

ORF522 – Linear and Nonlinear Optimization

3. Geometry and polyhedra

Ed forum

Questions and notes

- Ed Forum: put questions/comments for each lecture under “Notes - Lecture XX” instead of General “Questions”.
- SVM setting: why from $a^T v_i - b > 0 (< 0)$ to $s_i(a^T v_i - b_i) \geq 1$? (numerical issues + ambiguity)
- $\ell_1 - \ell_0$ equivalence? Not always and difficult to verify! (Compressed sensing literature).
- Why is ℓ_0 penalty not linear? It can be modeled with mixed-integer constraints. We will see it in the branch-and-bound lecture.

Today's agenda

Readings [Chapter 2, Bertsimas and Tsitsiklis]

- Polyhedra and linear algebra
- Corners: extreme points, vertices, basic feasible solutions
- Constructing basic solutions
- Existence and optimality of extreme points

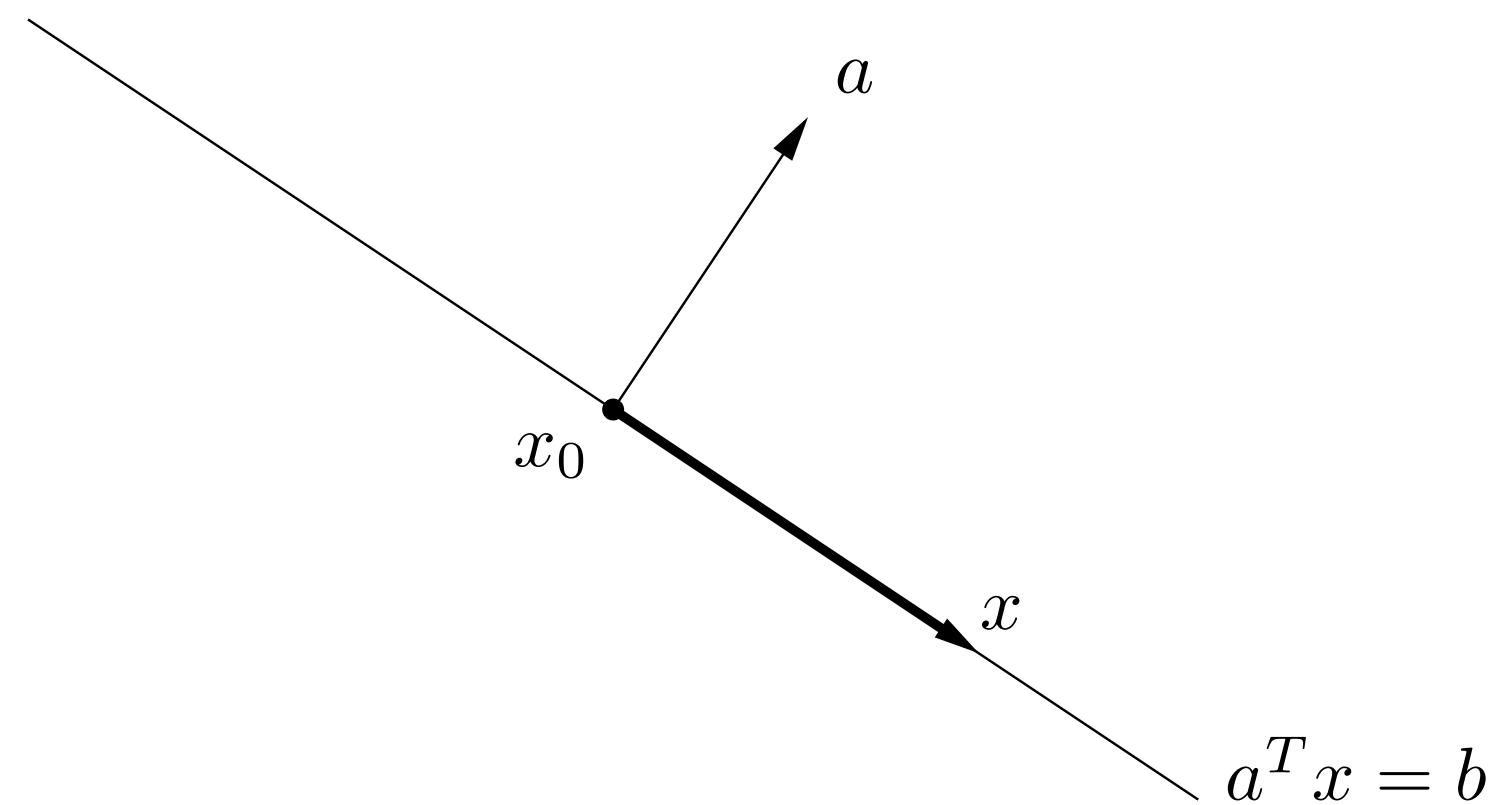
Polyhedra and linear algebra

Hyperplanes and halfspaces

Definitions

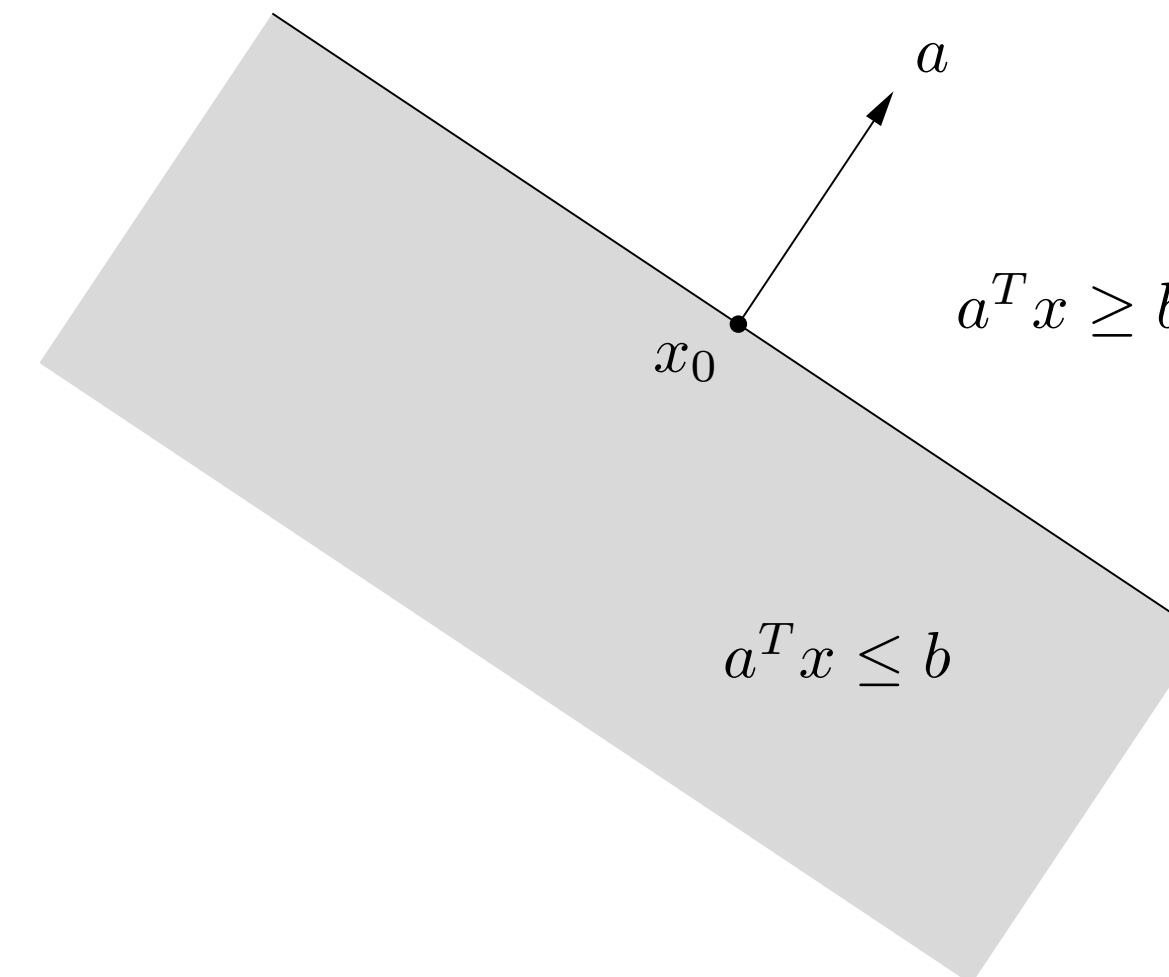
Hyperplane

$$\{x \mid a^T x = b\}$$



Halfspace

$$\{x \mid a^T x \leq b\}$$

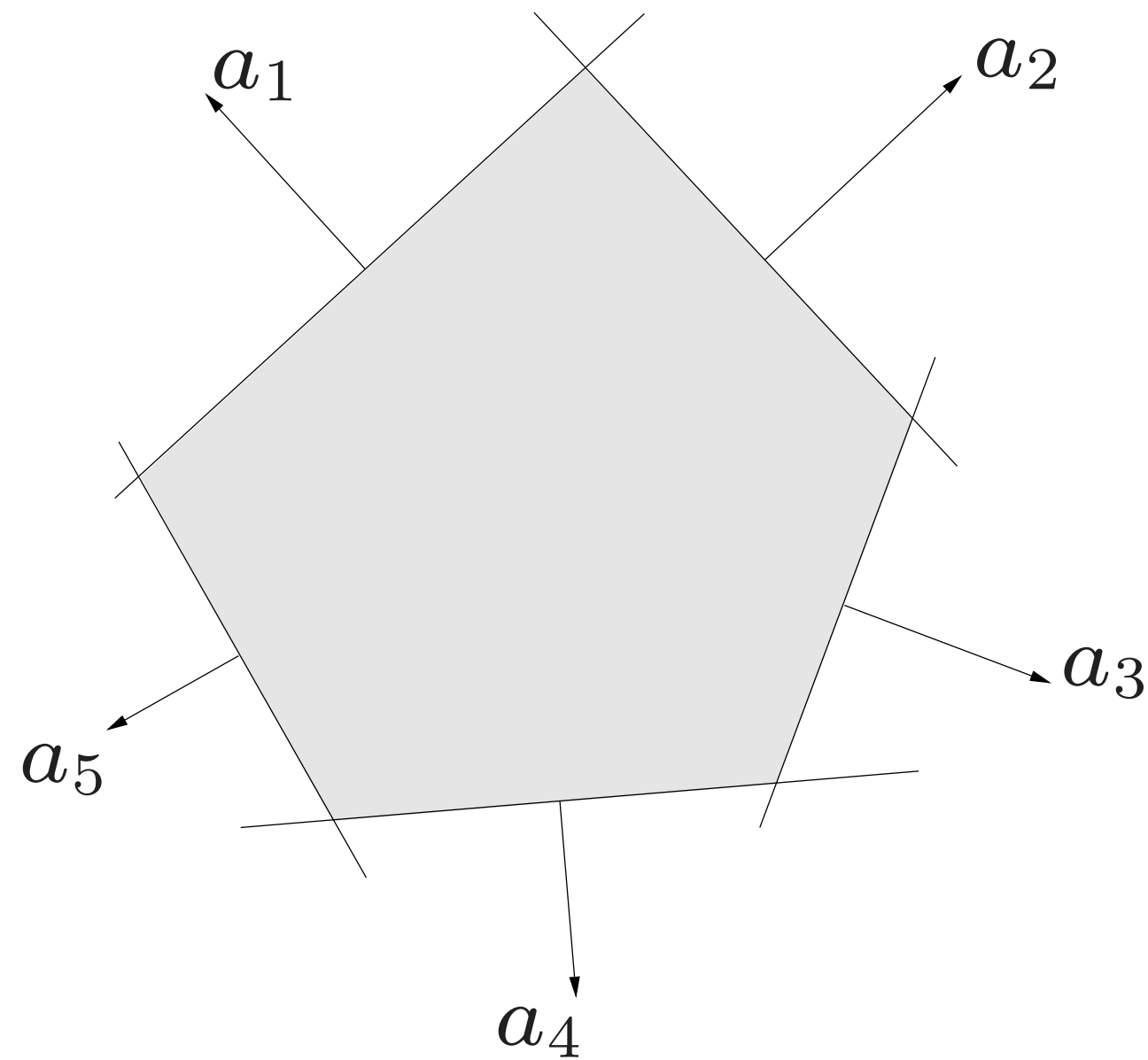


- x_0 is a specific point in the hyperplane
- For any x in the hyperplane defined by $a^T x = b$, $x - x_0 \perp a$
- The halfspace determined by $a^T x \leq b$ extends in the direction of $-a$

Polyhedron

Definition

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\} = \{x \mid Ax \leq b\}$$



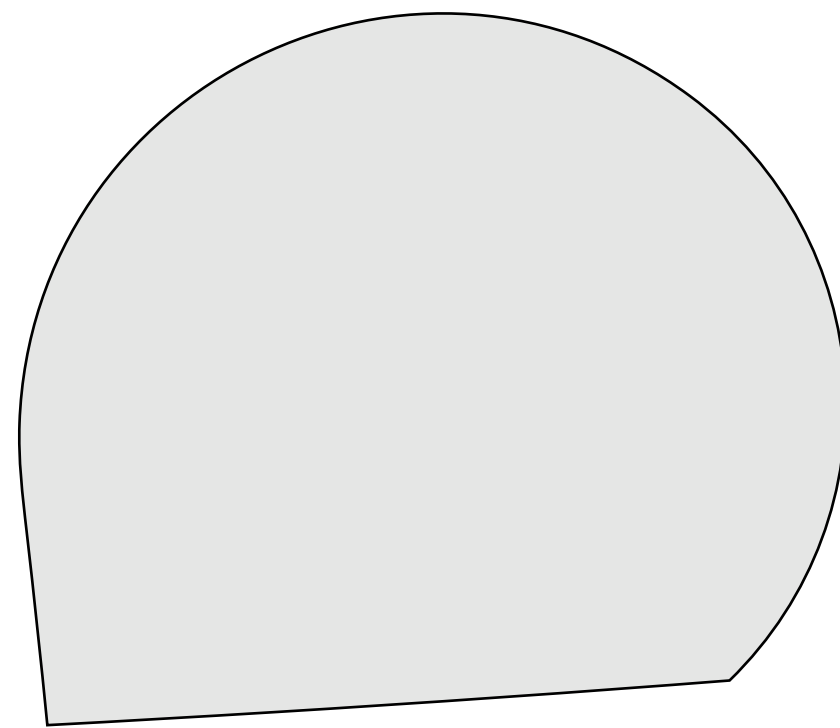
- Intersection of finite number of halfspaces
- Can include equalities

