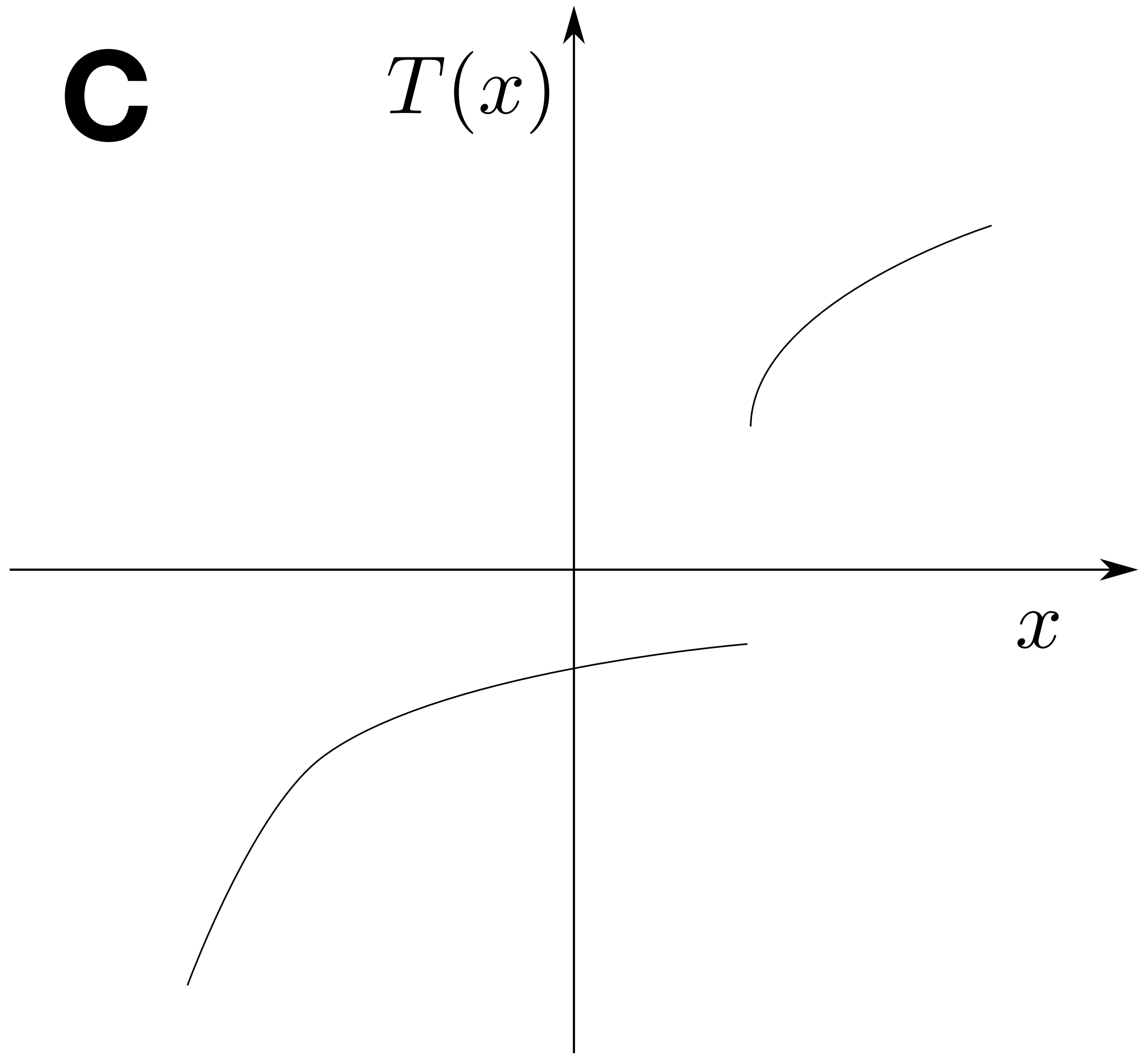
A



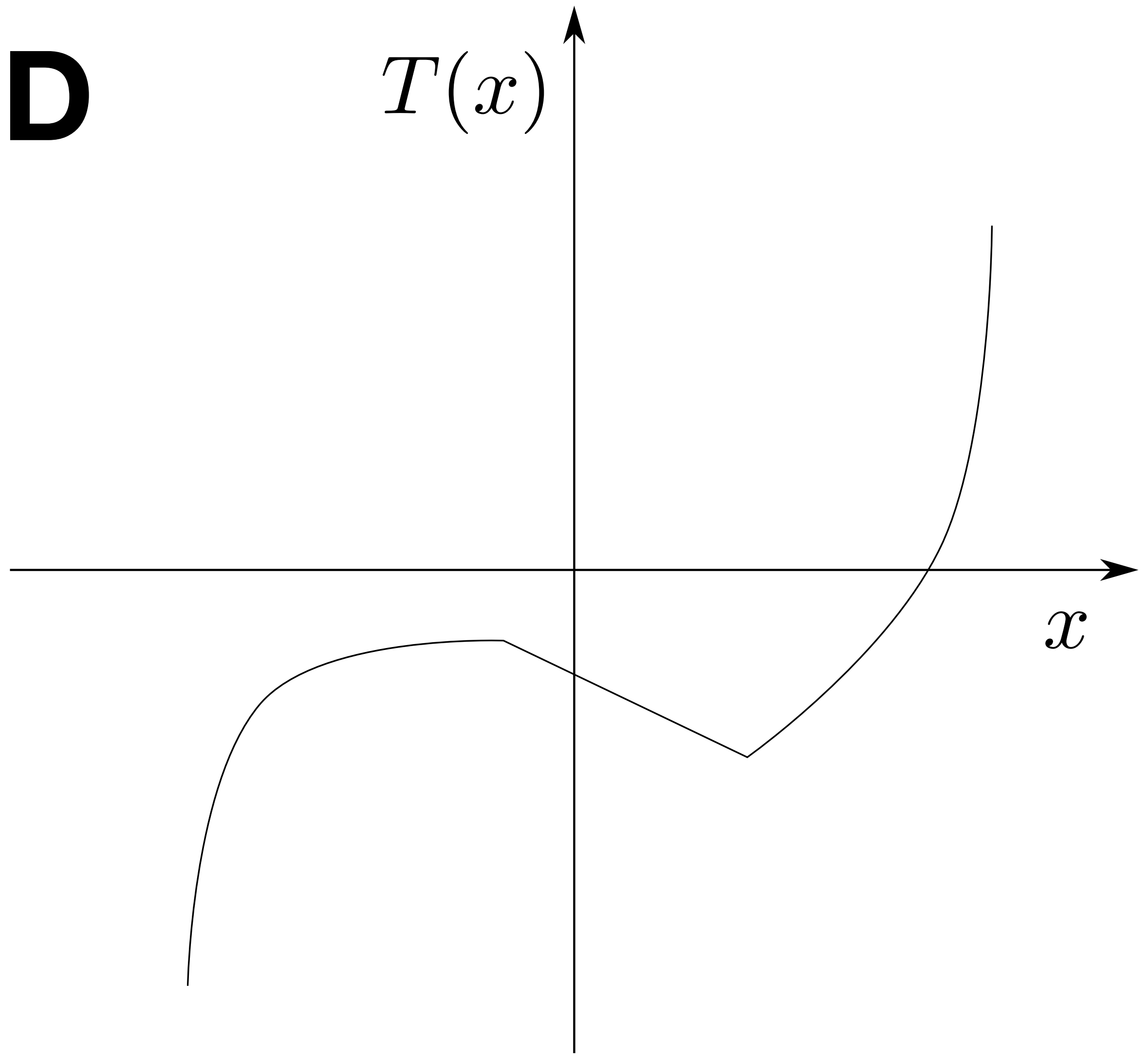
B



C



D



Let's fill the table

	Monotone	Max Monotone
A		
B		
C		
D		

Monotonicity

$$y > x \quad \Rightarrow \quad T(y) \geq T(x)$$

Continuity

If T single-valued, continuous and monotone, then it's maximal monotone

Monotone operator properties

- **sum** $T + R$ is monotone
- **nonnegative scaling** αT with $\alpha \geq 0$ is monotone
- **inverse** T^{-1} is monotone
- **congruence** for $M \in \mathbf{R}^{n \times m}$, then $M^T T(Mz)$ is monotone on $\mathbf{R}^m$

