# ORF522 – Linear and Nonlinear Optimization

## 4. The simplex method

**Bartolomeo Stellato — Fall 2021**

# Ed Forum

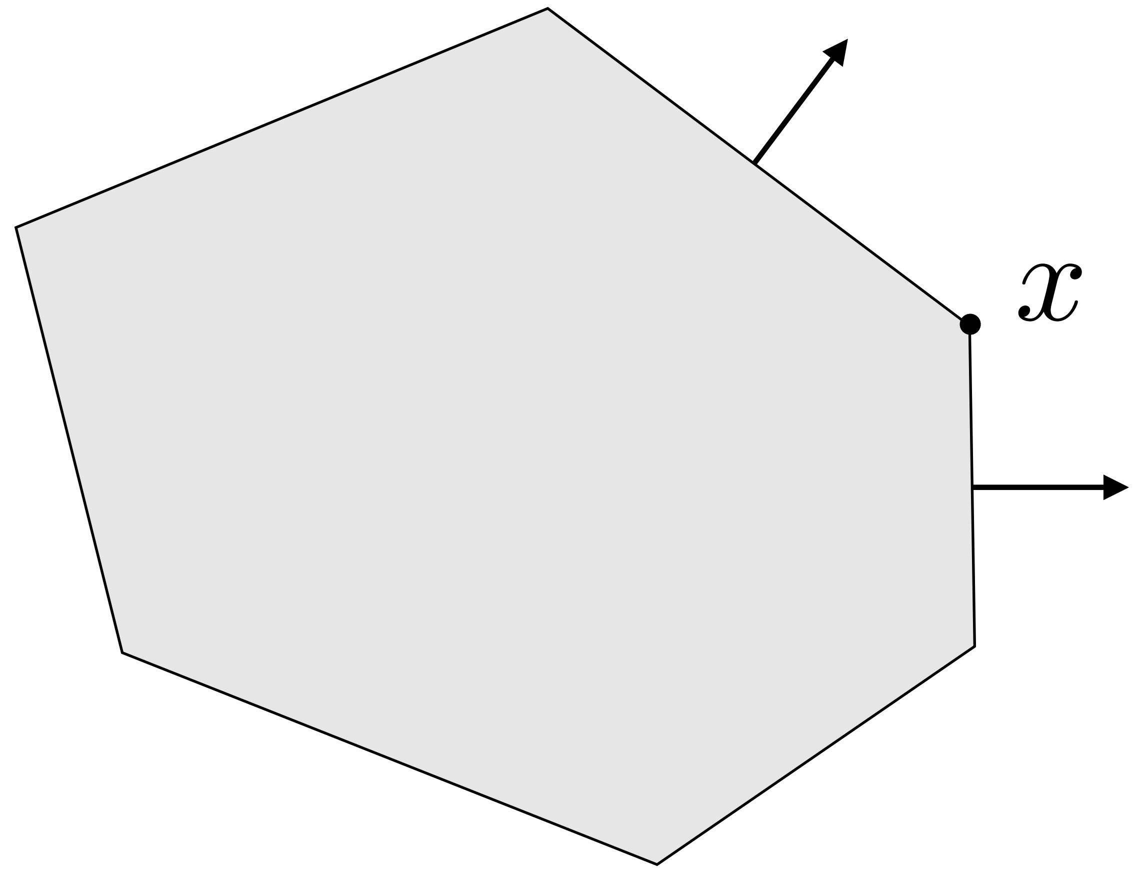
- Notebooks on GitHub: https://github.com/ORF522/companion
- Office hours change:
  Prof. Stellato: Thu 3:30pm-5:30pm
  Scander Mustapha: Mon: 1:30pm-3:30pm
- 10% Participation. The note should **summarize what you learned** in the last lecture, and **highlight the concepts that were most confusing** or that you would like to review. A note will receive full credit if: it is **submitted before the beginning of next lecture**, it is **related to the content** of the lecture, and it is **understandable** and coherent.
- Question: connection between geometry and standard form?
  Yes, they are equivalent (more in the next slides)

# Recap

# Equivalence

## Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$




Let $x \in P$

$x$ is a **vertex** $\iff$ $x$ is an **extreme point** $\iff$ $x$ is a **basic feasible solution**

# Basic feasible solution

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\}$$

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**Active constraints** at $\bar{x}$

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \dots, m\} \mid a_i^T \bar{x} = b_i\}$$

Index of all the constraints satisfied as **equality**

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**Basic solution** $\bar{x}$

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has $n$ linearly independent vectors

# Basic feasible solution

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**Basic solution** $\bar{x}$

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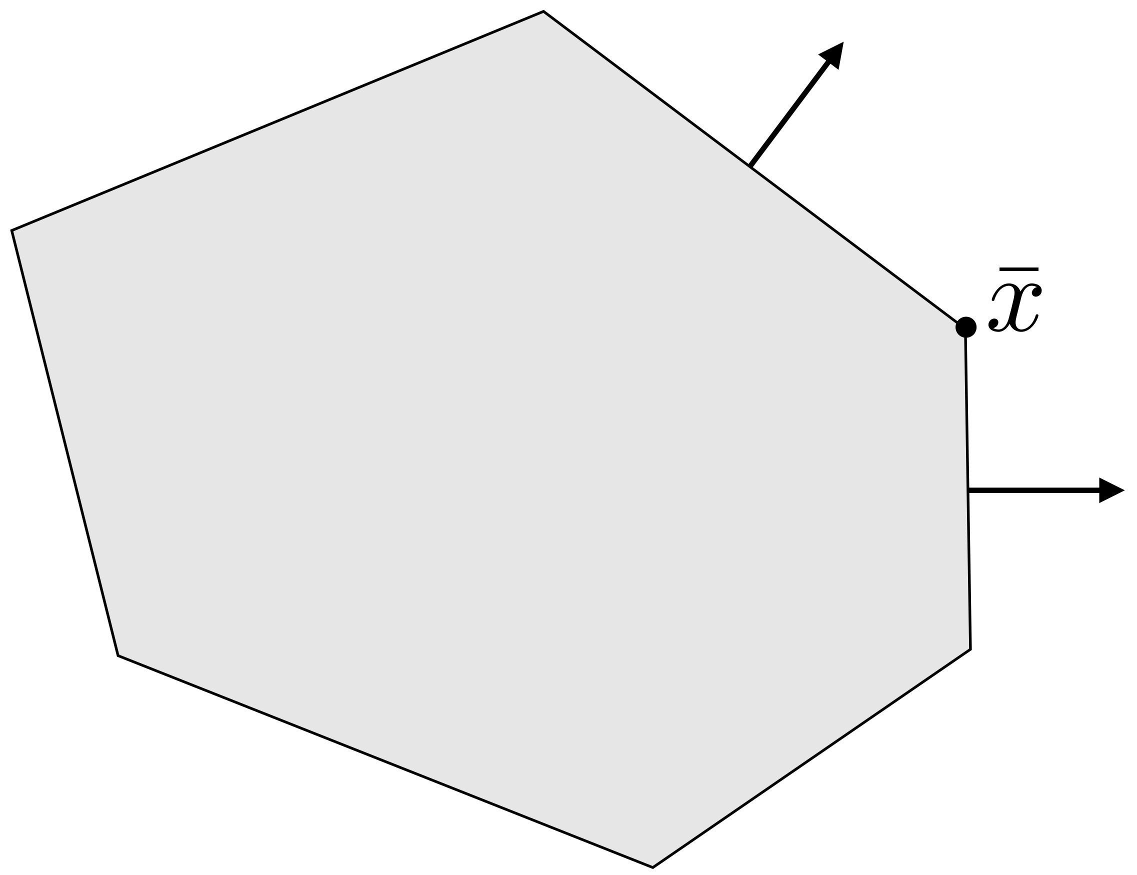
**Basic feasible solution** $\bar{x}$

- $\bar{x} \in P$
- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has $n$ linearly independent vectors