Convex set

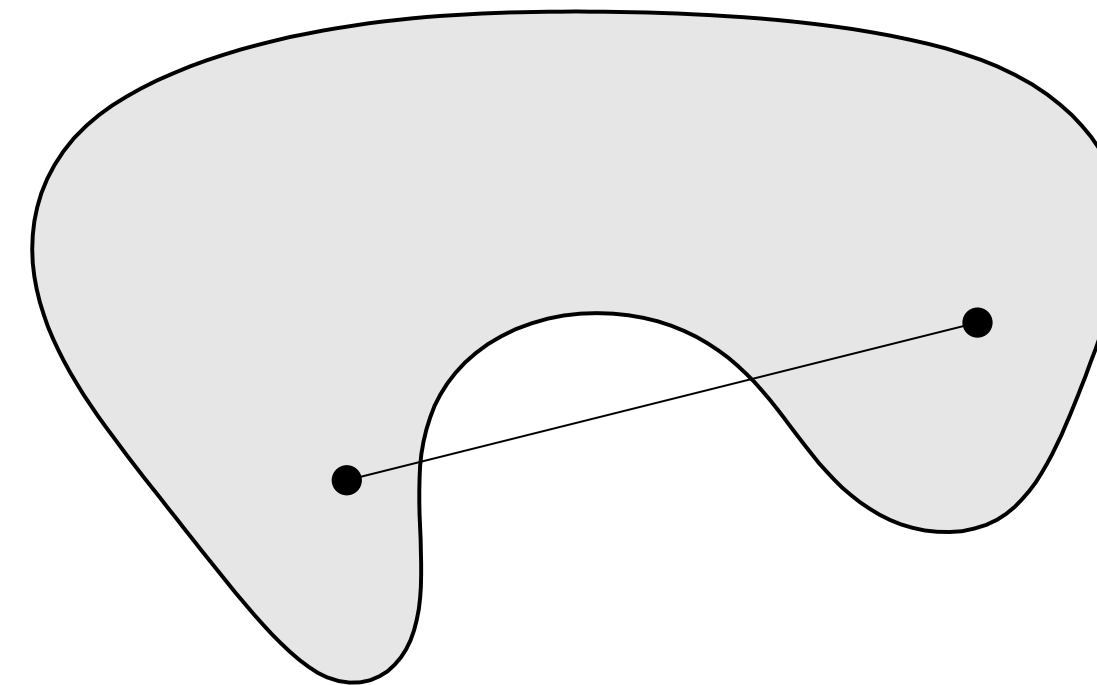
Definition

For any $x, y \in C$ and any $\alpha \in [0, 1]$

$$\alpha x + (1 - \alpha)y \in C$$



Convex



Not convex

Examples

- $\mathbf{R}^n$
- Hyperplanes
- Halfspaces
- Polyhedra

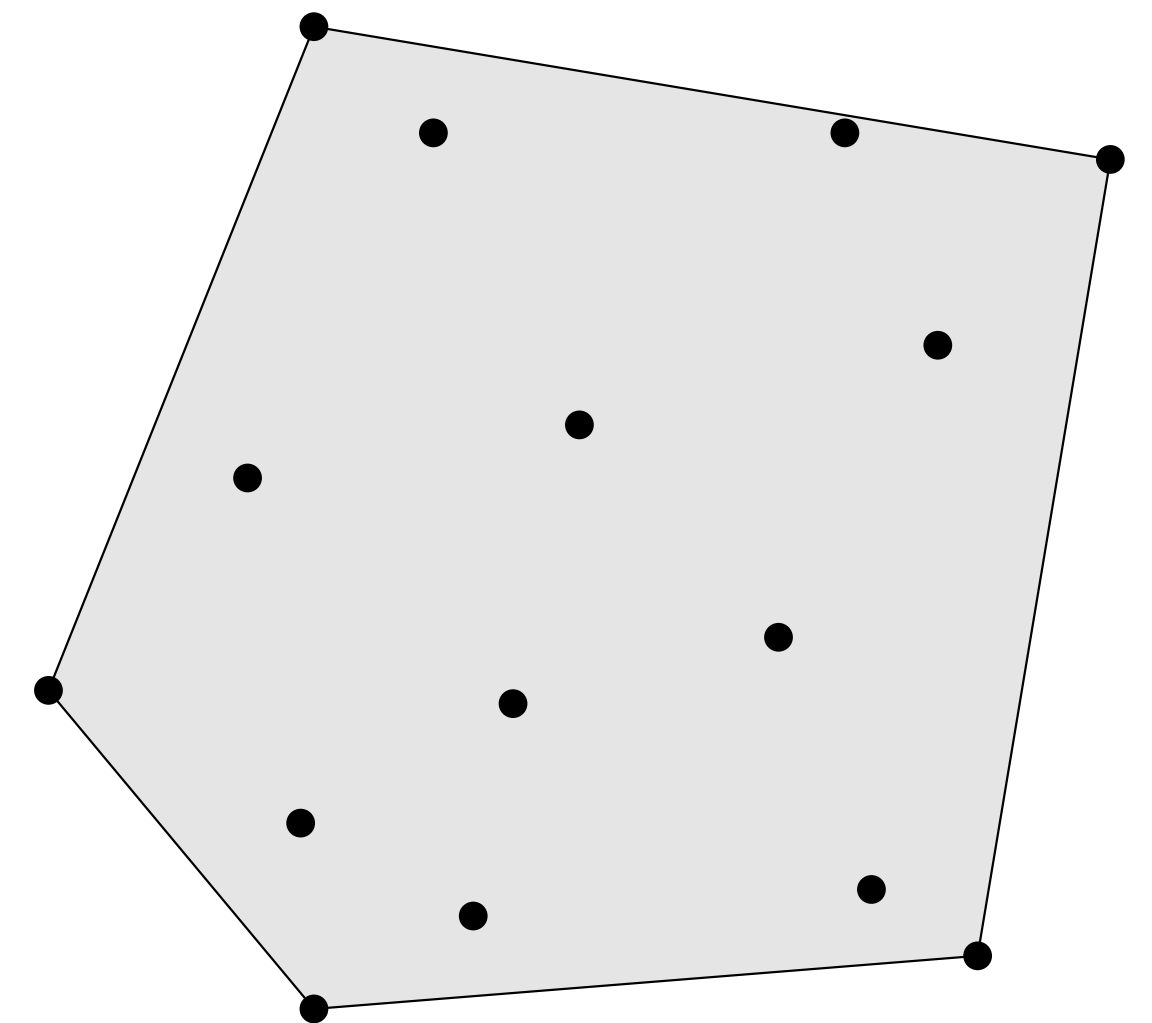
Convex combinations

Convex combination

$\alpha_1 x_1 + \cdots + \alpha_k x_k$ for any $x_1, \dots, x_k$ and $\alpha_1, \dots, \alpha_k$ such that $\alpha_i \geq 0$, $\sum_{i=1}^k \alpha_i = 1$

Convex hull

$$\text{conv } C = \left\{ \sum_{i=1}^k \alpha_i x_i \mid x_i \in C, \alpha_i \geq 0, \mathbf{1}^T \alpha = 1 \right\}$$



Linear independence

a nonempty set of vectors $\{v_1, \dots, v_k\}$ is **linearly independent** if

$$\alpha_1 v_1 + \dots + \alpha_k v_k = 0$$

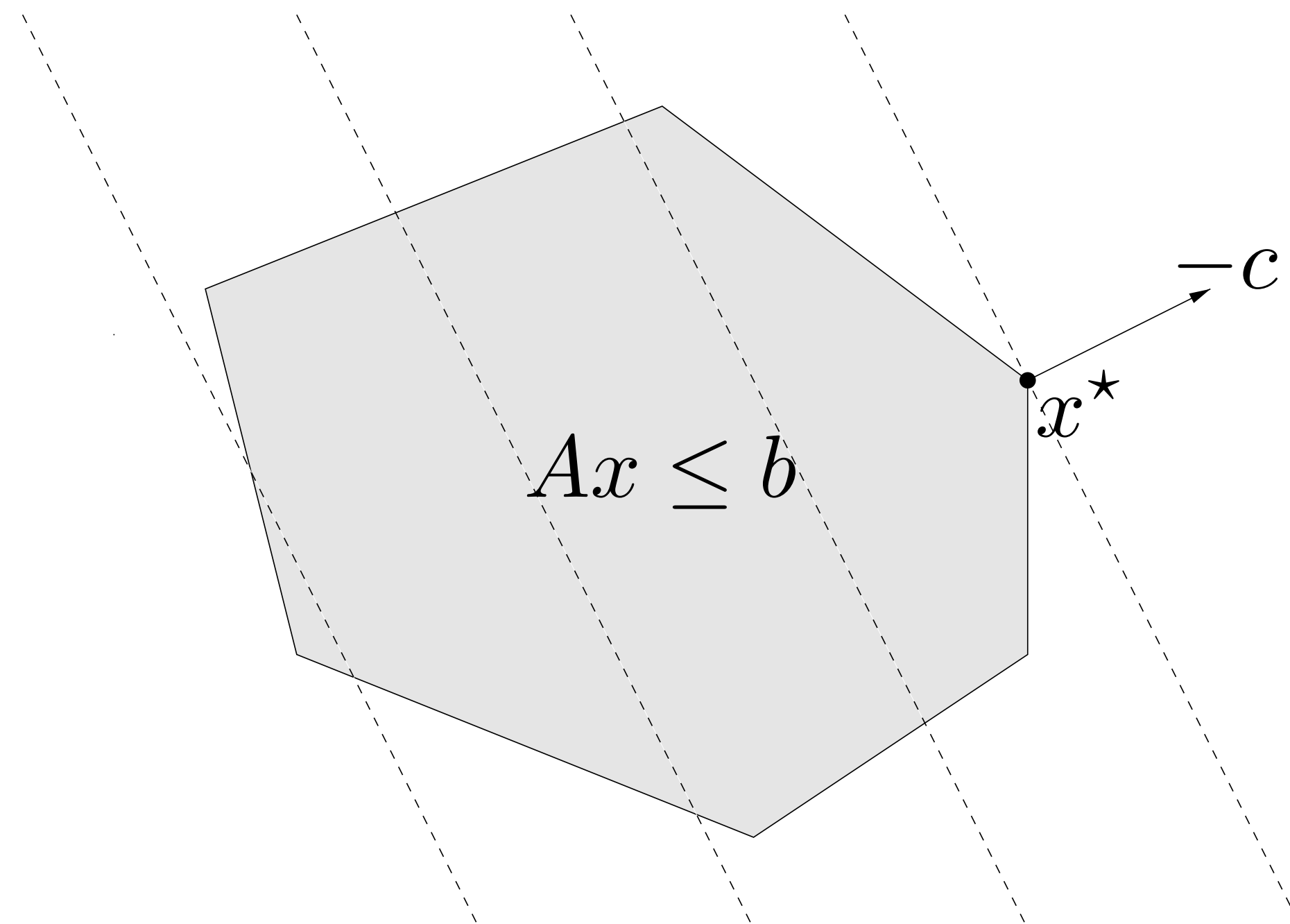
holds only for $\alpha_1 = \dots = \alpha_k = 0$

Properties

- The coefficients α_k in a linear combination $x = \alpha_1 v_1 + \dots + \alpha_k v_k$ are unique
- None of the vectors v_i is a linear combination of the other vectors

Geometrical interpretation of linear optimization

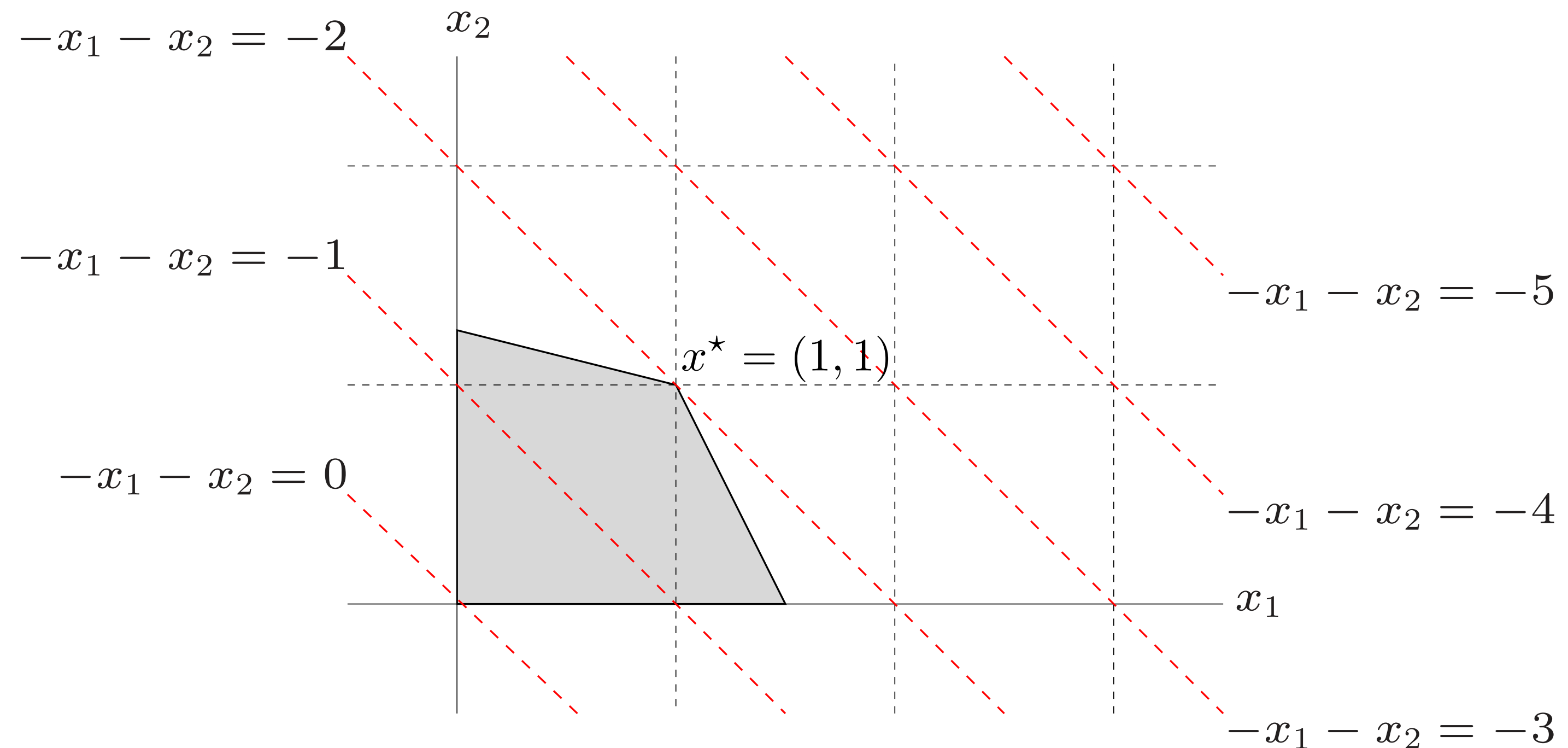
$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$