Affine function $T(x) = Ax + b$ is maximal monotone

$$\iff \quad A + A^T \succeq 0$$

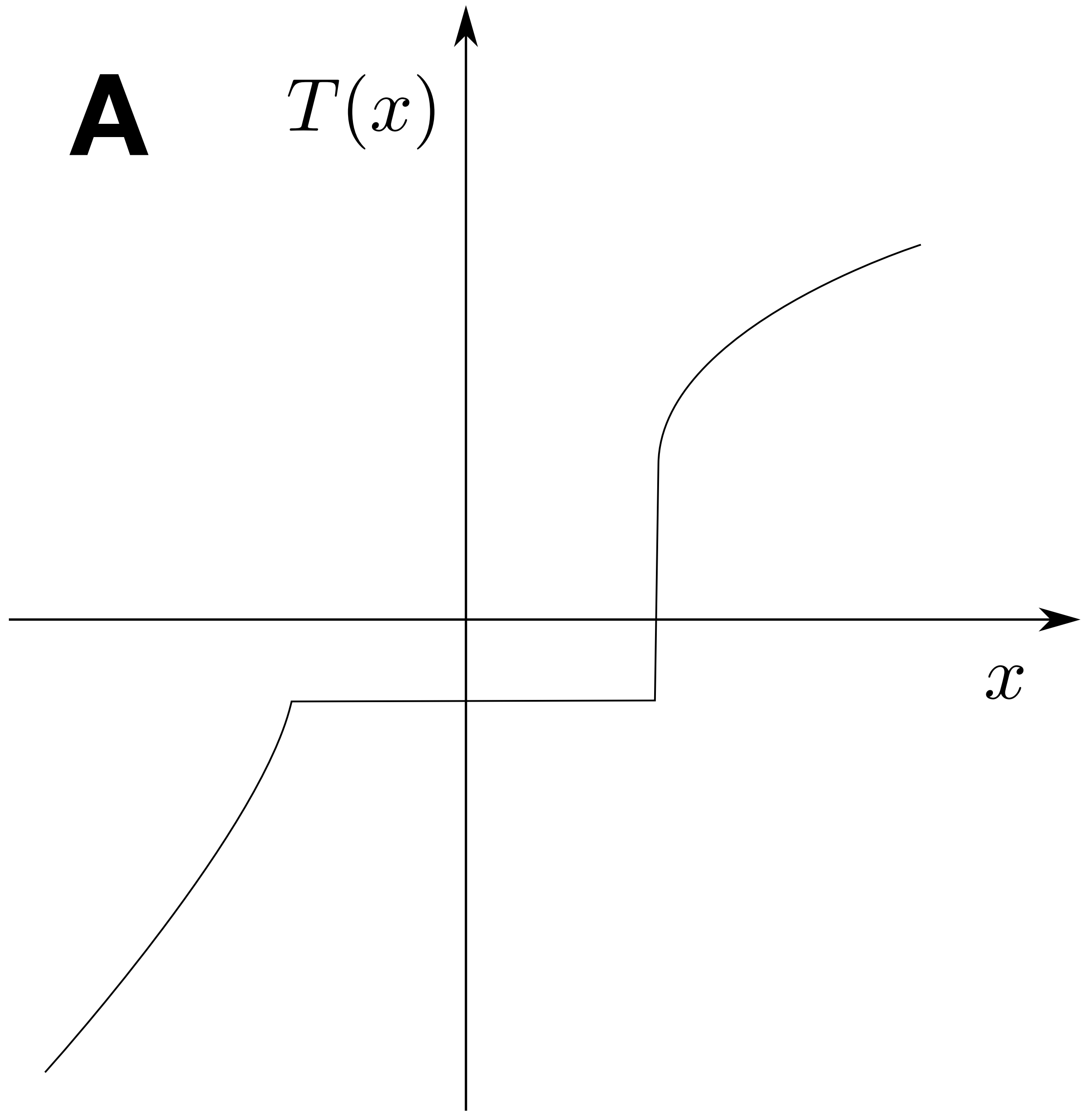
Strongly monotone operators

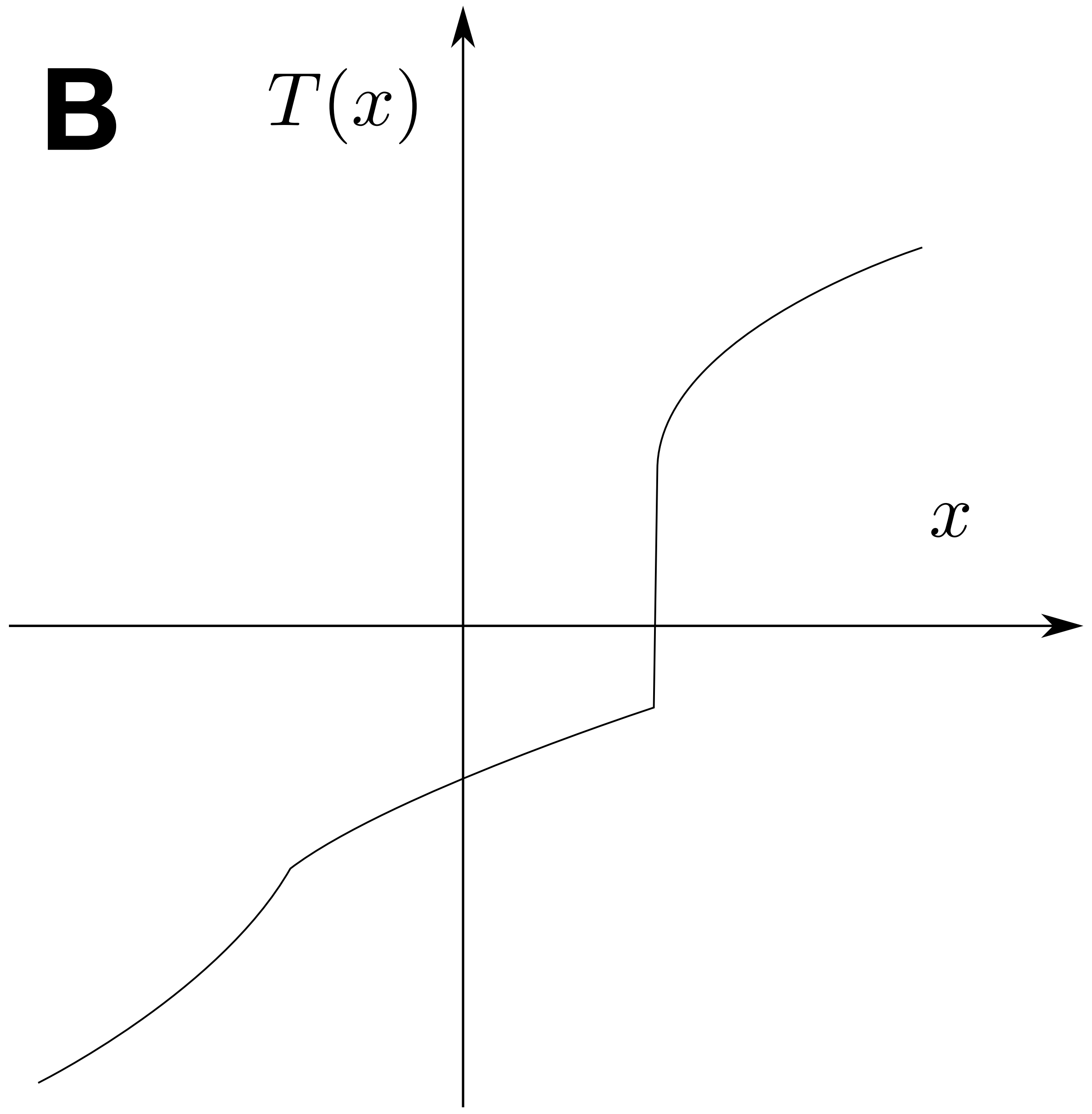
An operator T on $\mathbf{R}^n$ is μ-**strongly monotone** if

$$(u-v)^T(x-y) \geq \mu\|x-y\|^2, \quad \mu > 0$$

(also called μ-**coercive**)

