ORF522 – Linear and Nonlinear Optimization

9. Sensitivity analysis for linear optimization

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Ed Forum

- Why does limiting it to the vertices make these problems deterministic strategies and no longer random? Is it just something to do with the problem having the same amount of variables as equations so everything can just be solved?
- What are the advantages and disadvantages of the dual simplex method over the simplex method? For any linear optimization problem, is it always okay to use both the simplex and dual simplex methods? In what cases is it better to use the dual?
- Dual simplex questions:

1. How can we prove if the primal problem is feasible and the duality gap is zero then the dual problem is also feasible?

2. Under non-degenerate assumption, why $\bar{c}_N > 0$?

3. What does "The dual simplex is equivalent to the primal simplex applied to the dual problem" means?

4. We use the dual simplex method to solve the dual problem. So why the example in the slides finally output the optimal solution x^*?

Recap

Optimal mixed strategies

P1: optimal strategy $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & \max_{y \in P_n} x^T A y \\ \text{subject to} & x \in P_m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \max_{j=1,\dots,n} (A^T x)_j \\ \text{subject to} & x \in P_m \end{array}$$

Inner problem over deterministic strategies (**vertices**)

P2: optimal strategy $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & \min_{x \in P_m} x^T A y \\ \text{subject to} & y \in P_n \end{array} \longrightarrow \begin{array}{ll} \text{maximize} & \min_{i=1,\dots,m} (A y)_i \\ \text{subject to} & y \in P_n \end{array}$$

Optimal strategies $x^\star$ and $y^\star$ can be computed using **linear optimization**

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

The optimal $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & A y \geq w\mathbf{1} \\ & \mathbf{1}^T y = 1 \\ & y \geq 0 \end{array}$$

The two LPs are **duals** and by **strong duality** the equality follows. ■

General forms

Standard form LP

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

Inequality form LP

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c = 0 \\ & y \geq 0 \end{aligned}$$

Today's lecture

[Chapter 5, Bertsimas and Tsitsiklis]

Sensitivity analysis in linear optimization

- Adding new constraints and variables
- Change problem data
- Differentiable optimization

Adding new constraints and variables

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Solution $(x^\star, 0), y^\star$ **optimal** for the new problem?

Adding new variables

Optimality conditions

$$\begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

$\longrightarrow$ Solution $(x^\star, 0)$ is still **primal feasible**

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Yes

$(x^\star, 0)$ still **optimal** for new problem

Otherwise

Primal simplex

Adding new variables

Example

$$\begin{array}{lll} \text{minimize} & -60x_1 - 30x_2 - 20x_3 & \text{-profit} \\ \text{subject to} & 8x_1 + 6x_2 + x_3 \le 48 & \text{material} \\ & 4x_1 + 2x_2 + 1.5x_3 \le 20 & \text{production} \\ & 2x_1 + 1.5x_2 + 0.5x_3 \le 8 & \text{quality control} \\ & x \ge 0 & \end{array}$$

$$c = (-60, -30, -20, 0, 0, 0)$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \ge 0 \end{array}$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Adding new variables

Example: add new product?

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

minimize	$c^T x + c_{n+1} x_{n+1}$	
subject to	$Ax + A_{n+1} x_{n+1} = b$	
	$x, x_{n+1} \geq 0$	

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Previous solution

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Still optimal

$$A_{n+1}^T y^\star + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

Shall we add a new product?

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + a_{m+1} y_{m+1} + c \geq 0 \end{array}$$

Solution $x^\star, (y^\star, 0)$ **optimal** for the new problem?