Dashed lines (hyperplanes) are level sets $c^T x = \alpha$ for different α

Example of linear optimization

minimize $-x_1 - x_2$
subject to $2x_1 + x_2 \leq 3$
 $x_1 + 4x_2 \leq 5$
 $x_1 \geq 0, x_2 \geq 0$



Optimal solutions tend to be at a “**corner**” of the feasible set

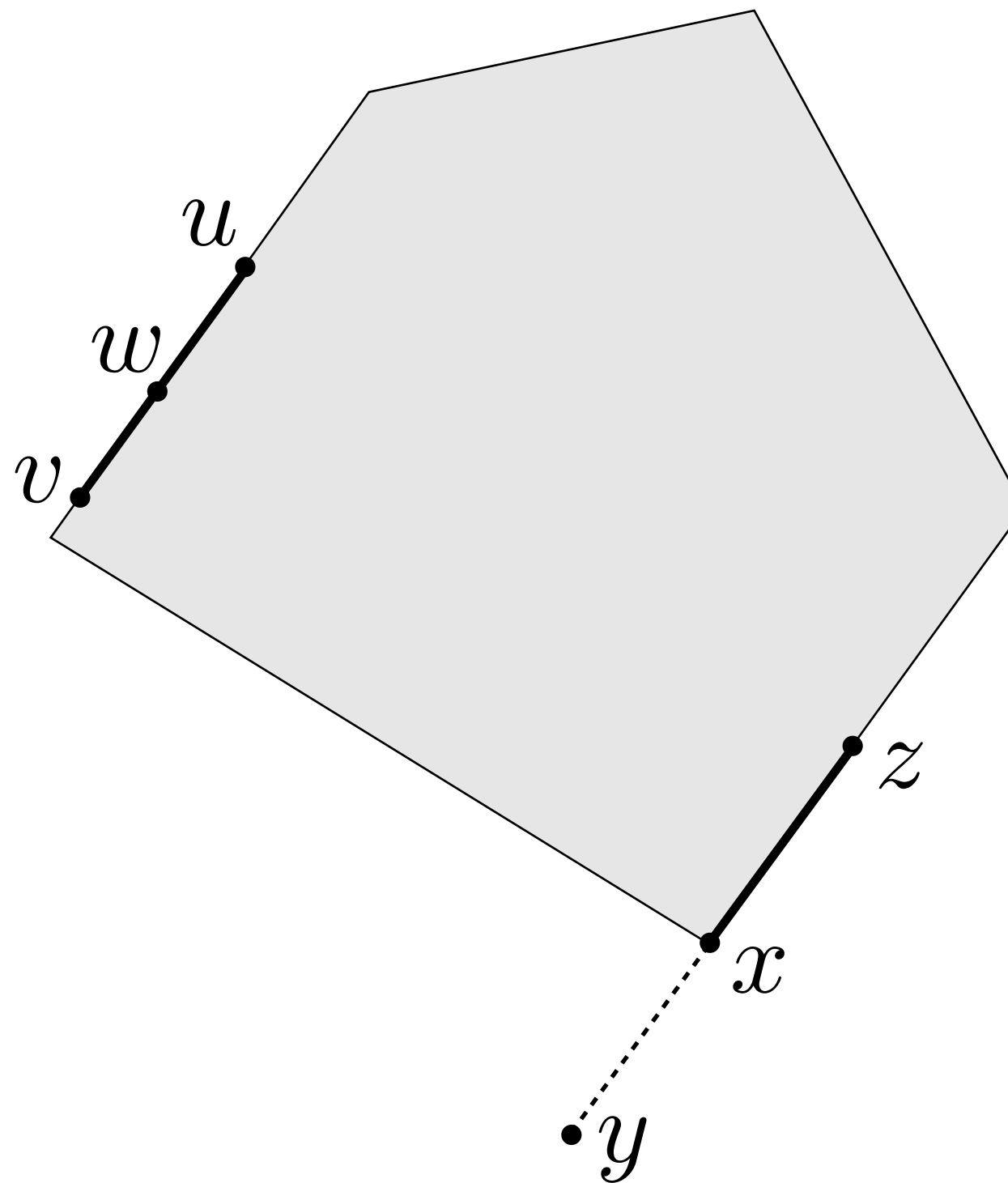
How do we formalize it?

Corners of linear optimization

Extreme points

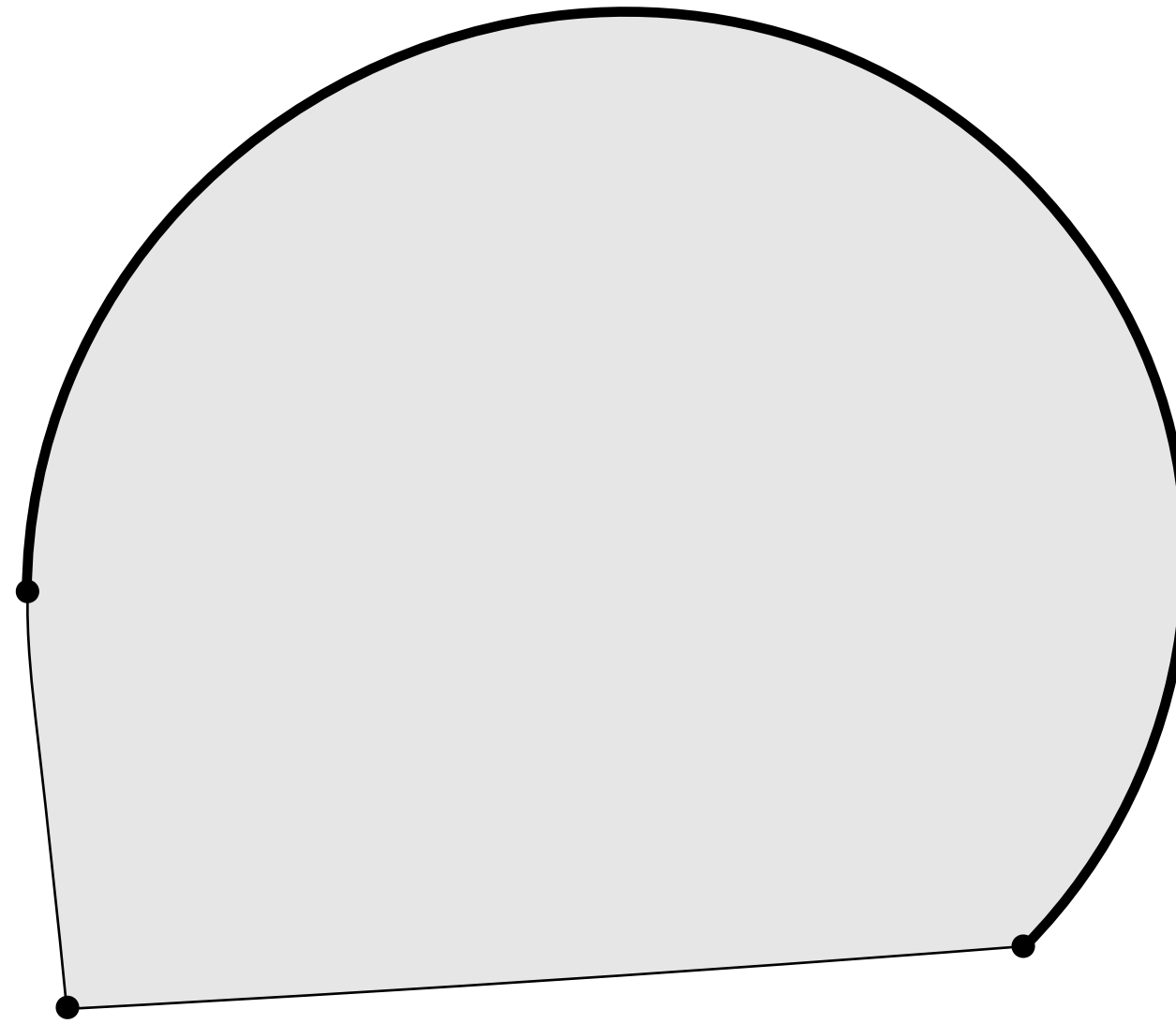
Definition

$x \in P$ is said to be an **extreme point** of P if
 $\nexists y, z \in P$ ($y \neq x, z \neq x$) and $\alpha \in [0, 1]$ such that $x = \alpha y + (1 - \alpha)z$



Extreme points

Convex sets



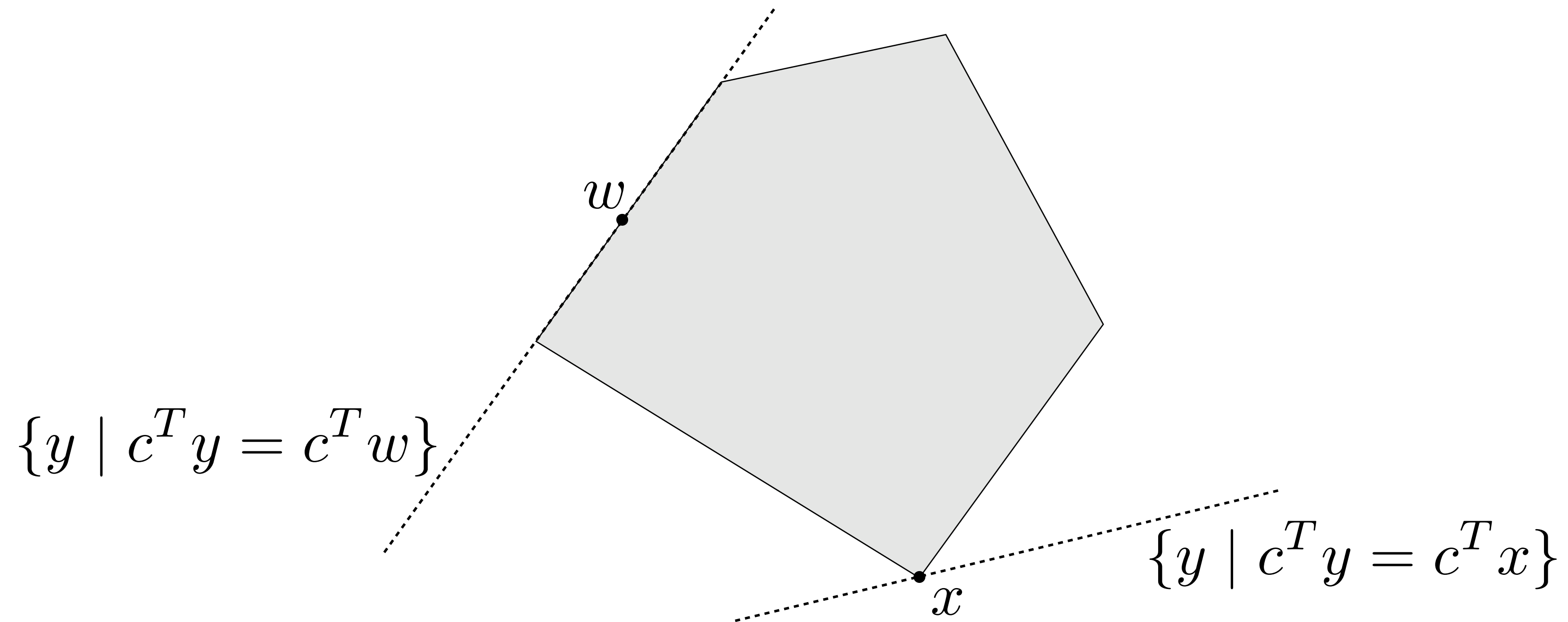
- Convex sets can have an infinite number of extreme points
- Polyhedra are convex sets with a finite number of extreme points