$\forall (x,u),(y,v) \in \mathbf{gph} T$





Let's fill the table

	Monotone	Strongly Monotone
A		
B		

The slope is at least μ

Cococercive operators

An operator T is β**-cocoercive**, $\beta > 0$, if

$$(T(x) - T(y))^T(x - y) \geq \beta\|T(x) - T(y)\|^2$$

If T is β**-cocoercive**, then T is $(1/\beta)$**-Lipschitz**

Proof $\beta\|T(x) - T(y)\|^2 \leq (T(x) - T(y))^T(x - y) \leq \|T(x) - T(y)\|\|x - y\|$

$$\implies \|T(x) - T(y)\| \leq (1/\beta)\|x - y\| \qquad \blacksquare$$

If T is μ**-strongly monotone** if and only if T^{-1} is μ**-cocoercive**

Proof $(T(x) - T(x))^T(x - y) \geq \mu\|x - y\|^2$

Inverse: $u = T(x)$ and $v = T(y)$ if and only if $x \in T^{-1}(u)$ and $y \in T^{-1}(v)$

$$(u - v)^T(T^{-1}(u) - T^{-1}(v)) \geq \mu\|T^{-1}(u) - T^{-1}(v)\|^2 \qquad \blacksquare$$

Cocoercive and nonexpansive operators

If T is β-**cococercive** if and only if $I - 2\beta T$ is **nonexpansive**

Proof $\|(I-2\beta T)(y) - (I - 2\beta T)(x)\|^2 =$

$$= \|y - 2\beta T(y) - x - 2\beta T(x)\|^2$$

$$= \|y - x\|^2 - 4\beta (T(y) - T(x))^T (y - x) + 4\beta^2 \|T(y) - T(x)\|^2$$

$$= |y - x\|^2 - 4\beta \left((T(y) - T(x))^T (y - x) - \beta \|T(y) - T(x)\|^2\right)$$

$$\leq \|y - x\|^2 \quad \blacksquare$$

(cocoercive)

Summary of monotone and cocoercive operators

$$\begin{tikzcd}
\begin{array}{c}\textbf{Monotone}\\ (T(x)-T(y))^T(x-y)\geq 0\end{array} & \begin{array}{c}\textbf{Lipschitz}\\ \|F(x)-F(y)\|\leq L\|x-y\|\end{array} \\
\begin{array}{c}\textbf{Strongly monotone}\\ (T(x)-T(y))^T(x-y)\geq \mu\|x-y\|^2\end{array} \arrow[u, "\mu=0"'] \arrow[r, leftrightarrow, "F=T^{-1}"'] & \begin{array}{c}\textbf{Cocoercive}\\ (F(x)-F(y))^T(x-y)\geq \mu\|F(x)-F(y)\|^2\end{array} \arrow[u, "L=1/\mu"'] \arrow[d, leftrightarrow, "G=I-2\mu F"'] \\
 & \begin{array}{c}\textbf{Nonexpansive}\\ \|G(x)-G(y)\|\leq\|x-y\|\end{array}
\end{tikzcd}$$

Fixed point iterations

Fixed point iteration

Apply operator

$$x^{k+1} = T(x^k)$$

until you reach $\bar{x} \in \mathbf{fix}\, T$

Main approach

1. Find a suitable T such that $\bar{x} \in \mathbf{fix}\, T$ solve your problem
2. Show that the fixed point iteration converges