Adding new constraints

Optimality conditions

maximize $-b^T y$

subject to $A^T y + a_{m+1} y_{m+1} + c \geq 0$ $\longrightarrow$ Solution $(y^\star, 0)$ is still **dual feasible**

Is $x^\star$ still **primal feasible**?

$$\begin{aligned} &Ax = b \\ &a_{m+1}^T x = b_{m+1} \\ &x \geq 0 \end{aligned}$$

Yes

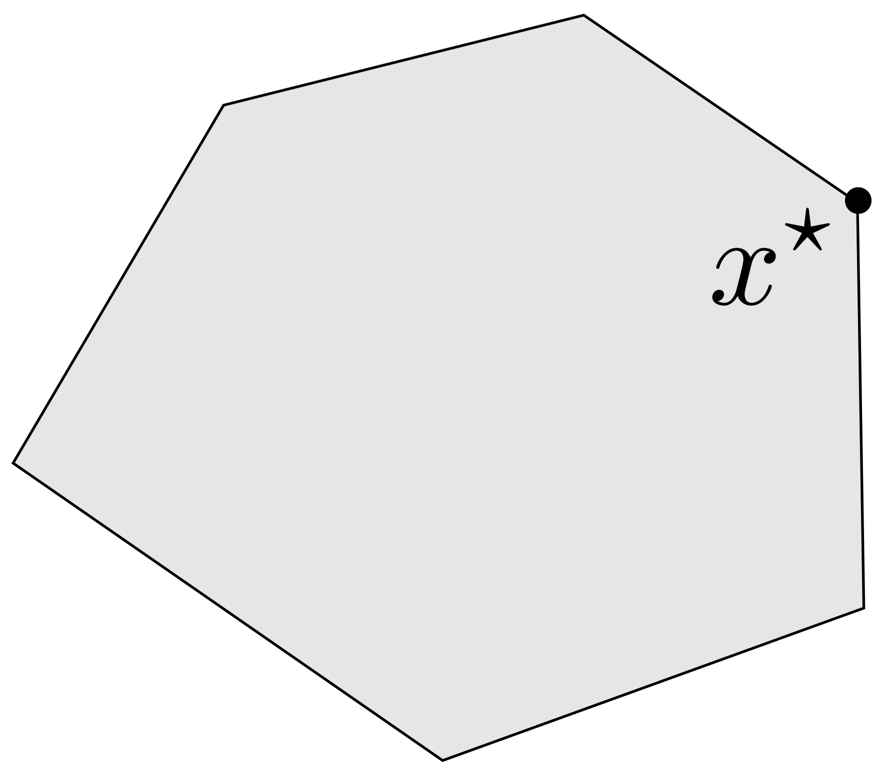
$x^\star$ still **optimal** for new problem