Vertices

Definition

$x \in P$ is a **vertex** if $\exists c$ such that x is the unique optimum of

$$\begin{array}{ll} \text{minimize} & c^T y \\ \text{subject to} & y \in P \end{array}$$



Basic feasible solution

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$$

Active constraints at $\bar{x}$

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \dots, m\} \mid a_i^T \bar{x} = b_i\}$$

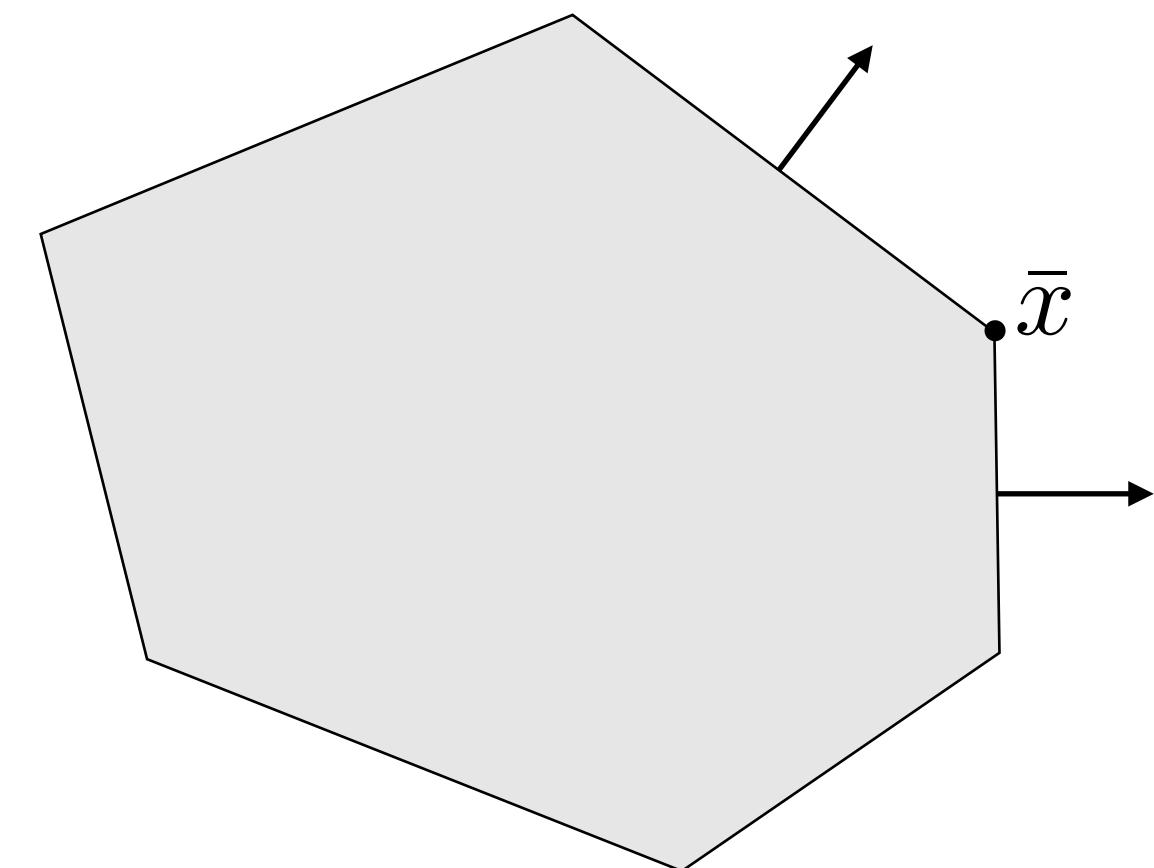
Index of all the constraints
satisfied as **equality**