# Standard form polyhedra

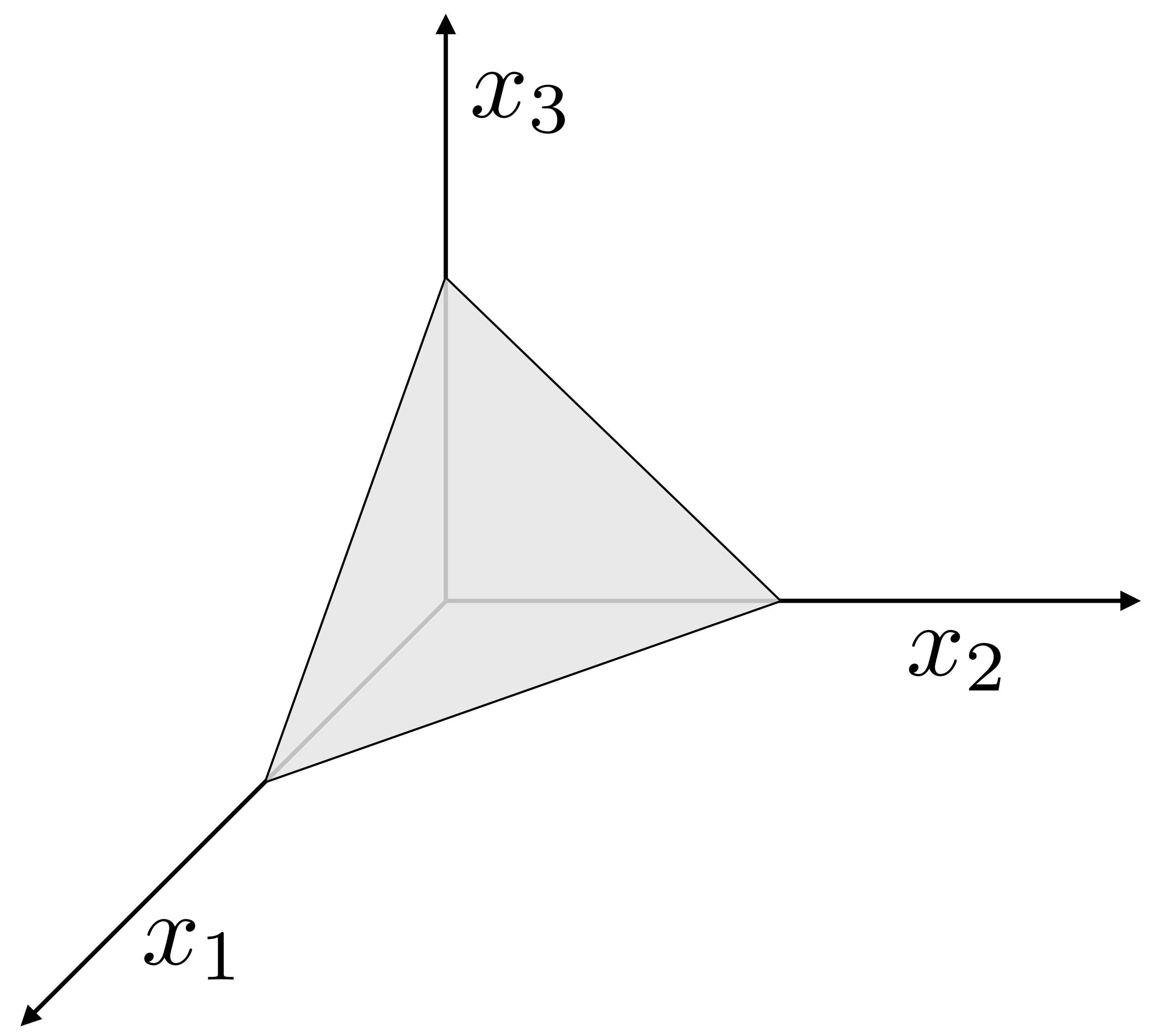
## Definition

**Standard form LP**

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

**Standard form polyhedron**

$$P = \{x \mid Ax = b,\ x \geq 0\}$$




## Assumption

$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

## Interpretation

$P$ lives in $(n - m)$-dimensional subspace

# Basic solutions

## Standard form polyhedra

$P = \{x \mid Ax = b,\ x \geq 0\}$ with $A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

$x$ is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \ldots, B(m)$ such that
  - columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent
  - $x_i = 0$ for $i \neq B(1), \ldots, B(m)$

$x$ is a **basic feasible solution** if $x$ is a **basic solution** and $x \geq 0$

# From geometry to standard form

minimize $\quad c^T x$

subject to $\quad Ax \leq b$

# From geometry to standard form

minimize $c^T x$

subject to $Ax \leq b$ ⟶

minimize $c^T(x^+ - x^-)$

subject to $\begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b$

$(x^+, x^-, s) \geq 0$

$\tilde{c} = (c, -c, 0)$

# From geometry to standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T(x^+ - x^-) \\ \text{subject to} & \underbrace{\begin{bmatrix} A & -A & I \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b \\ & \underbrace{(x^+, x^-, s)}_{\tilde{x}} \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} = b \\ & \tilde{x} \geq 0 \end{array}$$

$x \in \mathbb{R}^n$

m ineq

Variables: $\tilde{n} = 2n + m$

(Equality) constraints: $\tilde{m} = m \Longrightarrow$ **active**

# From geometry to standard form

minimize $c^T x$
subject to $Ax \leq b$

$\longrightarrow$

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For a **basic solution** $\longrightarrow$ We need $\tilde{n} - \tilde{m} = 2n$
active inequalities $\Rightarrow \tilde{x}_i = 0$ (non basic)

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Variables: $\tilde{n} = 2n + m$

(Equality) constraints: $\tilde{m} = m \implies$ **active**

For a **basic solution** $\longrightarrow$ We need $\tilde{n} - \tilde{m} = 2n$ active inequalities $\Rightarrow \tilde{x}_i = 0$ (non basic)

Which corresponds to $m$ inequalities inactive $\Rightarrow \tilde{x}_i > 0$ (basic)

# From geometry to standard form

minimize $c^T x$
subject to $Ax \leq b$ $\longrightarrow$

minimize $c^T(x^+ - x^-)$
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$(x^+, x^-, s) \geq 0$ $\longrightarrow$

minimize $\tilde{c}^T \tilde{x}$
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$\tilde{x} \geq 0$

Variables: $\tilde{n} = 2n + m$
(Equality) constraints: $\tilde{m} = m \Longrightarrow$ **active**

Formal proof at
Theorem 2.4 LO book

For a **basic solution** $\longrightarrow$ We need $\tilde{n} - \tilde{m} = 2n$
active inequalities $\Rightarrow \tilde{x}_i = 0$ (non basic)

Which corresponds to $m$ inequalities inactive $\Rightarrow \tilde{x}_i > 0$ (basic)

# Constructing basic solution

1. Choose any $m$ independent columns of $A$: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

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Basis matrix — Basis columns — Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \ldots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

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Basis matrix

Basis columns

Basic variables

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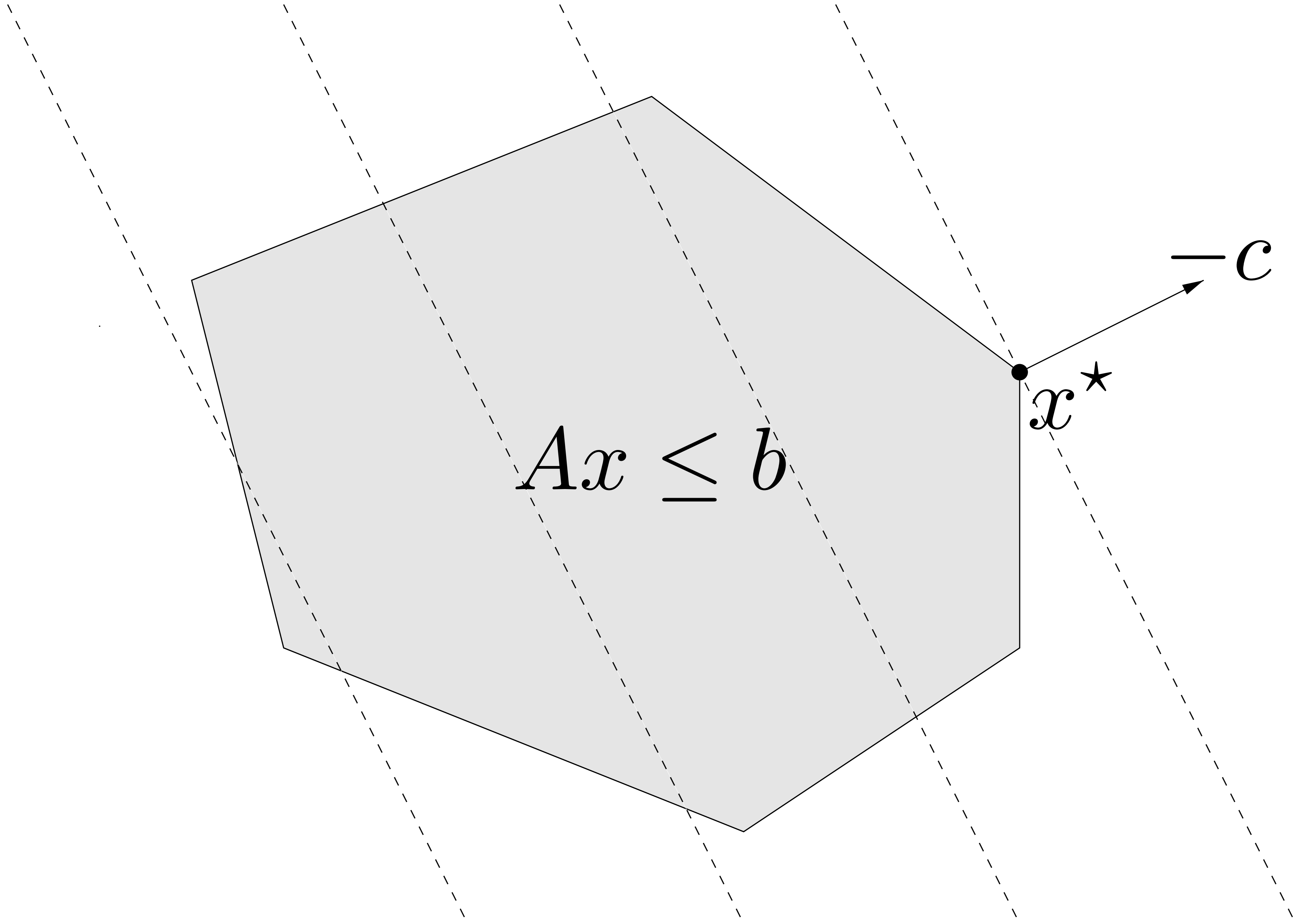
Solve $A_B x_B = b$