Otherwise

Dual simplex

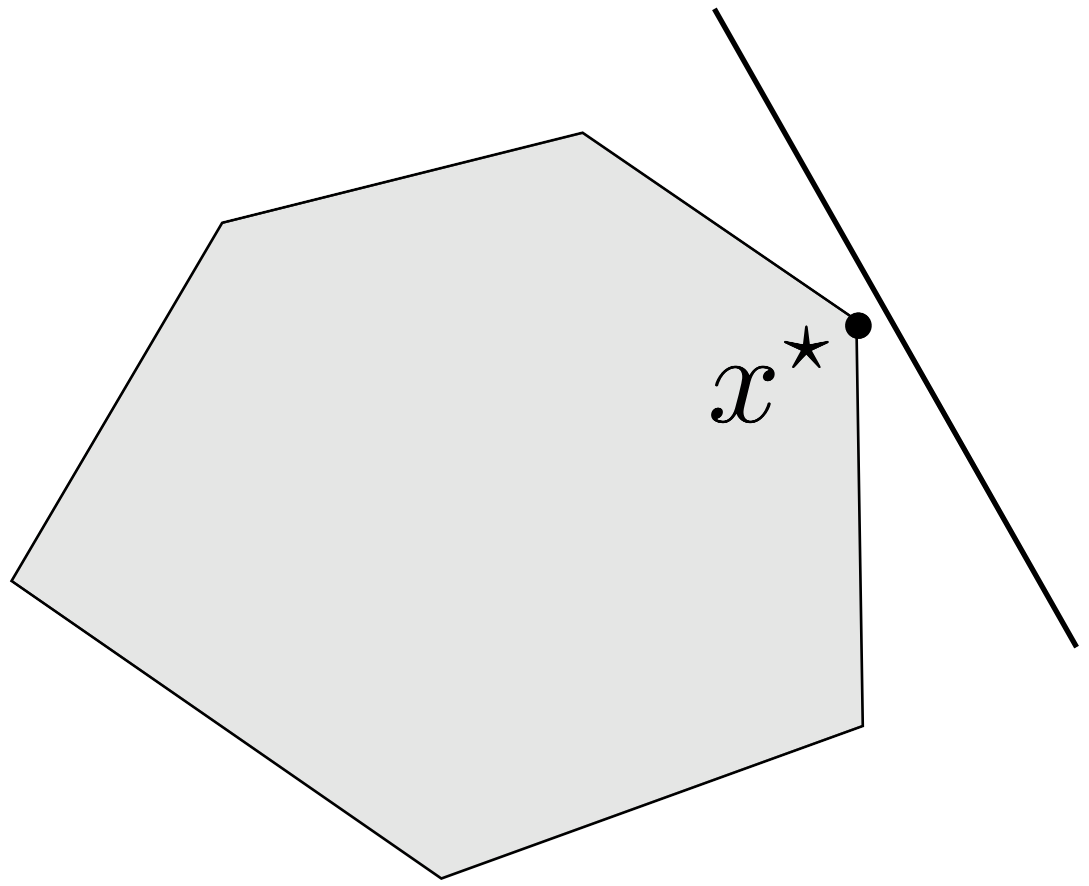
Adding new constraints

Example

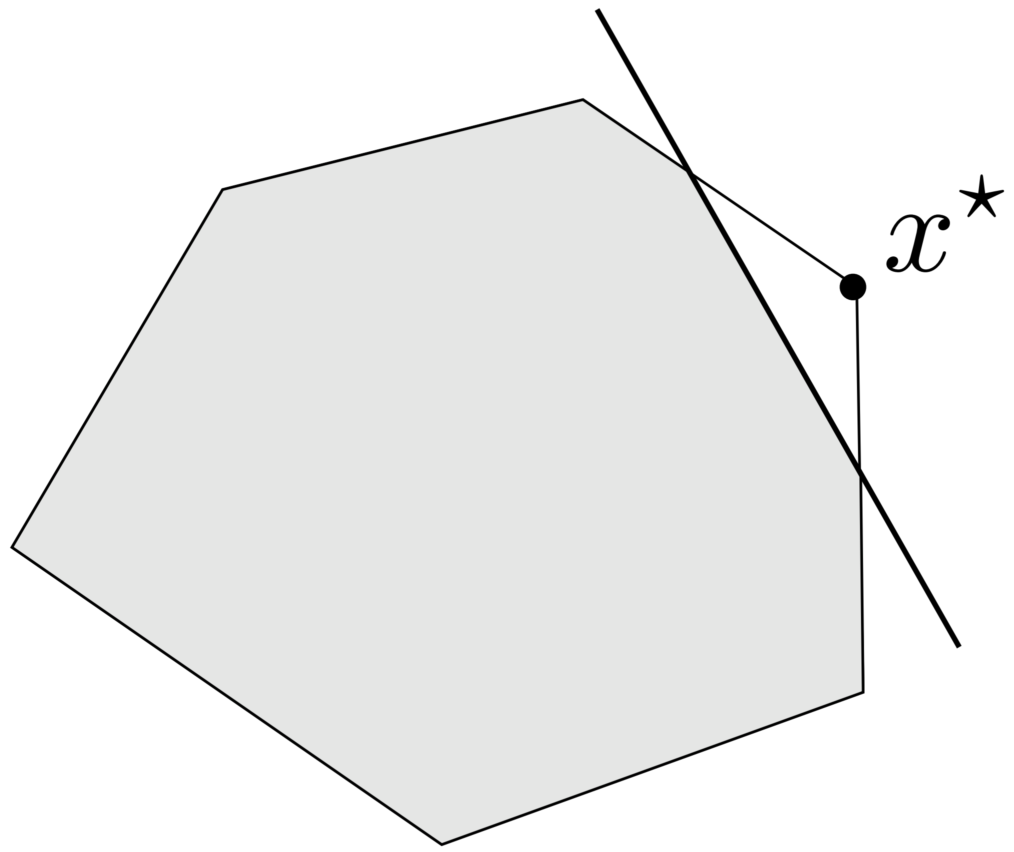


Add new constraint

$x^\star$ still **feasible**



$x^\star$ **infeasible**



Global sensitivity analysis

Information from primal-dual solution

Goal: extract information from $x^\star, y^\star$ about their sensitivity with respect to changes in problem data