Basic solution $\bar{x}$

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

Basic feasible solution $\bar{x}$

- $\bar{x} \in P$
- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

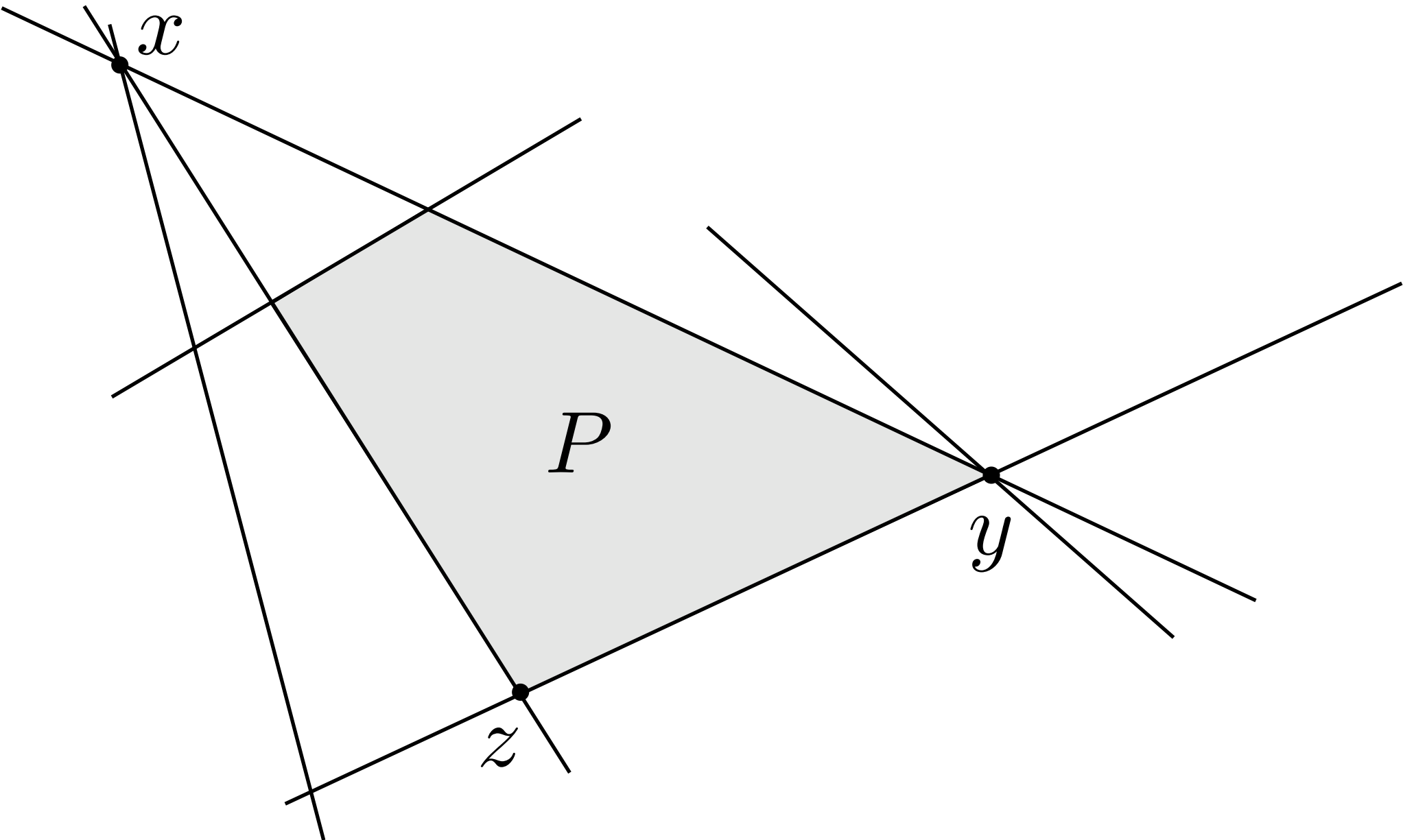


Degenerate basic feasible solutions

A solution $\bar{x}$ is degenerate if $|\mathcal{I}(\bar{x})| > n$

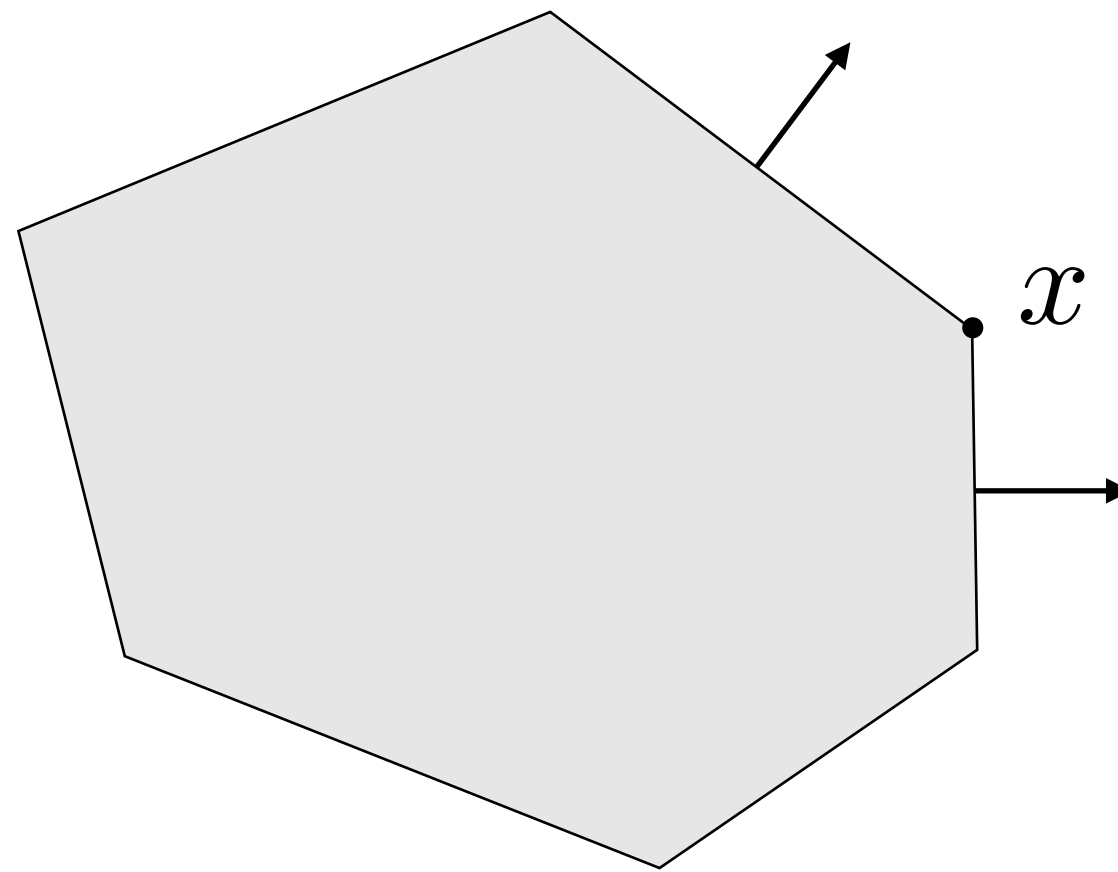
True or False?

	Basic	Feasible	Degenerate
x			
y			
z			



Equivalence Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



Let $x \in P$

x is a **vertex** $\iff x$ is an **extreme point** $\iff x$ is a **basic feasible solution**

Equivalent theorem proof

Vertex $\rightarrow$ Extreme point

If x is a vertex, $\exists c$ such that $c^T x < c^T y, \quad \forall y \in P, y \neq x$

Let's assume x is not an extreme point:

$\exists y, z \neq x$ such that $x = \lambda y + (1 - \lambda)z$

Since x is a vertex, $c^T x < c^T y$ and $c^T x < c^T z$

Therefore, $c^T x = \lambda c^T y + (1 - \lambda)c^T z > \lambda c^T x + (1 - \lambda)c^T x = c^T x$