If $x_B \geq 0$, then $x$ is a **basic feasible solution**

# Optimality of extreme points

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$



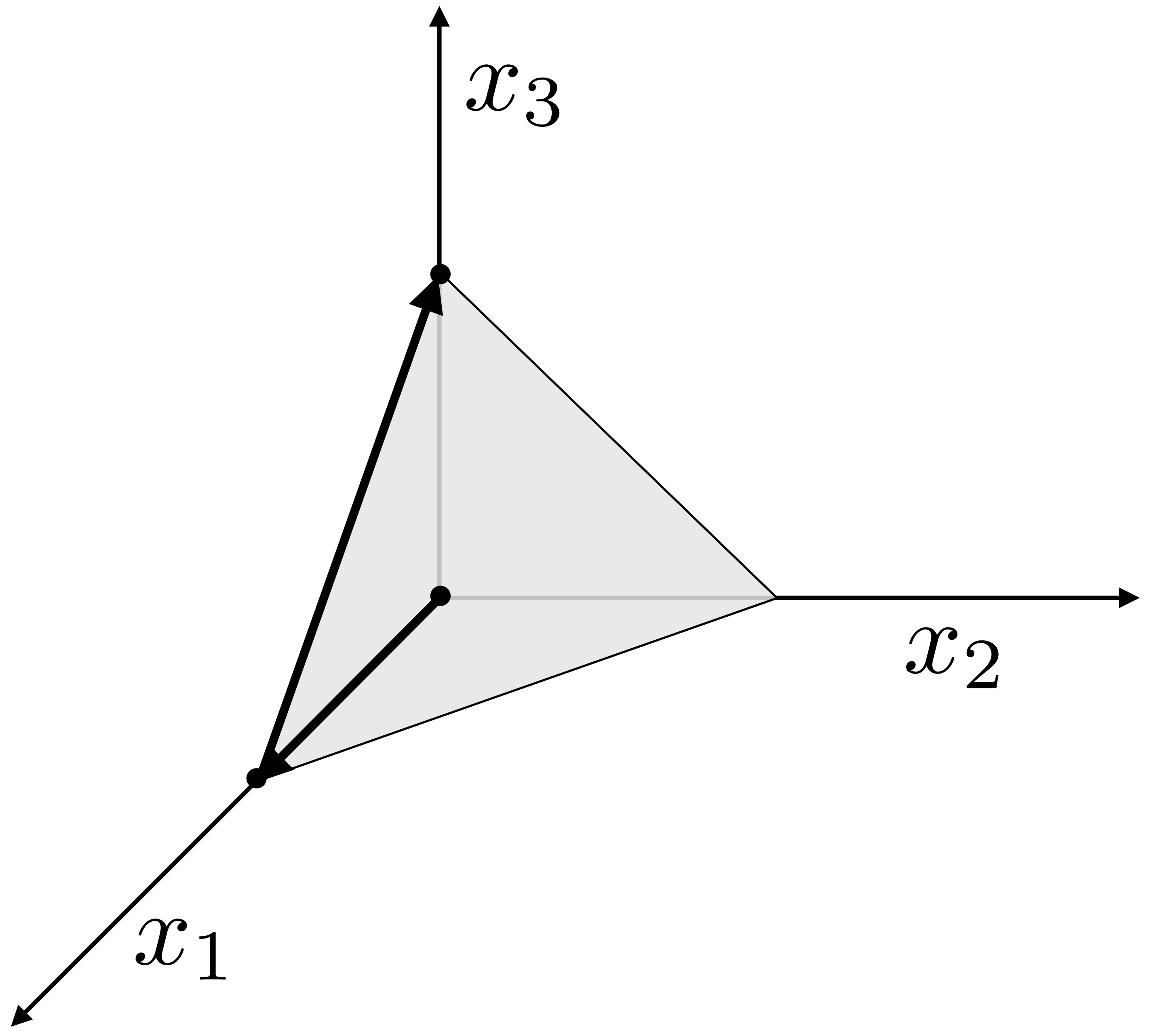

If
- $P$ has at least one extreme point
- There exists an optimal solution $x^\star$

Then, there exists an optimal solution which is an **extreme point** of $P$

We only need to search between **extreme points**

# Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



# Today's agenda
## Readings: [Chapter 3, LO]

**Simplex method**

- Iterate between neighboring basic solutions
- Optimality conditions
- Simplex iterations

# The simplex method

## Top 10 algorithms of the 20th century

1946: Metropolis algorithm
**1947: Simplex method**
1950: Krylov subspace method
1951: The decompositional approach to matrix computations
1957: The Fortran optimizing compiler
1959: QR algorithm
1962: Quicksort
1965: Fast Fourier transform
1977: Integer relation detection
1987: Fast multipole method

[SIAM News (2000)]

# The simplex method

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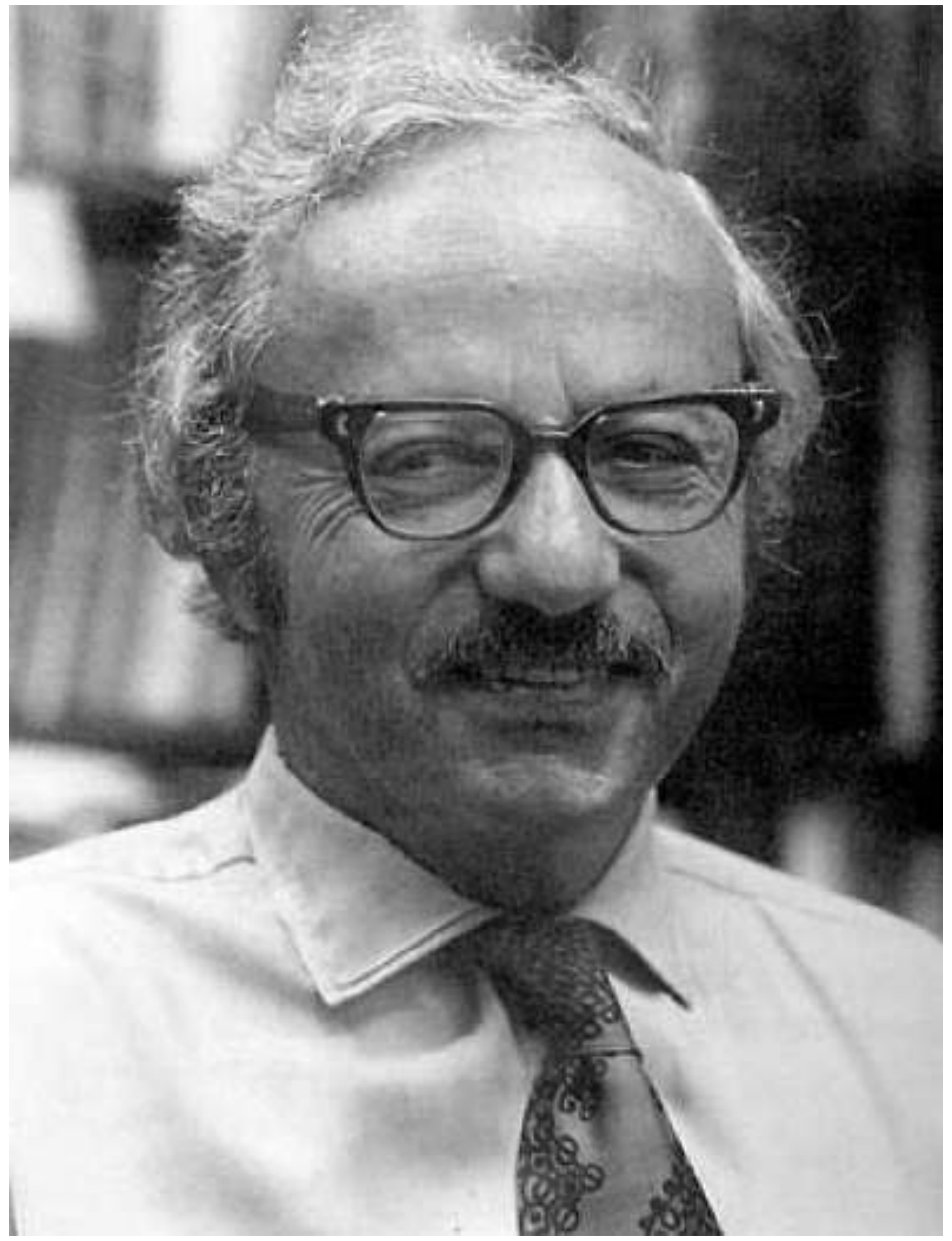
**George Dantzig**