Modified LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + u \\ & x \geq 0 \end{array}$$

Optimal cost $p^\star(u)$

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Global lower bound

Given $y^\star$ a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T y^\star \\ &= p^\star(0) - u^T y^\star \end{aligned}$$

(from weak duality and dual feasibility)

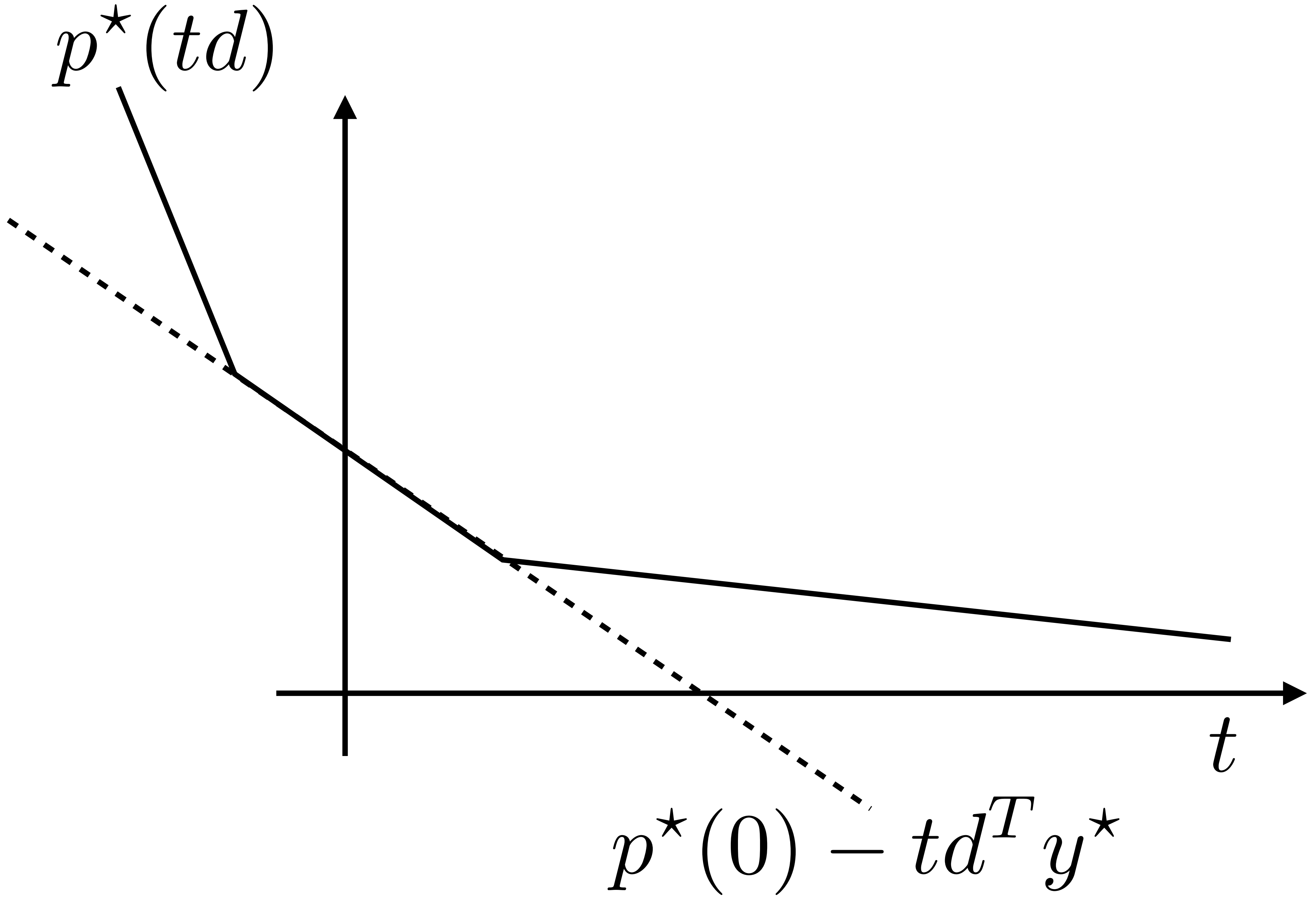
It holds for any u

Global sensitivity

Example

Take $u = td$ with $d \in \mathbf{R}^m$ fixed

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + td \\ & x \geq 0 \end{array}$$