$\Rightarrow$ **contradiction**



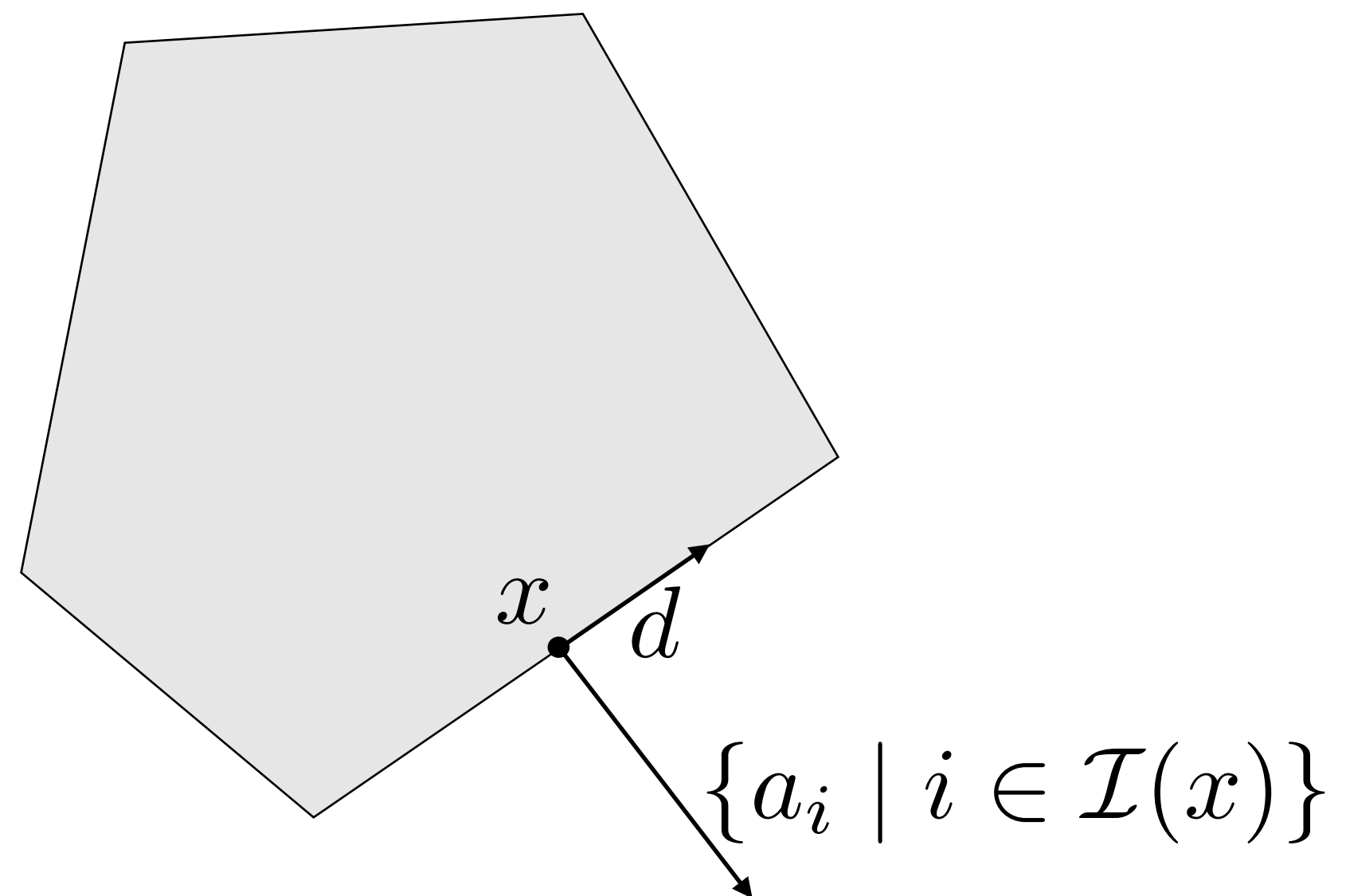
Equivalent theorem proof

Extreme point $\rightarrow$ Basic feasible solution (proof by contraposition)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span $\mathbf{R}^n$

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$



Equivalent theorem proof

Extreme point $\rightarrow$ Basic feasible solution (proof by contraposition)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span $\mathbf{R}^n$

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$

Let $\epsilon > 0$ and define $y = x + \epsilon d$ and $z = x - \epsilon d$

For $i \in \mathcal{I}(x)$ we have $a_i^T y = b_i$ and $a_i^T z = b_i$

For $i \notin \mathcal{I}(x)$ we have $a_i^T x < b_i \Rightarrow a_i^T (x + \epsilon d) < b_i$ and $a_i^T (x - \epsilon d) < b_i$

Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda)z$ with $\lambda = 0.5$.

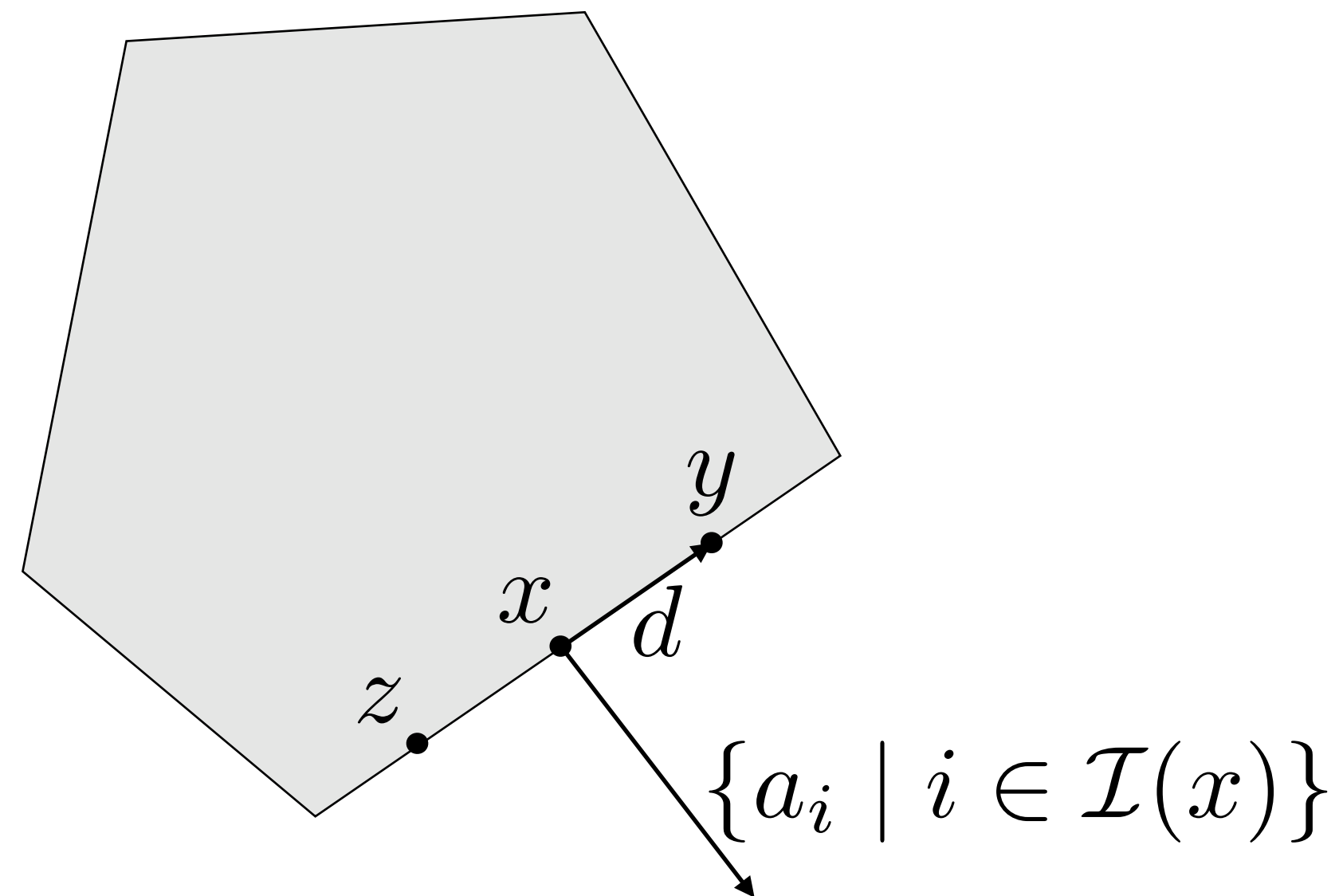
$\Rightarrow x$ is **not an extreme point**