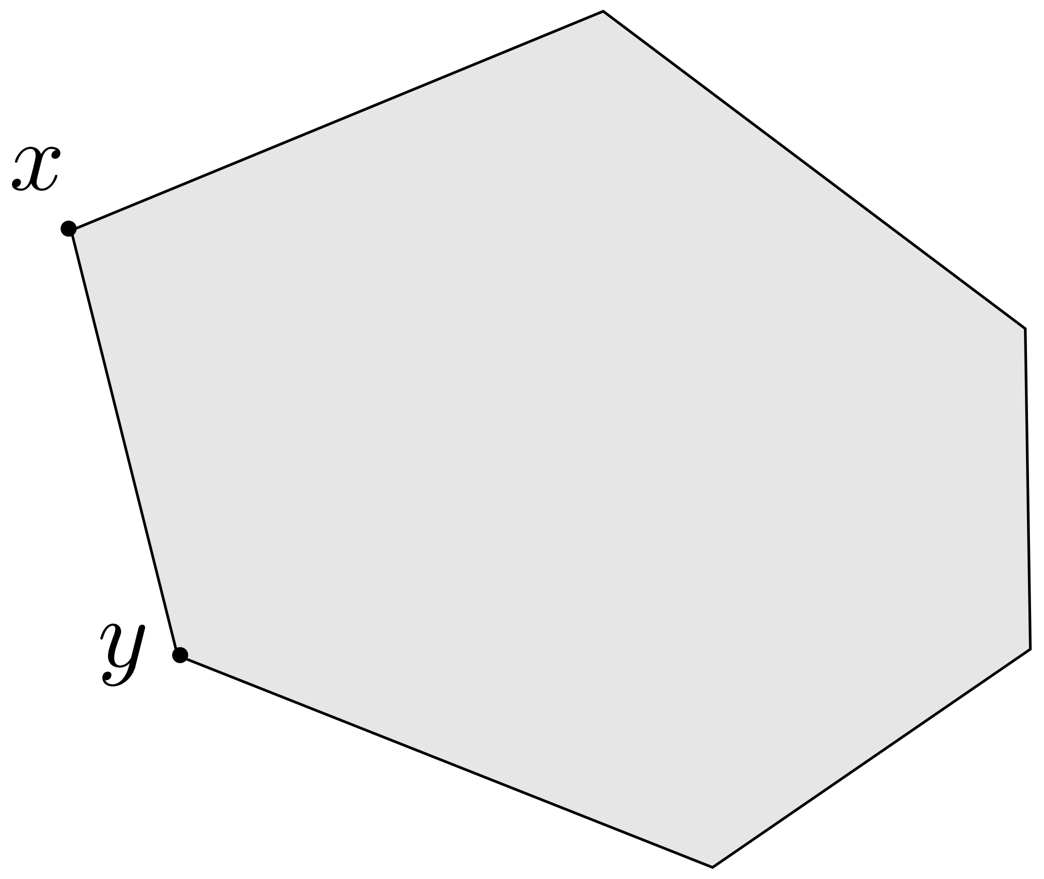


[SIAM News (2000)]

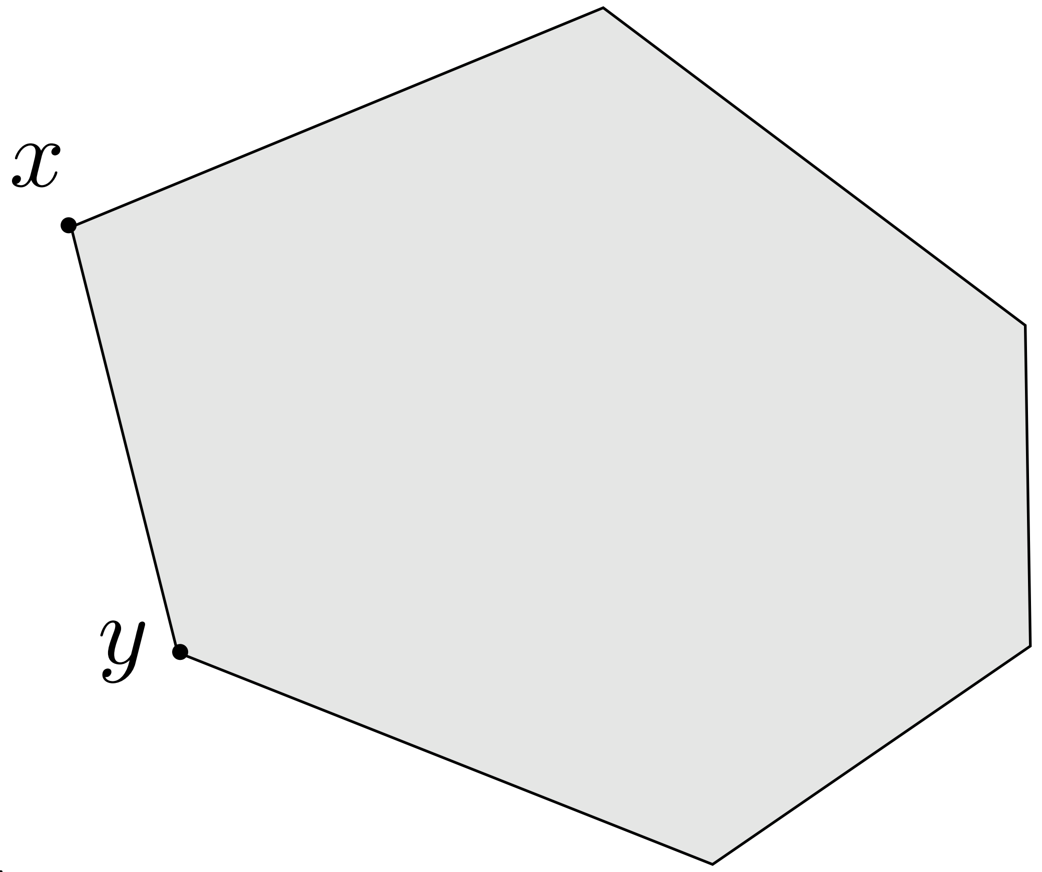
# Neighboring basic solutions

# Neighboring solutions

Two basic solutions are **neighboring** if their basic indices differ by exactly one variable



# Neighboring solutions

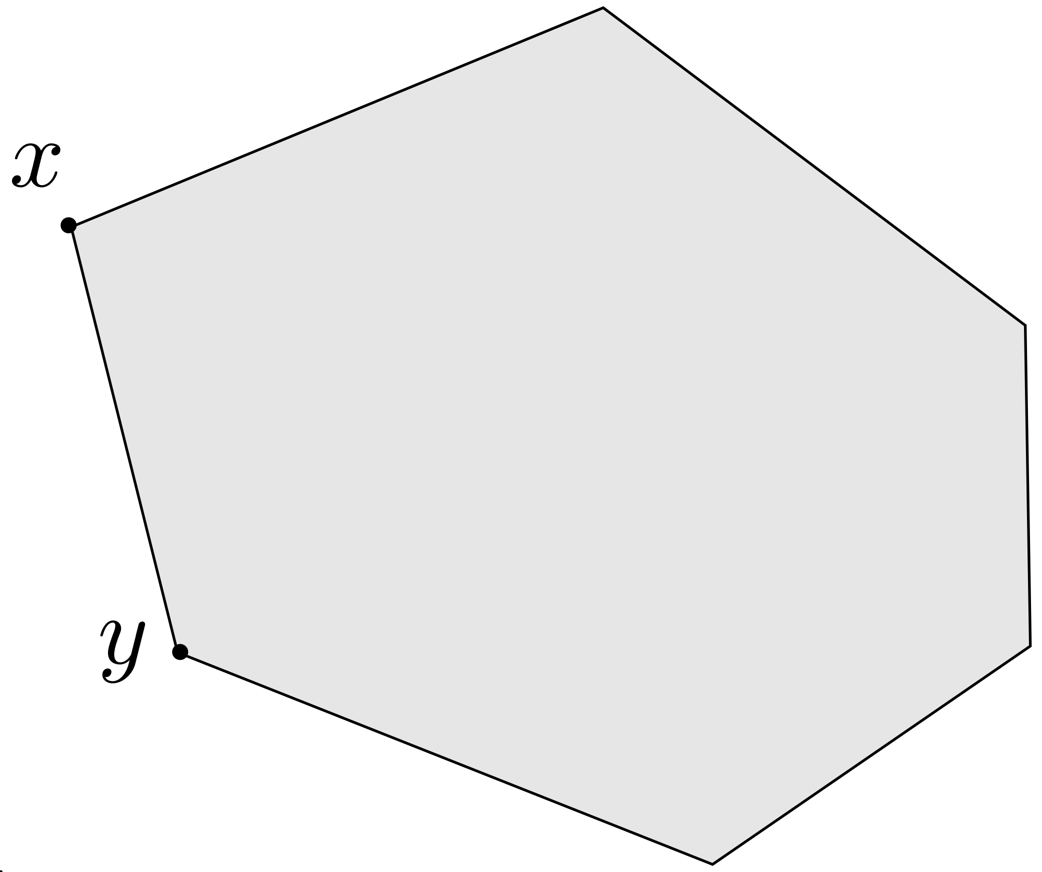



Two basic solutions are **neighboring** if their basic indices differ by exactly one variable