$p^\star(td)$ is the optimal value as a function of t

Sensitivity information (assuming $d^T y^\star \geq 0$)

- $t < 0$ the optimal value increases
- $t > 0$ the optimal value decreases (not so much if t is small)

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Assumption: $p^\star(0)$ is finite

Properties

- $p^\star(u) > -\infty$ everywhere (from global lower bound)
- the domain $\{u \mid p^\star(u) < +\infty\}$ is a polyhedron
- $p^\star(u)$ is piecewise-linear on its domain

Optimal value function is piecewise linear

Proof

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Dual feasible set

$$D = \{y \mid A^T y + c \geq 0\}$$

Assumption: $p^\star(0)$ is finite

If $p^\star(u)$ finite

$$p^\star(u) = \max_{y \in D} -(b-u)^T y = \max_{k=1,\ldots,r} -y_k^T u - b^T y_k$$

$y_1, \ldots, y_r$ are the extreme points of D

Local sensitivity analysis

Local sensitivity

u in neighborhood of the origin

Original LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
$$

→

Optimal solution

Primal
$$x_i = 0, \quad i \notin B$$
$$x_B^\star = A_B^{-1} b$$

Dual
$$y^\star = -A_B^{-T} c_B$$

Modified LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b + u \\
 & x \geq 0
\end{array}
$$

Modified dual

$$
\begin{array}{ll}
\text{maximize} & -(b+u)^T y \\
\text{subject to} & A^T y + c \geq 0
\end{array}
$$

Optimal basis does not change

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1} u$$
$$y^\star(u) = y^\star$$

Derivative of the optimal value function

Modified optimal solution

$$x^\star_B(u) = A_B^{-1}(b+u) = x^\star_B + A_B^{-1}u$$
$$y^\star(u) = y^\star$$

Optimal value function

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c_B^T A_B^{-1} u \\ &= p^\star(0) - y^{\star T} u \end{aligned}$$

(affine for small u)

Local derivative

$$\frac{\partial p^\star(u)}{\partial u} = -y^\star$$

($y^\star$ are the **shadow prices**)

Sensitivity example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
	$4x_1 + 2x_2 + 1.5x_3 \leq 20$	production
	$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$	quality control
	$x \geq 0$	

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

What does $y_3^\star = 10$ mean?

Let's increase the quality control budget by 1, i.e., $u = (0, 0, 1)$

$$p^\star(10) = p^\star(0) - y^{\star T} u = -280 - 10 = -290$$

Differentiable optimization

Training a neural network

Single layer model

$$x = f(\theta)$$

Training

minimize $\quad \mathcal{L}(\theta)$

Gradient descent (more on this later)

$$\theta \leftarrow \theta - t \nabla_\theta \mathcal{L}(\theta)$$

Sensitivity

$$\nabla_\theta \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \theta}\right)^T = \left(\frac{\partial \mathcal{L}}{\partial x}\frac{\partial x}{\partial \theta}\right)^T = \left(\frac{\partial x}{\partial \theta}\right)^T \nabla_x \mathcal{L}$$

Can f be an **optimization problem**?

Implicit layers

https://implicit-layers-tutorial.org/

find $x(\theta)$

subject to $r(\theta, x(\theta)) = 0$

($x(\theta)$ is implicitly defined by r)

How do we compute derivatives?

$$\frac{\partial x(\theta)}{\partial \theta}$$

Implicit function theorem

Under mild assumptions (non-singularity),

$$\frac{\partial r(\theta, x(\theta))}{\partial x} \frac{\partial x(\theta)}{\partial \theta} + \frac{\partial r(\theta, x(\theta))}{\partial \theta} = 0 \longrightarrow \frac{\partial x(\theta)}{\partial \theta} = -\left(\frac{\partial r(\theta, x(\theta))}{\partial x}\right)^{-1} \frac{\partial r(\theta, x(\theta))}{\partial \theta}$$