Equivalent theorem proof

Extreme point $\rightarrow$ Basic feasible solution (proof by contraposition)

Suppose $x \in P$ is not basic feasible solution



Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda)z$ with $\lambda = 0.5$.

$\implies x$ is not an extreme point



Equivalent theorem proof

Basic feasible solution $\rightarrow$ Vertex

Left as exercise

Hint

Define $c = \sum_{i \in \mathcal{I}(x)} a_i$

Constructing basic solutions

Standard form polyhedra

Definition

Standard form LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

Assumption

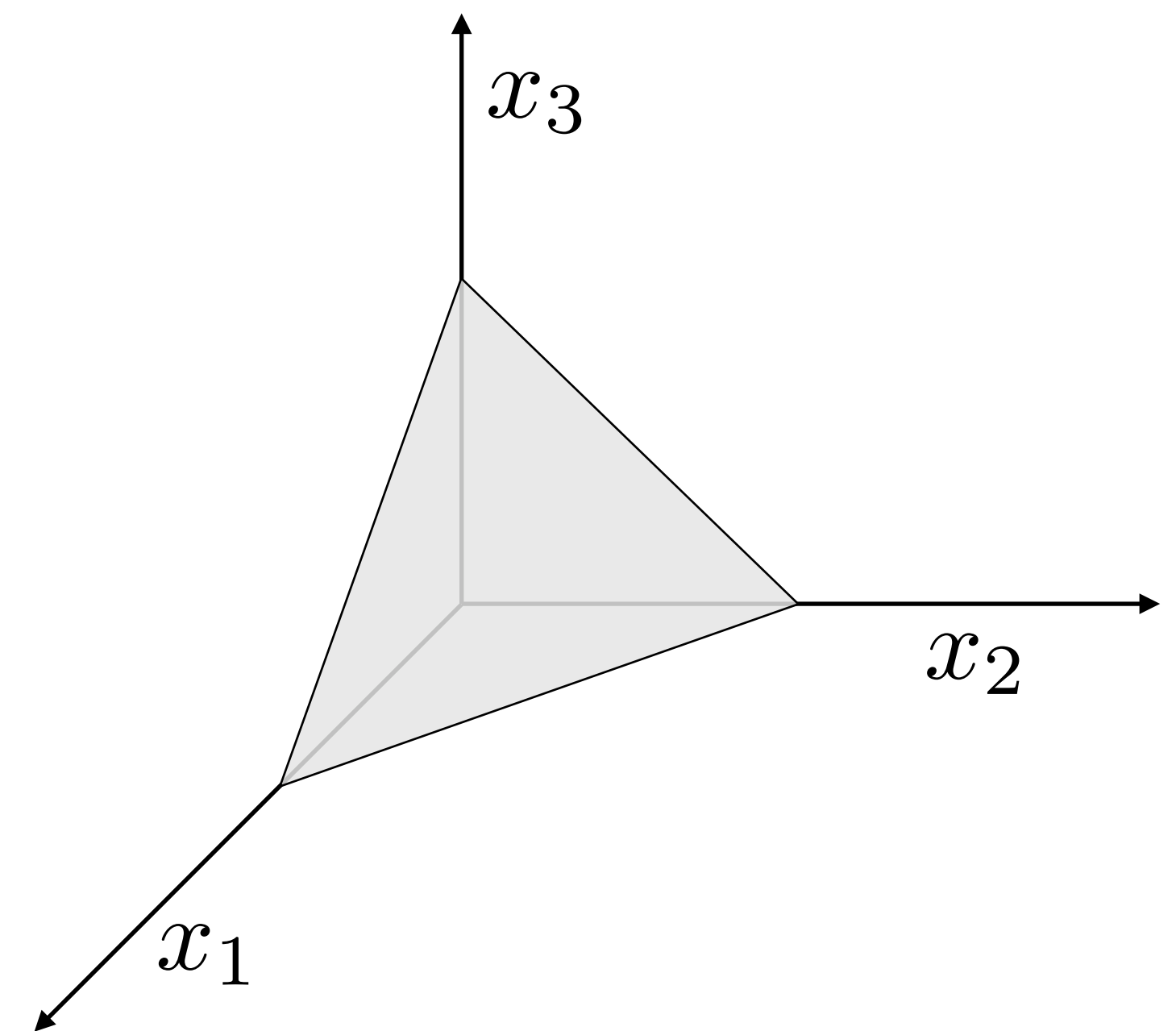
$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P lives in $(n - m)$ -dimensional subspace

Standard form polyhedron

$$P = \{x \mid Ax = b, x \geq 0\}$$



Basic solutions

Standard form polyhedra

$$P = \{x \mid Ax = b, x \geq 0\} \quad \text{with} \quad A \in \mathbf{R}^{m \times n} \text{ has full row rank } m \leq n$$

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
 - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
 - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

Constructing basic solution

1. Choose any m independent columns of A : $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

$$\begin{array}{c} \text{Basis} \\ \text{matrix} \end{array} \quad \begin{array}{c} \text{Basis columns} \end{array} \quad \begin{array}{c} \text{Basic variables} \end{array}$$

$$A_B = \left[\begin{array}{c|c|c|c} | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{array} \right], \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

Finding a basic solution

$$\begin{bmatrix} 1 & \boxed{0} & 1 & \boxed{0} & \boxed{1} \\ 2 & \boxed{-1} & -3 & \boxed{0} & \boxed{0} \\ 0 & \boxed{2} & 8 & \boxed{1} & \boxed{2} \end{bmatrix} \begin{bmatrix} x_1 \\ \boxed{x_2} \\ x_3 \\ \boxed{x_4} \\ \boxed{x_5} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

$A_{B(1)}$ $A_{B(2)}$ $A_{B(3)}$

Solve

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \geq 0$$

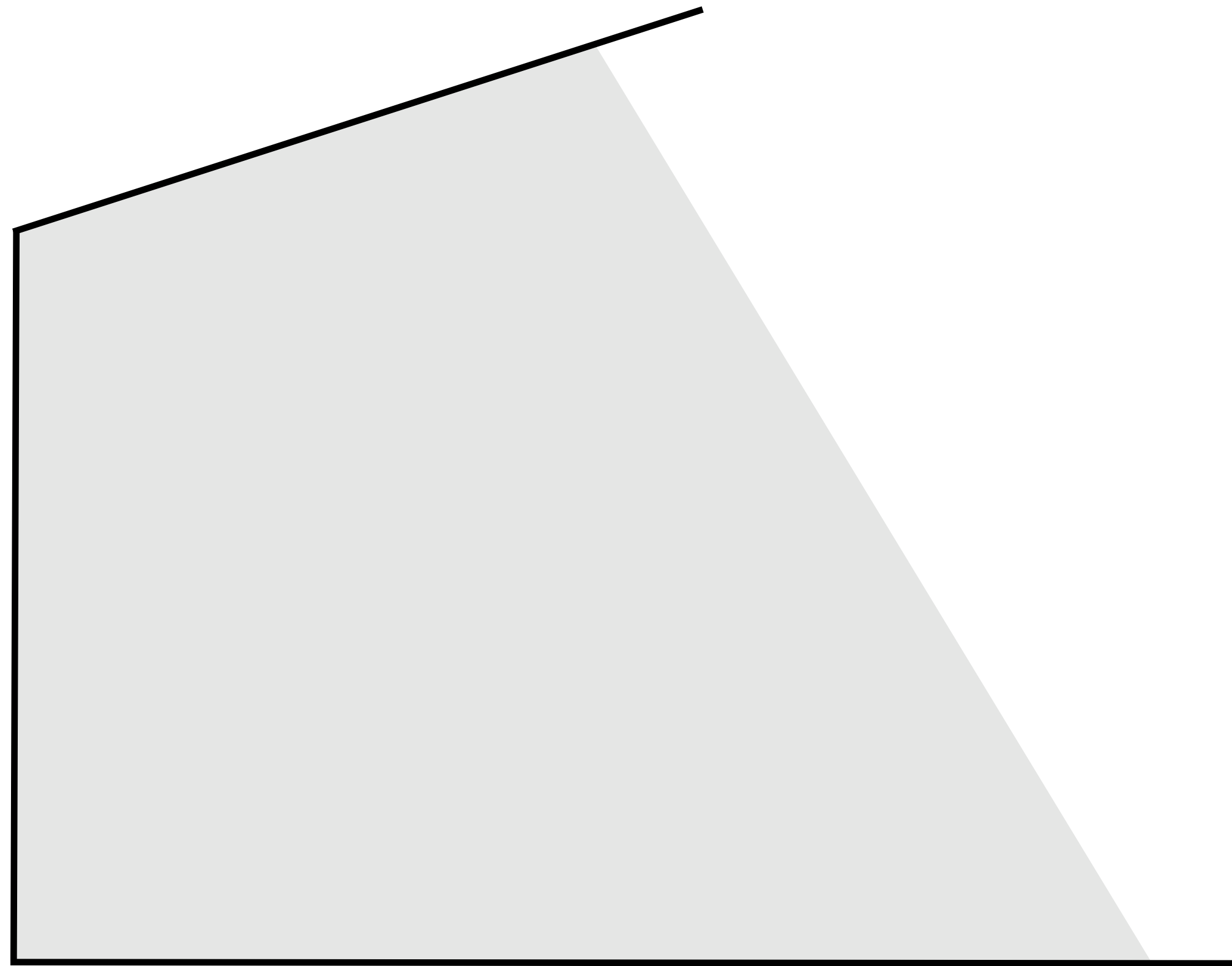
Existence and optimality of extreme points

Existence of extreme points

Example



No extreme points



Extreme points

Existence of extreme points

Characterization

A polyhedron P **contains a line** if

$\exists x \in P$ and a nonzero vector d such that $x + \lambda d \in P, \forall \lambda \in \mathbf{R}$.