**Example**

$$\overset{A}{\begin{bmatrix} 1 & -1 & 0 & 3 & -2 \\ 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 4 & -1 & 4 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \overset{b}{\begin{bmatrix} -5 \\ -1 \\ 14 \end{bmatrix}}$$

# Neighboring solutions

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$B = \{1, 3, 5\} \qquad x_2 = x_4 = 0$

$$A_B x_B = b \longrightarrow x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2.5 \end{bmatrix}$$

# Neighboring solutions

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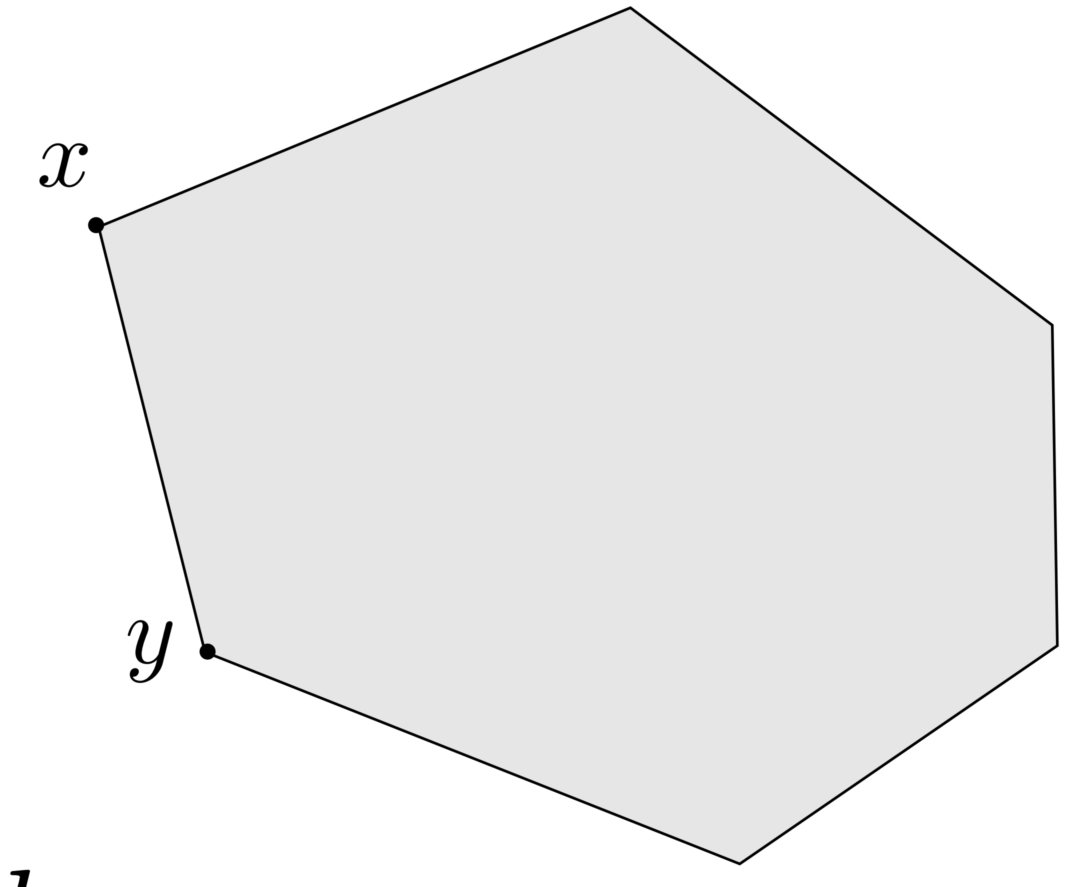


**Example**

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$$\bar{B} = \{1, 3, 4\} \qquad y_2 = y_5 = 0$$

$$A_{\bar{B}} y_{\bar{B}} = b \longrightarrow y_{\bar{B}} = \begin{bmatrix} y_1 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 3.0 \\ -1.7 \end{bmatrix}$$

# Feasible directions

## Conditions

$$P = \{x \mid Ax = b,\ x \geq 0\}$$

Given a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

we have basic feasible solution $x$:

- $x_B$ solves $A_B x_B = b$
- $x_i = 0,\ \forall i \neq B(1), \dots, B(m)$

# Feasible directions

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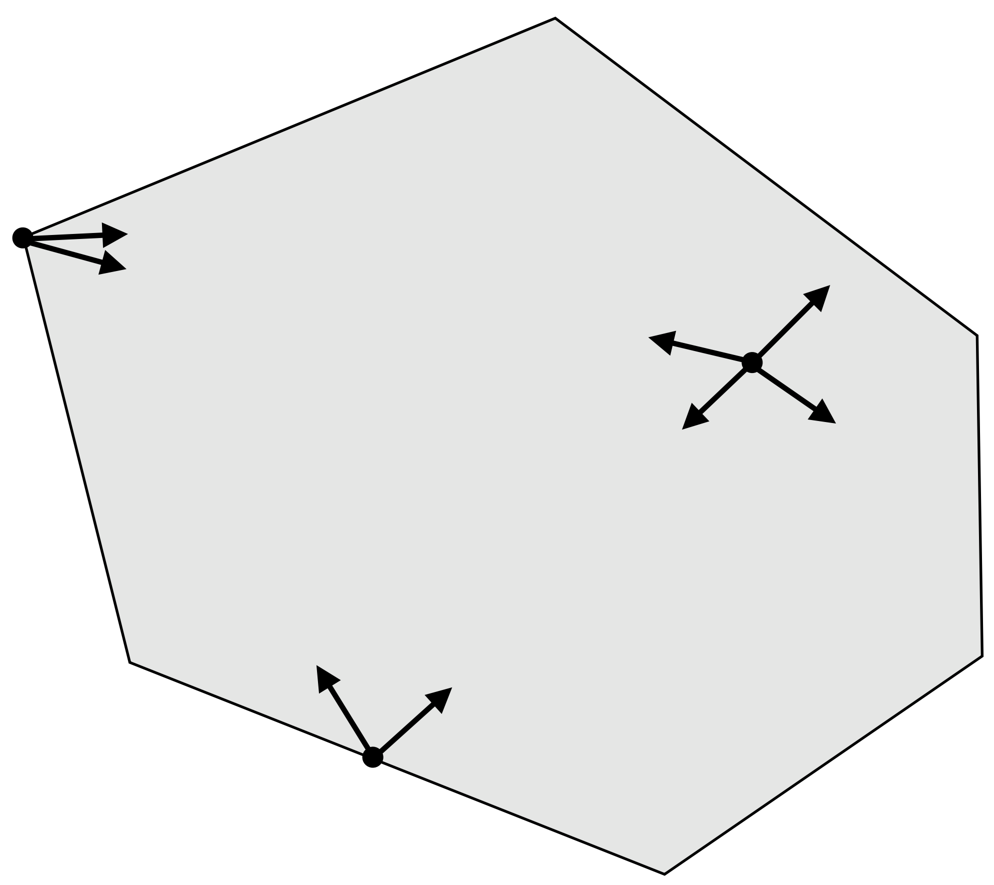
- $x_i = 0,\ \forall i \neq B(1), \dots, B(m)$

Let $x \in P$, a vector $d$ is a **feasible direction** at $x$ if $\exists \theta > 0$ for which $x + \theta d \in P$



**Feasible direction** $d$

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

$Ax = b$

# Feasible directions

## Computation

**Feasible direction** $d$

- $A(x + \theta d) = b \Longrightarrow Ad = 0$
- $x + \theta d \geq 0$

**Nonbasic indices** $(x_j = 0)$

- $d_j = 1$ ⟶ **Basic direction**
- $d_k = 0,\ \forall k \notin \{j, B(1), \ldots, B(m)\}$

# Feasible directions

## Computation

**Feasible direction** $d$

- $A(x + \theta d) = b \Longrightarrow Ad = 0$
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- $d_j = 1$ ⟶ **Basic direction**
- $d_k = 0, \ \forall k \notin \{j, B(1), \ldots, B(m)\}$

**Basic indices**

$$Ad = 0 \Rightarrow \sum_{i=1}^{n} A_i d_i = A_B d_B + A_j = 0 \Longrightarrow d_B = -A_B^{-1} A_j$$

# Feasible directions

## Computation

**Feasible direction $d$**

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

**Nonbasic indices**

- $d_j = 1$ → **Basic direction**
- $d_k = 0, \ \forall k \notin \{j, B(1), \dots, B(m)\}$