Fixed point residual to terminate

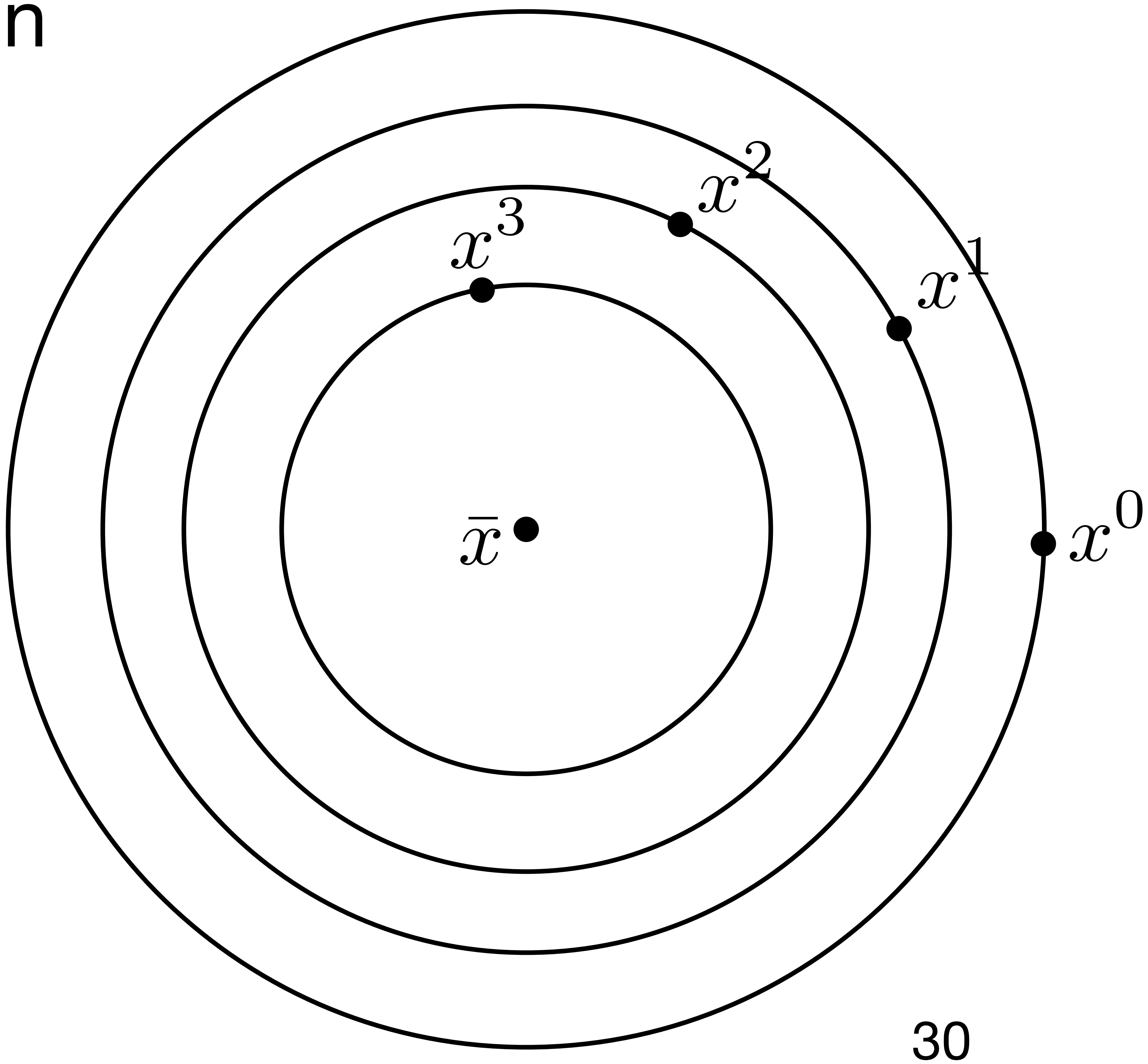
$$r^k = T(x^k) - x^k$$

Contractive fixed point iterations

Contraction mapping theorem
If T is L-Lipschitz with $L < 1$ (contraction), the iteration

$$x^{k+1} = T(x^k)$$

converges to $\bar{x}$, the unique fixed point of T



Properties

- Distance to $\bar{x}$ decreases at each step

  $$\|x^{k+1} - \bar{x}\| \leq L\|x^k - \bar{x}\|$$

  (iteration is **Fejer monotone**)

- Linear convergence rate L

Contraction mapping theorem

Proof

The sequence x^k is Cauchy

$$
\begin{aligned}
\|x^{k+\ell} - x^k\| &\leq \|x^{k+\ell} - x^{k+\ell-1}\| + \cdots + \|x^{k+1} - x^k\| \\
&\leq (L^{\ell-1} + \cdots + 1)\|x^{k+1} - x^k\| && \text{(Lipschitz constant)} \\
&\leq \frac{1}{1-L}\|x^{k+1} - x^k\| && \text{(geometric series)} \\
&\leq \frac{L^k}{1-L}\|x^1 - x^0\| && \text{(Lipschitz constant)}
\end{aligned}
$$

Therefore it converges to a point $\bar{x}$ which must be the (unique) fixed point of T

The convergence is linear (geometric) with rate L

$$\|x^k - \bar{x}\| = \|T(x^{k-1}) - T(\bar{x})\| \leq L\|x^{k-1} - \bar{x}\| \leq L^k\|x^0 - x^\star\|$$