[Theorem 1. B.1, Dontchev and Rockafellar 2009]

Optimization layers

$$x^\star(\theta) = \underset{x}{\text{argmin}} \quad c^T x \qquad \text{subject to} \quad Ax \leq b$$

Parameters: $\theta = \{c, A, b\}$
Solution $x^\star(\theta)$

Features

- Add **domain knowledge** and **hard constraints**
- **End-to-end** training and optimization
- Nice theory and algorithms for general **convex optimization**
- **Applications** in RL, control, meta-learning, game theory, etc.

Goal

Compute $\dfrac{\partial x^\star(\theta)}{\partial \theta}$

Optimality conditions

minimize $c^T x$
subject to $Ax \leq b$

Parameters: $\theta = \{c, A, b\}$
Solution $x^\star(\theta)$

Solve and obtain primal-dual pair $x^\star, y^\star$ (forward-pass)

Optimality conditions

$$A^T y + c = 0$$
$$\mathbf{diag}(y)(Ax - b) = 0$$
$$y \geq 0,\ b - Ax \geq 0$$

Mapping $r(\theta, x(\theta)) = 0$

Computing derivatives

Take differentials

$$A^T y^\star + c = 0 \qquad \longrightarrow \qquad \mathrm{d}A^T y^\star + A^T \mathrm{d}y = 0$$

$$\mathbf{diag}(y^\star)(Ax - b) = 0 \qquad \longrightarrow \qquad \mathbf{diag}(Ax - b)\mathrm{d}y + \mathbf{diag}(y^\star)(\mathrm{d}Ax^\star + A\mathrm{d}x - \mathrm{d}b) + \mathrm{d}c = 0$$

Linear system

$$\begin{bmatrix} 0 & A^T \\ \mathbf{diag}(y^\star)A & \mathbf{diag}(Ax^\star - b) \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = - \begin{bmatrix} \mathrm{d}A^T y^\star + \mathrm{d}c \\ \mathbf{diag}(y^\star)(\mathrm{d}Ax^\star - \mathrm{d}b) \end{bmatrix}$$

Example: How does $x^\star$ change with b_1?

Set $\mathrm{d}b = e_1, \mathrm{d}A = 0, \mathrm{d}c = 0$ and solve the linear system.

The solution $\mathrm{d}x$ will correspond to $\dfrac{\partial x}{\partial b_1}$

Is it always differentiable?

The linear system matrix must be invertible
(the problem must have unique solution)

$$\underbrace{\begin{bmatrix} 0 & A^T \\ \mathbf{diag}(y^\star)A & \mathbf{diag}(Ax^\star - b) \end{bmatrix}}_{M} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = -\underbrace{\begin{bmatrix} \mathrm{d}A^T y^\star + \mathrm{d}c \\ \mathbf{diag}(y^\star)(\mathrm{d}Ax^\star - \mathrm{d}b) \end{bmatrix}}_{q}$$

Remember. implicit function theorem

$$\frac{\partial x(\theta)}{\partial \theta} = -\left(\frac{\partial r(\theta, x(\theta))}{\partial x}\right)^{-1} \frac{\partial r(\theta, x(\theta))}{\partial \theta}$$

If not, **least squares “**subdifferential”

$$\text{minimize} \quad \left\| M \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} + q \right\|_2^2$$