**Basic indices**

$$Ad = 0 = \sum_{i=1}^{n} A_i d_i = B d_B + A_j = 0 \implies d_B = -B^{-1} A_j$$

**Non-negativity (non-degenerate assumption)**

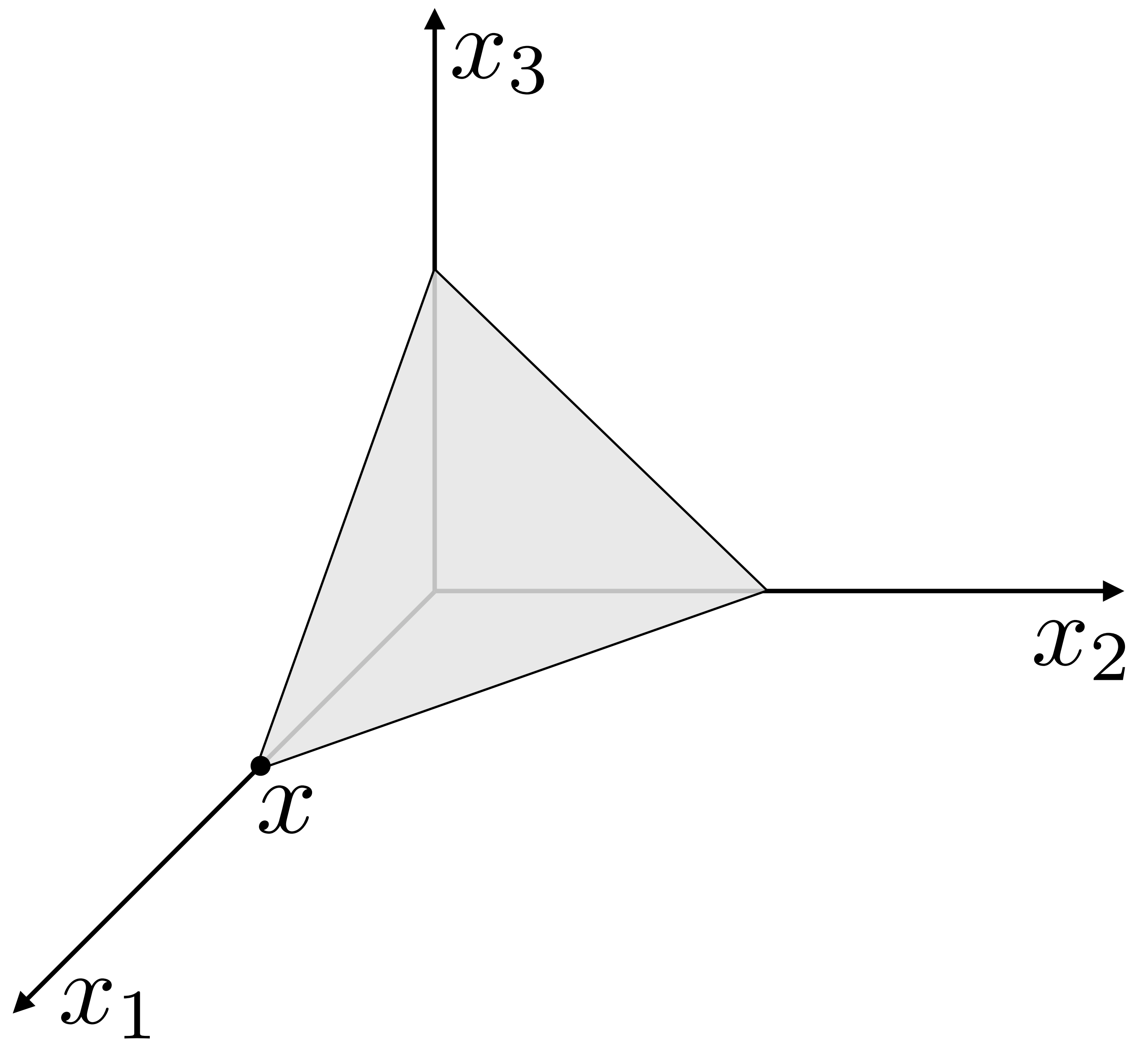
- Non-basic variables: $x_i = 0$. Nonnegative direction $d_i \geq 0$
- Basic variables: $x_B > 0$. Therefore $\exists \theta > 0$ such that $x_B + \theta d_B \geq 0$

# Feasible directions

## Example

$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

$$x = (2, 0, 0) \qquad B = \{1\}$$



$A = [1 \ 1 \ 1]$

# Feasible directions

## Example

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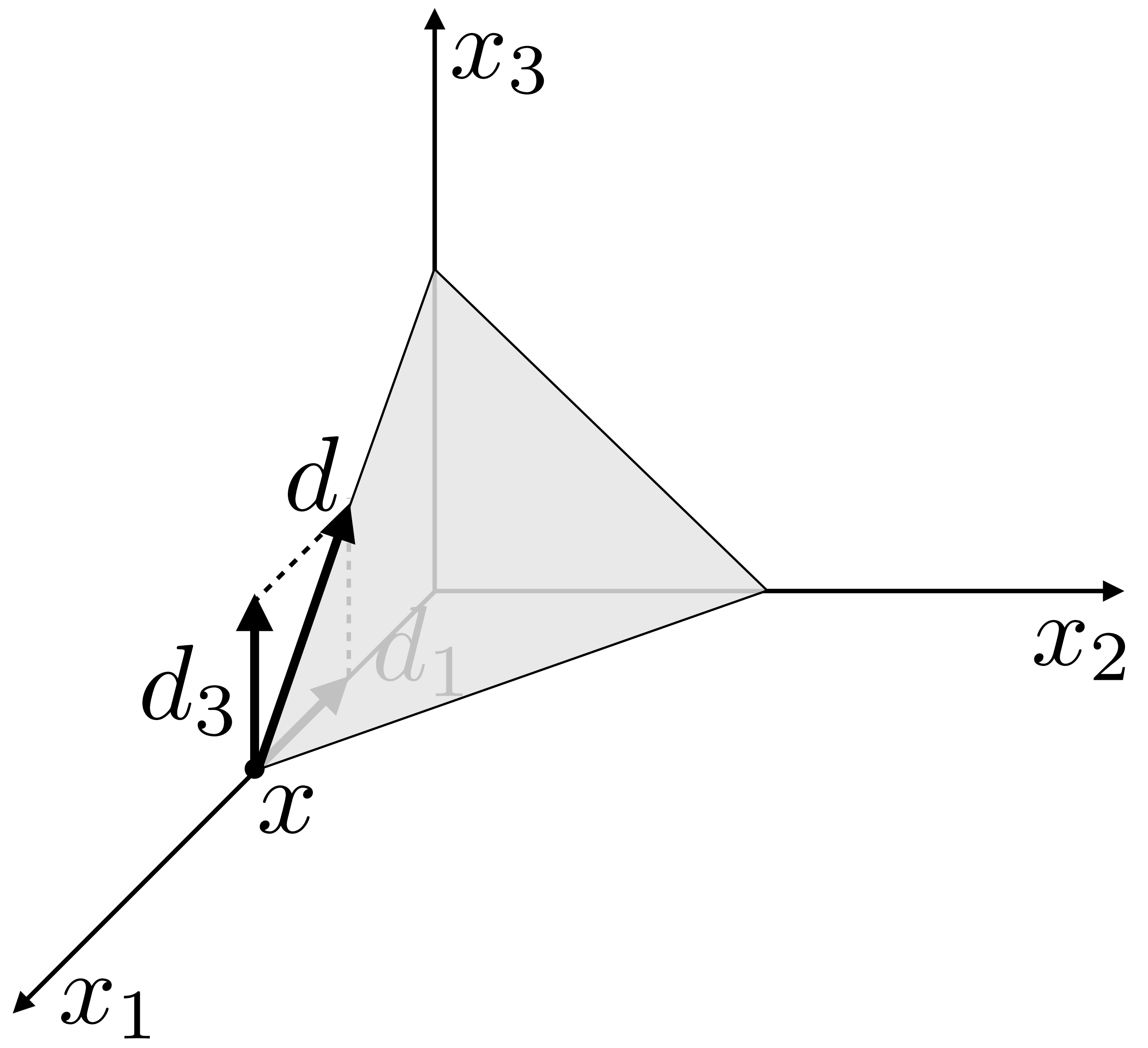


Non

**Basic index** $j = 3 \longrightarrow d = (-1, 0, 1)$

$$d_j = 1$$

$$A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1$$

# How does the cost change?

**Cost improvement**

$$c^T(x + \theta d) - c^T x = \theta c^T d$$

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**Cost improvement**

$$\underbrace{c^T(x+\theta d)}_{\text{New cost}} - \underbrace{c^T x}_{\text{Old cost}} = \theta c^T d$$

**New cost**

**Old cost**

We call $\bar{c}_j$ the **reduced cost** of (introducing) variable $x_j$ in the basis

$$\bar{c}_j = c^T d = \sum_{i=1}^{n} c_i d_i = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j$$

# Reduced costs

**Interpretation**

Change in objective/marginal cost of adding $x_j$ to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\bar{c}_j < 0$: adding $x_j$ will decrease the objective (good)

# Reduced costs

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Change in objective/marginal cost of adding $x_j$ to the basis

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Cost per-unit increase of variable $x_j$

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# Reduced costs

**Interpretation**

Change in objective/marginal cost of adding $x_j$ to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Cost per-unit increase of variable $x_j$ (refers to $c_j$)

Cost to change other variables compensating for $x_j$ to enforce $Ax = b$ (refers to $- c_B^T A_B^{-1} A_j$)

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# Reduced costs

**Interpretation**

Change in objective/marginal cost of adding $x_j$ to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Cost per-unit increase of variable $x_j$ (the $c_j$ term)

Cost to change other variables compensating for $x_j$ to enforce $Ax = b$ (the $c_B^T A_B^{-1} A_j$ term)

- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\bar{c}_j < 0$: adding $x_j$ will decrease the objective (good)

**Reduced costs for basic variables is 0**

$$\bar{c}_{B(i)} = c_{B(i)} - c_B^T A_B^{-1} A_{B(i)} = c_{B(i)} - c_B^T (A_B^{-1} A_B) e_i$$

$$= c_{B(i)} - c_B^T e_i = c_{B(i)} - c_{B(i)} = 0$$

$e_i = (0, 0 \ldots, 1, 0, \ldots)$

# Vector of reduced costs

**Reduced costs**

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

**Full vector in one shot?**

$$\bar{c} = (\bar{c}_1, \ldots, \bar{c}_n)$$

# Vector of reduced costs

**Reduced costs**

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Isolate basis $B$-related components $p$
(they are the same across $j$)

$$\bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T p$$

**Full vector in one shot?**

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# Vector of reduced costs

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**Full vector in one shot?**

$$\bar{c} = (\bar{c}_1, \ldots, \bar{c}_n)$$

Obtain $p$ by solving linear system

$$p = (A_B^{-1})^T c_B \quad \Rightarrow \quad A_B^T p = c_B$$

Note: $(M^{-1})^T = (M^T)^{-1}$
for any square invertible $M$

# Vector of reduced costs

**Reduced costs**

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**Computing reduced cost vector**

1. Solve $A_B^T p = c_B$
2. $\bar{c} = c - A^T p$

# Optimality conditions

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## Theorem

Let $x$ be a basic feasible solution associated with basis matrix ~~$B$~~ $A_B$.
Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then $x$ is **optimal**

# Optimality conditions

## Theorem

Let $x$ be a basic feasible solution associated with basis matrix $B$
Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then $x$ is **optimal**

**Remark**

This is a **stopping criterion** for the simplex algorithm.

If the **neighboring solutions** do not improve the cost, we are done (because of convexity).

# Optimality conditions

## Proof

For a basic feasible solution $x$ with basis $B$ the reduced costs are $\bar{c} \geq 0$.

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Since $x$ and $y$ are feasible, then $Ax = Ay = b$ and $Ad = 0$

$$Ad = A_B d_B + \sum_{i \in N} A_i d_i = 0 \quad \Rightarrow \quad d_B = -\sum_{i \in N} A_B^{-1} A_i d_i$$

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The change in objective is

$$c^T d = c_B^T d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c_B^T A_B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$$

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Since $y \geq 0$ and $x_i = 0,\ i \in N$, then $d_i = y_i - x_i \geq 0, i \in N$

$$c^T d = c^T (y - x) \geq 0 \quad \Rightarrow \quad c^T y \geq c^T x.$$

■

# Simplex iterations

# Stepsize

What happens if some $\bar{c}_j < 0$?

We can decrease the cost by bringing $x_j$ into the basis

# Stepsize

max $\theta$

s.t. $x_i + \theta d_i \geq 0 \quad \forall i$

if $d_i < 0$

$\Rightarrow \theta \geq -\frac{x_i}{d_i}$

What happens if some $\bar{c}_j$ < 0?

We can decrease the cost by bringing $x_j$ into the basis

**How far can we go?**

$\theta^\star = \max\{\theta \mid \theta \geq 0$ and $x + \theta d \geq 0\}$

$d$ is the $j$-th basic direction

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**Unbounded**

If $d \geq 0$, then $\theta^\star = \infty$. The LP is unbounded.

**Bounded**

If $d_i < 0$ for some $i$, then

$$\theta^\star = \min_{\{i | d_i < 0\}} \left( -\frac{x_i}{d_i} \right) = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right)$$

(Since $d_i \geq 0,\ i \notin B$)

# Moving to a new basis

**Next feasible solution**

$$x + \theta^\star d$$

# Moving to a new basis

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Let $B(\ell) \in \{B(1), \ldots, B(m)\}$ be the index such that $\theta^\star = -\dfrac{x_{B(\ell)}}{d_{B(\ell)}}$. Then,

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**New solution**

- $x_{B(\ell)}$ becomes $0$ (exits)
- $x_j$ becomes $\theta^\star$ (enters)

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**New basis**

$$A_{\bar{B}} = \begin{bmatrix} A_{B(1)} & \dots & A_{B(\ell-1)} & A_j & A_{B(\ell+1)} & \dots & A_{B(m)} \end{bmatrix}$$

# An iteration of the simplex method

## First part

We start with

- a basic feasible solution $x$
- a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots, A_{B(m)} \end{bmatrix}$

1. Compute the reduced costs $\bar{c}$
    - Solve $A_B^T p = c_B$
    - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, $x$ **optimal**. **break**
3. Choose $j$ such that $\bar{c}_j < 0$

# An iteration of the simplex method

## Second part

4. Compute search direction $d$ with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right)$
7. Define $y$ such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ ($i$ exits and $j$ enters)

# Example

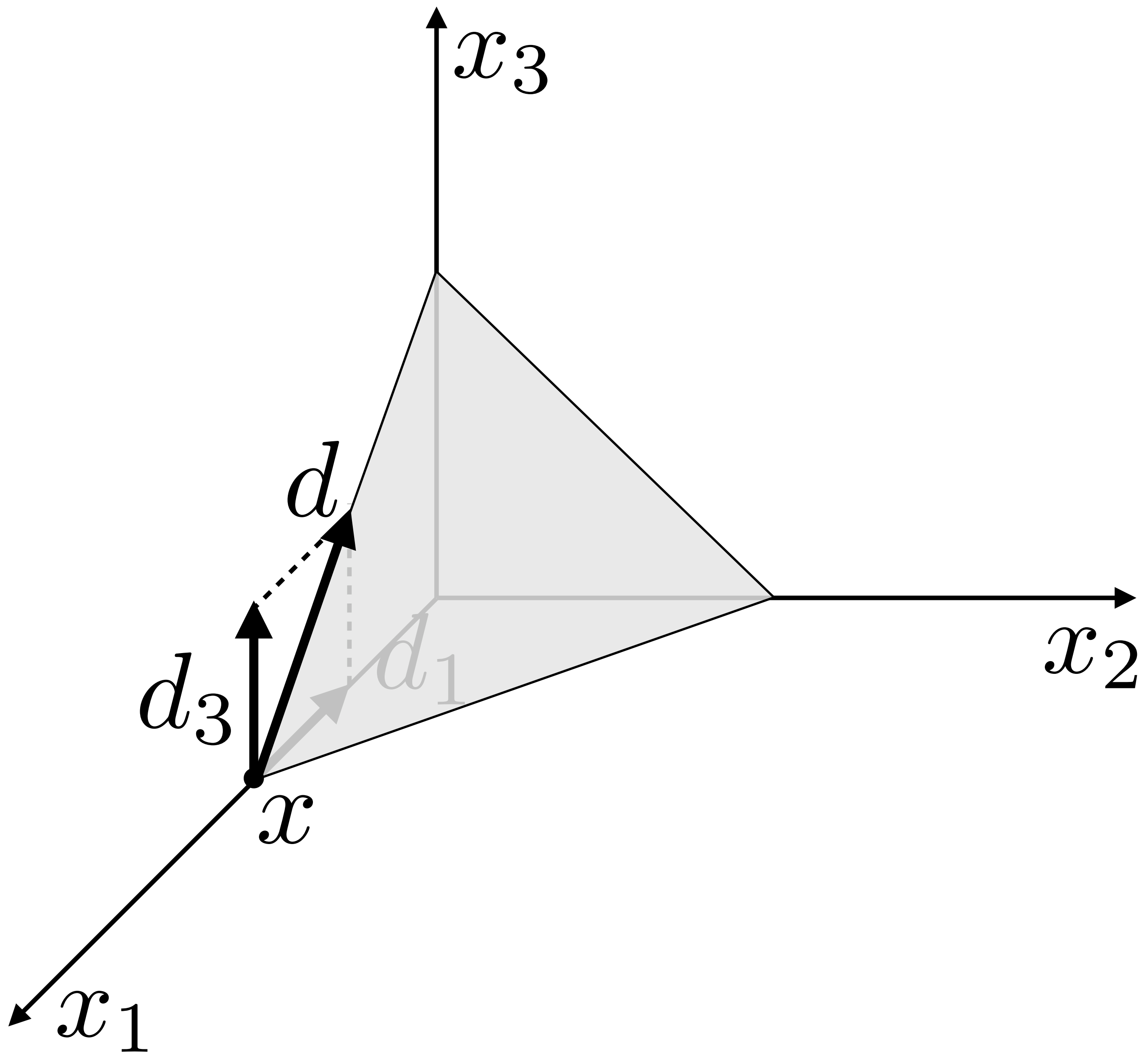
$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