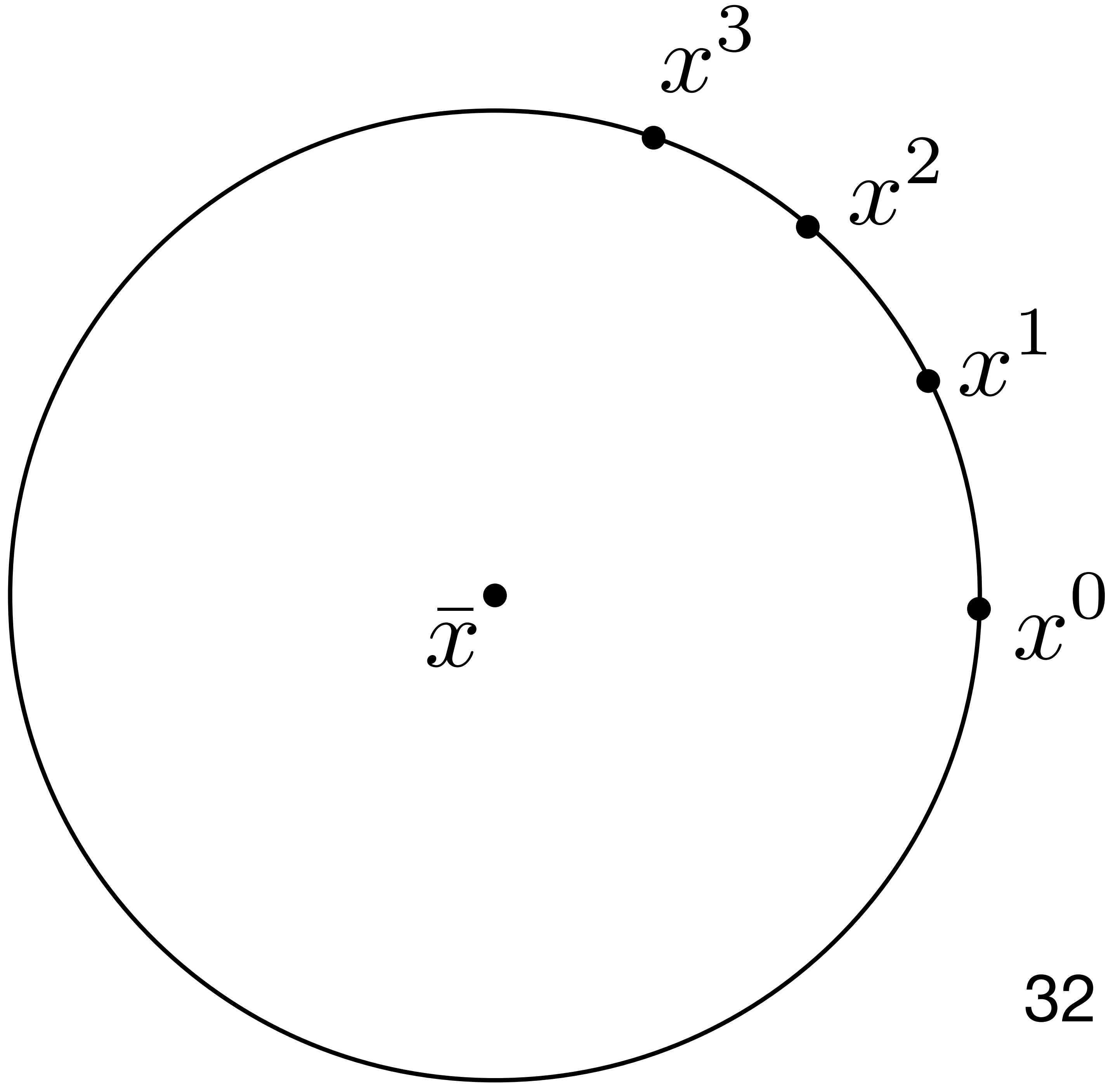
■

Nonexpansive fixed point iterations

If T is L-Lipschitz with $L = 1$ (nonexpansive), the iteration

$$x^{k+1} = T(x^k)$$

need not converge to a fixed point, even if one exists.



Example

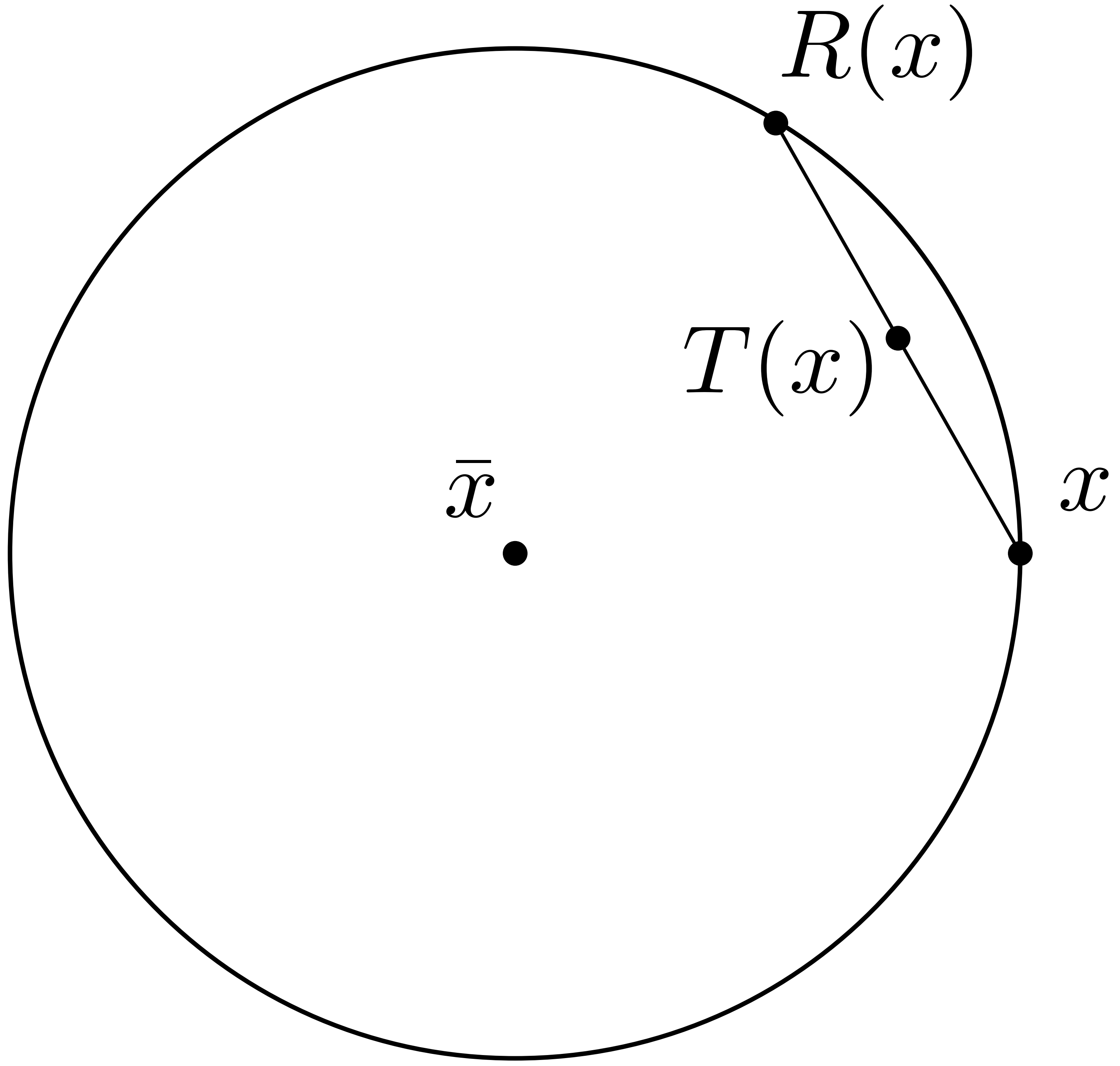
- Let T be a rotation around the origin
- T is nonexpansive and has a fixed point $\bar{x} = 0$
- $\|x^k\|$ never decreases

