# ORF522 – Linear and Nonlinear Optimization

## 7. Linear optimization duality

**Bartolomeo Stellato — Fall 2021**

# Ed Forum

- *How do we pick a permutation matrix?*
  Optimal way (intractable). In practice, we heuristics. A famous one is Approximate Minimum Degree ordering (AMD)

- *Why O(n^3) complexity for LU factorization?*
  It can be computed with an algorithm called: Gaussian Elimination with Partial pivoting. *Its complexity is O(n^3)*

- *Is there a standard number of times the same A_B can be used before needing to refactored?*
  Not standard because it depends on the problem dimensions. In practice, around 100 iterations.

# Recap

# Linear optimization formulations

**Standard form LP**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

**Inequality form LP**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

# Today's agenda

## Readings: [Chapter 4, Bertsimas, Tsitsiklis][Chapter 5, Vanderbei]

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

# Obtaining lower bounds

# Obtaining lower bounds

## A simple example

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + 3x_2 \geq 2 \end{array}$$

What is a **lower bound** on the optimal cost?

A lower bound is $2$ because $x_1 + 3x_2 \geq 2$

# Obtaining lower bounds

## Another example

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \end{array}$$

What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$\begin{aligned} & 1 \cdot (x_1 + x_2 \geq 2) \\ + & 2 \cdot (x_2 \geq 1) \\ = & x_1 + 3x_2 \geq 4 \end{aligned}$$

A lower bound is $4$

# Obtaining lower bounds

## A more interesting example

minimize $x_1 + 3x_2$

subject to
$$x_1 + x_2 \geq 2$$
$$x_2 \geq 1$$
$$x_1 - x_2 \geq 3$$

How can we obtain a lower bound?

**Add constraints**

$$y_1 \cdot (x_1 + x_2 \geq 2)$$
$$+ y_2 \cdot (x_2 \geq 1)$$
$$+ y_3 \cdot (x_1 - x_2 \geq 3)$$
$$= x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3$$

**Bound**

**Match cost coefficients**

$$y_1 + y_3 = 1$$
$$y_1 + y_2 - y_3 = 3$$
$$y_1, y_2, y_3 \geq 0$$

**Many options**

$y = (1, 2, 0) \Rightarrow$ Bound $4$

$y = (0, 4, 1) \Rightarrow$ Bound $7$

How can we get the **best one**?

# Obtaining lower bounds

## A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$\begin{array}{ll} \text{maximize} & 2y_1 + y_2 + 3y_3 \\ \text{subject to} & y_1 + y_3 = 1 \\ & y_1 + y_2 - y_3 = 3 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

This linear optimization problem is called the **dual problem**

# The dual problem

# Lagrange multipliers

Consider the LP in standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Relax the constraint

$$g(y) = \begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + y^T (Ax - b) \\ \text{subject to} & x \geq 0 \end{array}$$

Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) = c^T x^\star$$

Best lower bound

$$\underset{y}{\text{maximize}} \quad g(y)$$

# The dual

**Dual function**

$$g(y) = \underset{x \geq 0}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$= -b^T y + \underset{x \geq 0}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

**Dual problem** (find the best bound)

$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

# Primal and dual problems

**Primal problem**

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
$$

**Dual problem**

$$
\begin{array}{ll}
\text{maximize} & -b^T y \\
\text{subject to} & A^T y + c \geq 0
\end{array}
$$

Primal variable $x \in \mathbf{R}^n$

Dual variable $y \in \mathbf{R}^m$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

# Dual of inequality form LP

What if you find an LP with inequalities?