Example

Learning to play Sudoku

			3
1			
		4	
4			1

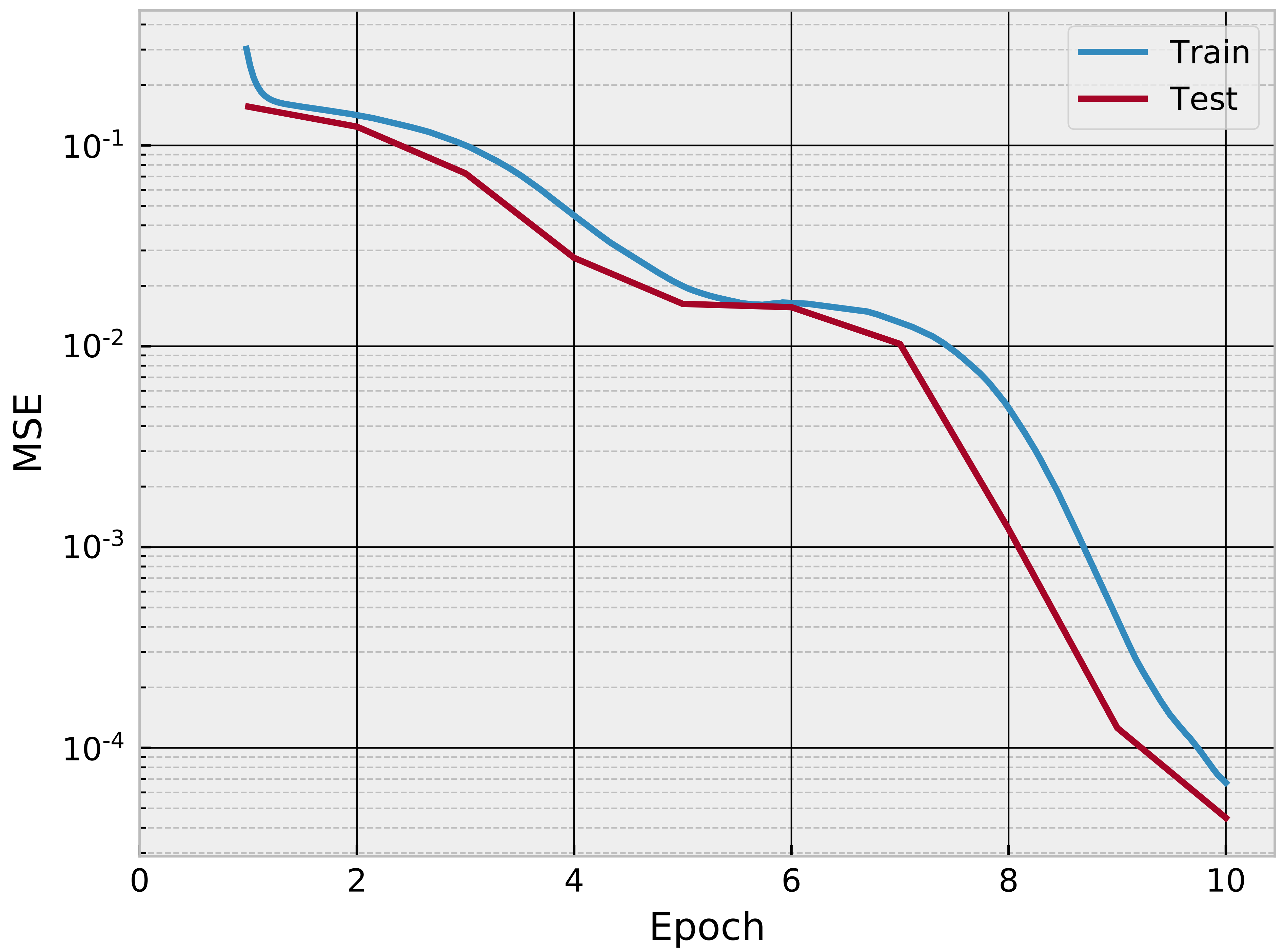
2	4	1	3
1	3	2	4
3	1	4	2
4	2	3	1

Sudoku constraint satisfaction problem

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax = b \\ & x \geq 0,\ x \in \mathbf{Z}^d \end{array}$$

Linear optimization layer (parameters $\theta = \{A, b\}$)

$$x^\star = \underset{x}{\text{argmin}} \quad 0 \quad \text{subject to} \quad Ax = b, \quad x \geq 0$$



[OptNet: Differentiable Optimization as a Layer in Neural Networks, B. Amos and J. Z. Kolter ICML 2017]

Sensitivity analysis in linear optimization

Today, we learned to:

- **Use** the most appropriate primal/dual simplex algorithm when variables and/or constraints are added
- **Analyze** sensitivity of the cost with respect to change in the data
- **Apply** sensitivity analysis to differentiable linear optimization layers

Next lecture

- Barrier methods for linear optimization