Averaged operators

We say that an operator T is $\alpha-$**averaged** with $\alpha \in (0, 1)$ if

$$T = (1 - \alpha)I + \alpha R$$

and R is nonexpansive.



Averaged operators fixed points

We say that an operator T is $\alpha-$**averaged** with $\alpha \in (0,1)$ if

$$T = (1-\alpha)I + \alpha R$$

Fact If T is α-averaged, then $\mathbf{fix}\, T = \mathbf{fix}\, R$

Proof

$$\begin{aligned} \bar{x} = T(\bar{x}) &= (1-\alpha)I(\bar{x}) + \alpha R(\bar{x}) \\ &= (1-\alpha)\bar{x} + \alpha R(\bar{x}) \\ \iff \quad & \alpha\bar{x} = \alpha R(\bar{x}) \\ \iff \quad & \bar{x} = R(\bar{x}) \end{aligned}$$

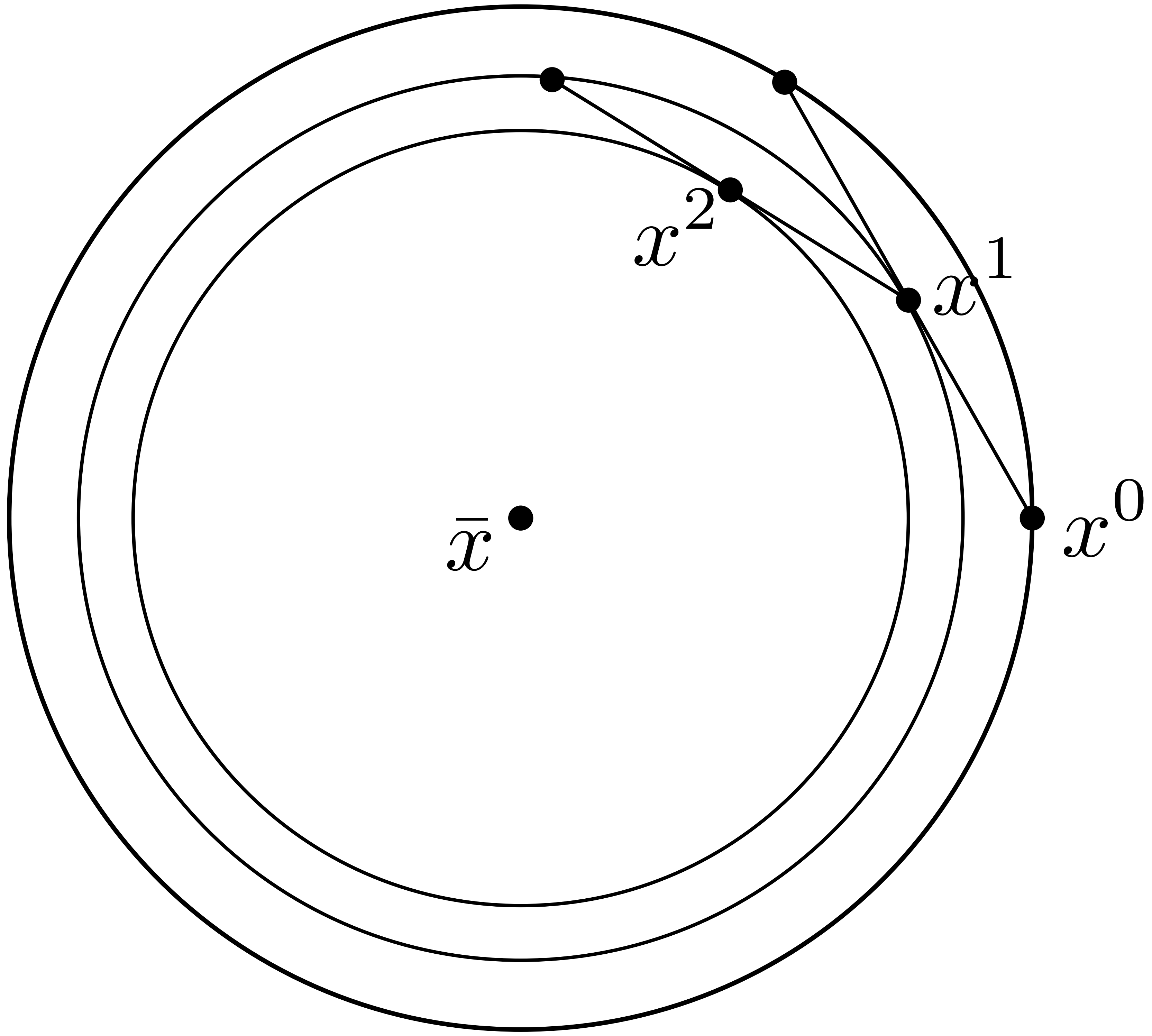
■

Averaged fixed point iterations

If $T = (1-\alpha)I + \alpha R$ is α-averaged
($\alpha \in (0,1)$ and R nonexpansive), the iteration

$$x^{k+1} = T(x^k)$$

converges to $\bar{x} \in \mathbf{fix}\, T$



(also called damped, averaged
or Mann-Krasnosel'skii iteration)

Properties

- Distance to $\bar{x}$ decreases at each step (**Fejer monotone**)
- Sublinear convergence to fixed-point residual

$$\|R(x^k) - x^k\| \leq \frac{1}{\sqrt{(k+1)\alpha(1-\alpha)}}\|x^0 - \bar{x}\|$$

Averaged fixed point iterations

Proof

Use the identity (proof by expanding)

$$\|(1-\alpha)a+\alpha b\|^2 = (1-\alpha)\|a\|^2 + \alpha\|b\|^2 - \alpha(1-\alpha)\|a-b\|^2$$

and apply it to

$$x^{k+1} - \bar{x} = (1-\alpha)(\underbrace{x^k - \bar{x}}_{a}) + \alpha(\underbrace{R(x^k) - \bar{x}}_{b})$$

obtaining

$$\begin{aligned}
\|x^{k+1} - \bar{x}\|^2 &= (1-\alpha)\|x^k - \bar{x}\|^2 + \alpha\|R(x^k) - \bar{x}\|^2 - \alpha(1-\alpha)\|x^k - R(x^k)\|^2 \\
&\leq (1-\alpha)\|x^k - \bar{x}\|^2 + \alpha\|x^k - \bar{x}\|^2 - \alpha(1-\alpha)\|x^k - R(x^k)\|^2 \quad \text{(nonexpansive)} \\
&= \|x^k - \bar{x}\|^2 \underbrace{- \alpha(1-\alpha)\|x^k - R(x^k)\|^2}_{\leq 0}
\end{aligned}$$

Iterations are Fejer monotone

Averaged fixed point iterations

Proof (continued)