$$x = (2, 0, 0) \qquad\qquad B = \{1\}$$

**Basic index** $j = 3 \longrightarrow d = (-1, 0, 1)$

$$d_j = 1$$

$$A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1$$



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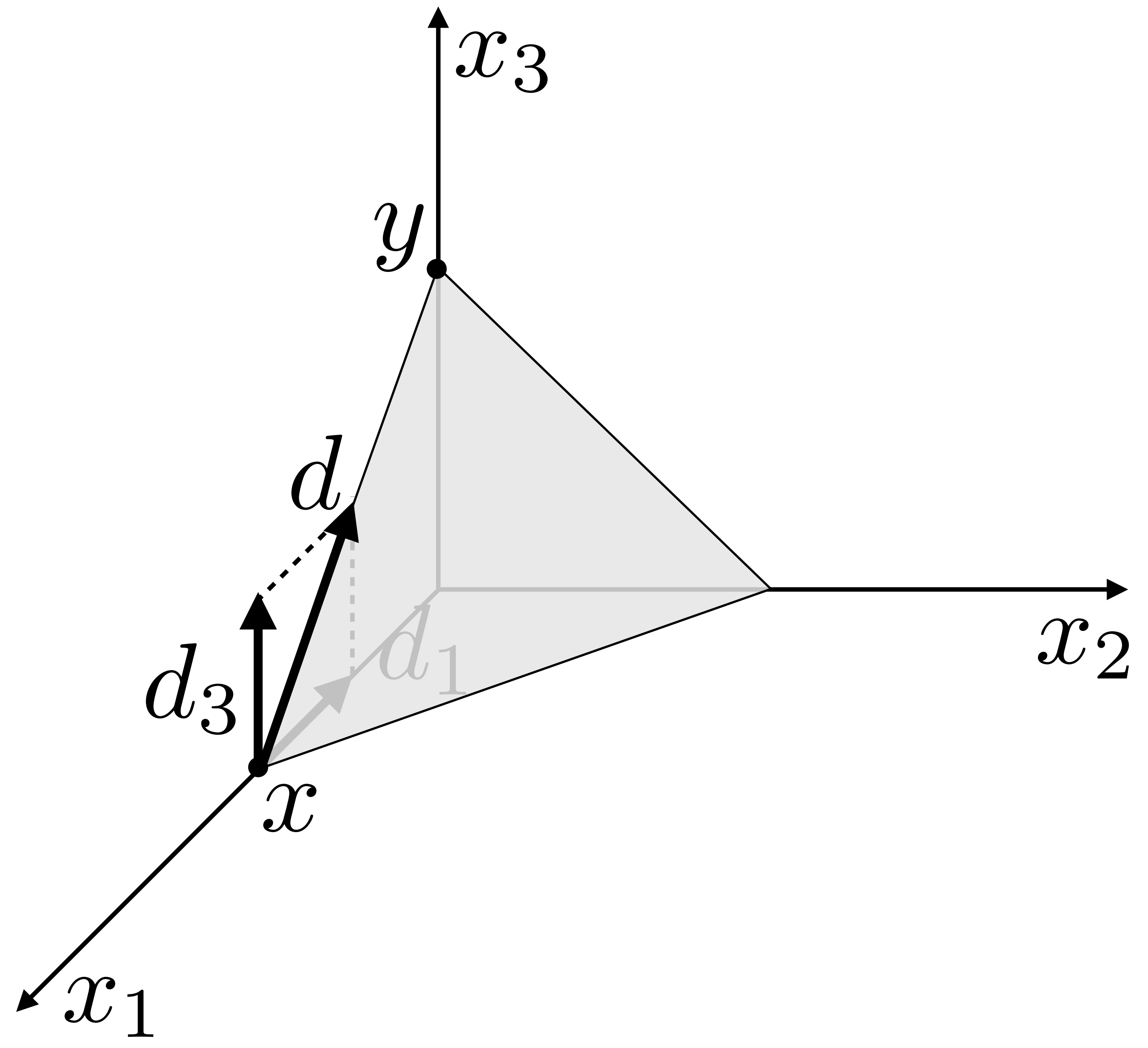

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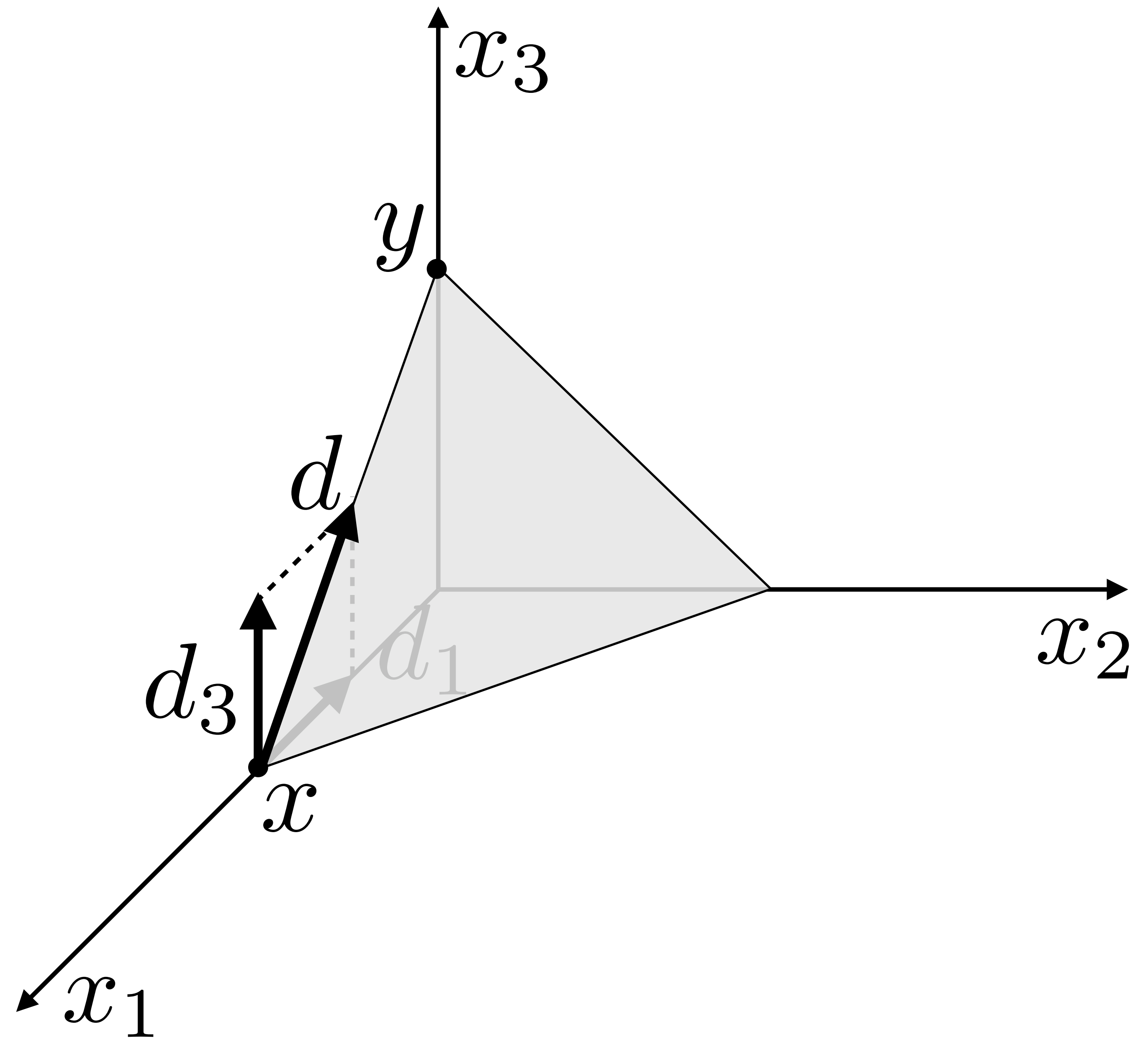
**Stepsize** $\theta^\star = -\dfrac{x_1}{d_1} = 2$

**New solution** $y = x + \theta^\star d = (0, 0, 2) \qquad \bar{B} = \{3\}$



# Finite convergence

**Assume** that

- $P = \{x \mid Ax = b, x \geq 0\}$ not empty
- Every basic feasible solution **non degenerate**

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**Assume** that

- $P = \{x \mid Ax = b, x \geq 0\}$ not empty
- Every basic feasible solution **non degenerate**

**Then**

- The simplex method **terminates after a finite number of iterations**
- At termination we either have one of the following
  - an **optimal basis** $B$
  - a **direction** $d$ such that $Ad = 0,\ d \geq 0,\ c^T d < 0$ and the optimal cost is $-\infty$

# Finite convergence

## Proof sketch

At each iteration the algorithm improves

- by a **positive** amount $\theta^\star$
- along the **direction** $d$ such that $c^T d < 0$

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Therefore

- The cost strictly decreases
- No basic feasible solution can be visited twice

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## Proof sketch

At each iteration the algorithm improves

- by a **positive** amount $\theta^\star$
- along the **direction** $d$ such that $c^T d < 0$

Therefore

- The cost strictly decreases
- No basic feasible solution can be visited twice

Since there is a **finite number of basic feasible solutions**

The algorithm **must eventually terminate**

■

# The simplex method

Today, we learned to:

- **Iterate** between basic feasible solutions
- **Verify** optimality and unboundedness conditions
- **Apply** a single iteration of the simplex method
- **Prove** finite convergence of the simplex method in the non-degenerate case

# Next lecture

- Finding initial basic feasible solution
- Degeneracy
- Complexity