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T(Ax - b)$$

Lower bound

$$g(y) \leq c^T x^\star + y^T(Ax^\star - b) \leq c^T x^\star$$

we must have $y \geq 0$

# Dual of LP with inequalities

## Derivation

**Dual function**

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$
$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y = 0 \quad \text{(and } y \geq 0\text{)} \\ -\infty & \text{otherwise} \end{cases}$$

**Dual problem** (find the best bound)

$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

# General forms

### Standard form LP

**Primal**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

**Dual**

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

### Inequality form LP

**Primal**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

**Dual**

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

### LP with inequalities and equalities

**Primal**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$

**Dual**

$$\begin{array}{ll} \text{maximize} & -b^T y - d^T z \\ \text{subject to} & A^T y + C^T z + c = 0 \\ & y \geq 0 \end{array}$$

# Example from before

minimize $x_1 + 3x_2$

subject to $x_1 + x_2 \geq 2$

$x_2 \geq 1$

$x_1 - x_2 \geq 3$

$\longrightarrow$

**Inequality form LP**

minimize $c^T x$

subject to $Ax \leq b$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$b = (-2, -1, -3)$$

**Dual**

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

$\longrightarrow$

maximize $2y_1 + y_2 + 3y_3$

subject to $-y_1 - y_3 = -1$

$-y_1 - y_2 + y_3 = -3$

$y_1, y_2, y_3 \geq 0$

# To memorize

**Ways to get the dual**

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

**Sanity-checks and signs convention**

- Consider constraints as $g(x) \leq 0$ or $g(x) = 0$
- Each dual variable is associated to a primal constraint
- $y$ free for primal equalities and $y \geq 0$ for primal inequalities

# Dual of the dual

**Theorem**

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

**Exercise**

Derive dual and dualize again

**Primal**

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b \\
& Cx = d
\end{array}
$$

**Dual**

$$
\begin{array}{ll}
\text{maximize} & -b^T y - d^T z \\
\text{subject to} & A^T y + C^T z + c = 0 \\
& y \geq 0
\end{array}
$$

**Theorem**

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

# Weak and strong duality

# Optimal objective values

**Primal**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

**Dual**

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

# Weak duality

**Theorem**

If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

**Proof**

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y \qquad \blacksquare$$

**Remark**

- Any dual feasible $y$ gives a **lower bound** on the primal optimal value
- Any primal feasible $x$ gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

# Weak duality

## Corollaries

**Unboundedness vs feasibility**

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

**Optimality condition**

If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then $x$ and $y$ are **optimal solutions** to the primal and dual problem respectively

# Strong duality

**Theorem**
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

# Strong duality

## Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
\longrightarrow
$$

- optimal basis $B$
- optimal solution $x^\star$ with $A_B x_B^\star = b$
- reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$

Define $y^\star$ such that $y^\star = -A_B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

$$-b^T y^\star = -b^T(-A_B^{-T} c_B) = c_B^T(A_B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

By weak duality theorem corollary, $y^\star$ is an optimal solution of the dual.
Therefore, $d^\star = p^\star$. ■

# Exception to strong duality

**Primal**

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

Optimal value is $p^\star = +\infty$

**Dual**

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & 0 \cdot y + 1 = 0 \\ & y \geq 0 \end{array}$$

Optimal value is $d^\star = -\infty$

Both **primal** and **dual infeasible**

# Relationship between primal and dual

| | $p^\star = +\infty$ | $p^\star$ finite | $p^\star = -\infty$ |
|---|---|---|---|
| $d^\star = +\infty$ | primal inf.<br>dual unb. | | |
| $d^\star$ finite | | optimal values equal | |
| $d^\star = -\infty$ | exception | | primal unb.<br>dual inf |

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**

# Example

# Production problem

maximize $x_1 + 2x_2$ ← Profits

subject to

$$x_1 \leq 100$$
$$2x_2 \leq 200$$
$$x_1 + x_2 \leq 150$$

← Resources

$$x_1, x_2 \geq 0$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

**Dualize**

1. Transform in inequality form

minimize $c^T x$

subject to $Ax \leq b$

2. Derive dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

# Production problem

## The dual

$$\begin{aligned} \text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

**Interpretation**

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product $1$ (price $1$, needs $1\times$ resource $1$ and $3$): $y_1 + y_3 \geq 1$
  - Product $2$ (price $2$, needs $2\times$ resource $2$ and $1\times$ resource $3$): $2y_2 + y_3 \geq 2$

# Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Interpret** simple dual optimization problems

# Next lecture

More on duality:

- Game theoretic interpretation
- Complementary slackness
- Alternative systems