iterate righthand side over k steps

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^0 - \bar{x}\|^2 - \alpha(1-\alpha)\sum_{i=0}^{k}\|x^i - R(x^i)\|^2$$

Since $\|x^{k+1} - \bar{x}\|^2 \geq 0$, we have $\displaystyle\sum_{i=0}^{k}\|x^i - R(x^i)\|^2 \leq \frac{1}{\alpha(1-\alpha)}\|x^0 - \bar{x}\|^2$

Using $\displaystyle\sum_{i=0}^{k}\|x^i - R(x^i)\|^2 \geq (k+1)\min_{i=0,\dots,k}\|x^i - R(x^i)\|^2$, we obtain

$$\min_{i=0,\dots,k}\|x^i - R(x^i)\|^2 \leq \frac{1}{(k+1)\alpha(1-\alpha)}\|x^0 - \bar{x}\|^2$$

(R is nonexpansive → $\min$ at k) $\displaystyle\|x^k - R(x^k)\|^2 \leq \frac{1}{(k+1)\alpha(1-\alpha)}\|x^0 - \bar{x}\|^2$ ■

Average fixed point iteration convergence rates

$$\|R(x^k) - x^k\| \leq \frac{1}{\sqrt{(k+1)\alpha(1-\alpha)}}\|x^0 - \bar{x}\|$$

Righthand side minimized when $\alpha = 1/2$

$$\|R(x^k) - x^k\| \leq \frac{2}{\sqrt{k+1}}\|x^0 - \bar{x}\|$$

Iterations

$$x^{k+1} = (1/2)x^k + (1/2)R(x^k)$$

Remarks

- Sublinear convergence (same as subgrad method), in general not the actual rate
- $\alpha = 1/2$ is very common for averaged operators

How to design an algorithm

Problem

minimize $\quad f(x)$

Algorithm (operator) construction

1. Find a suitable T such that $\bar{x} \in \mathbf{fix}\, T$ solve your problem
2. Show that the fixed point iteration converges

If T is contractive $\quad \Longrightarrow \quad$ **linear convergence**
If T is averaged $\quad \Longrightarrow \quad$ **sublinear convergence**

Most first order algorithms can be constructed in this way

Operator theory

Today, we learned to:

- **Define** and **evaluate** proximal operators for various common functions
- **Apply** proximal operators to generalize gradient descent (vanilla, projected, proximal)
- **Define** monotone and cocoercive operators and their relations
- **Use operator theory** to construct general fixed-point iterations and prove their convergence

Next lecture

- Operators in optimization algorithms