Given a polyhedron $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$, the following are **equivalent**

- P does not contain a line
- P has at least one extreme point
- n of the a_i vectors are linearly independent

Corollary

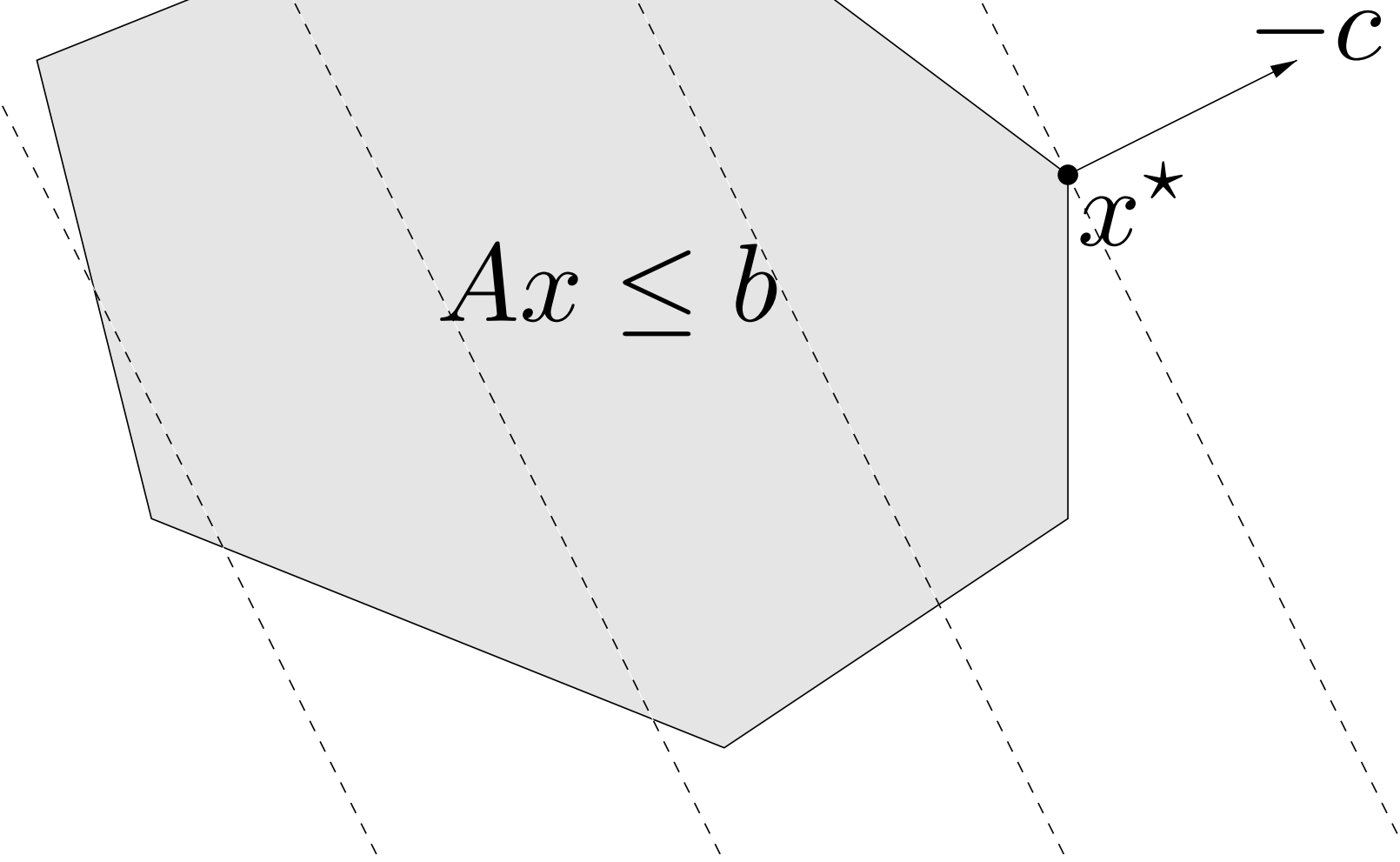
Every nonempty **bounded polyhedron** has
at least one basic feasible solution

Optimality of extreme points

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

- If
- P has at least one extreme point
 - There exists an optimal solution x^*

Then, there exists an optimal solution which is an **extreme point** of P



We only need to search between **extreme points**

How to search among basic feasible solutions?

Idea

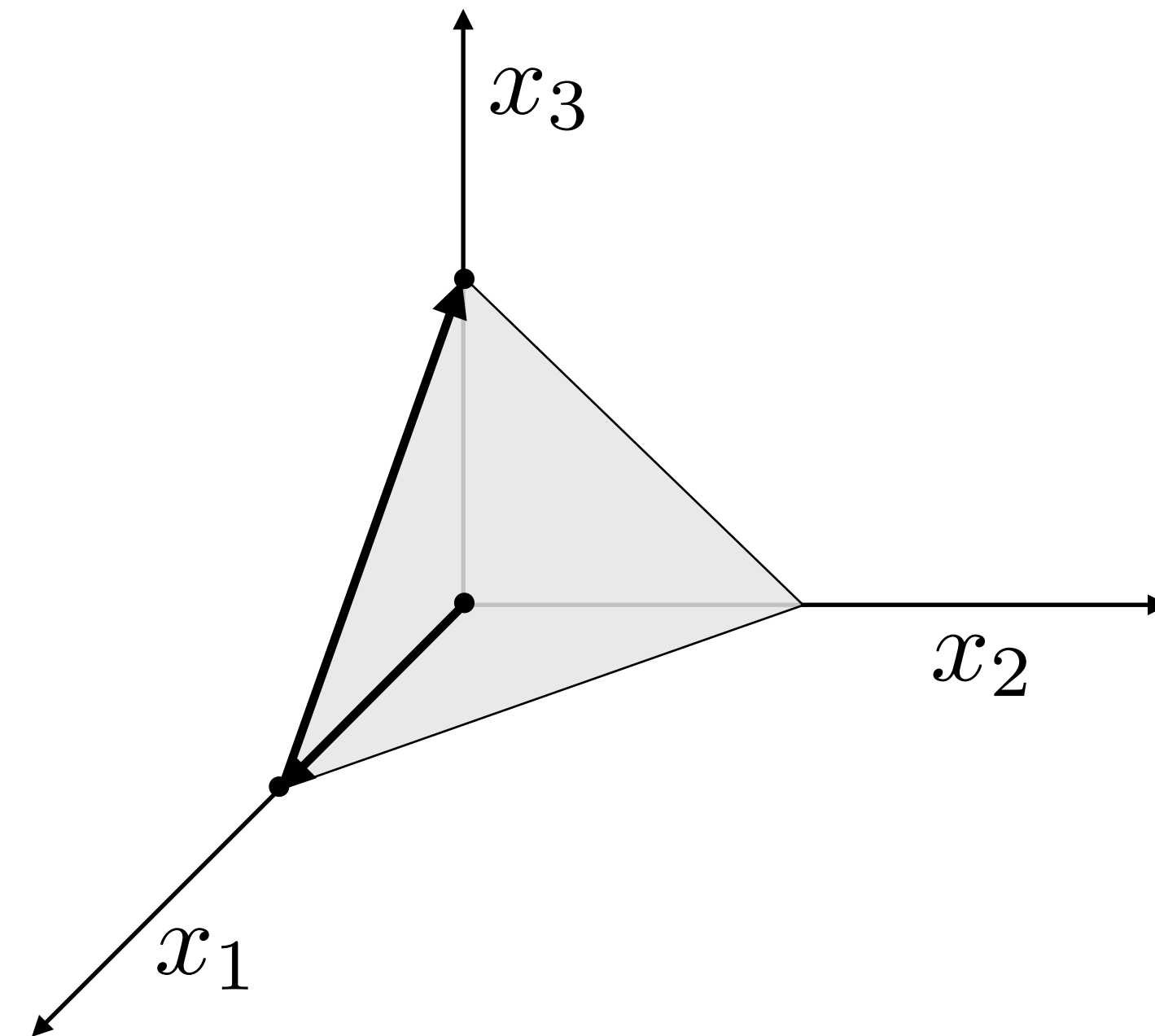
List all the basic feasible solutions, compare objective values and pick the best one.

Intractable!

If $n = 1000$ and $m = 100$, we have 10^{143} combinations!

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Geometry of linear optimization

Today, we learned to:

- **Apply geometric and algebraic properties** of polyhedra to characterize the “corners” of the feasible region.
- **Construct basic feasible solutions** by solving a linear system.
- **Recognize existence and optimality** of extreme points.

Next lecture

The simplex method

- Iterations
- Convergence
- Complexity