ORF522 – Linear and Nonlinear Optimization

2. Linear optimization

Bartolomeo Stellato — Fall 2021

Today's agenda

Readings: [Chapter 1, Bertsimas, Tsitsiklis]

- Linear optimization in inner-product and matrix notation
- Optimization terminology
- Standard form
- Piecewise-linear minimization
- Examples

Where does linear optimization appear?

Supply chain management

Assignment problems

Scheduling and routing problems

Finance

Optimal control problems

Network design and network operations

Many other domains…

Vector notations

By default, all vectors are column vectors and denoted by

$$x = (x_1, \ldots, x_n)$$

The transpose of a vector is x^T

$a^T x$ is the inner product between a and x

$$a^T x = a_1 x_1 + \cdots + a_n x_n = \sum_{i=1}^{n} a_i x_i$$

Linear optimization

Linear Programming (LP)

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{n} c_i x_i \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} d_{ij} x_j = f_i, \quad i = 1, \ldots, p
\end{array}
$$

Objective function and constraints are **linear in the decision variables**

Belongs to **continuous optimization**

Linear optimization

Inner product notation

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n c_i x_i \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ & \sum_{j=1}^n d_{ij} x_j = f_i, \quad i = 1, \ldots, p \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\ & d_i^T x = f_i, \quad i = 1, \ldots, p \end{array}$$

$c,\ a_i,\ d_i$ are n-vectors

$$c = (c_1, \ldots, c_n)$$
$$a_i = (a_{i1}, \ldots, a_{in})$$
$$d_i = (d_{i1}, \ldots, d_{in})$$

Linear optimization

Matrix notation

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{n} c_i x_i \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} d_{ij} x_j = f_i, \quad i = 1, \ldots, p
\end{array}
\longrightarrow
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b \\
& Dx = f
\end{array}
$$

A is $m \times n$-matrix with elements a_{ij} and rows a_i^T

D is $p \times n$-matrix with elements d_{ij} and rows d_i^T

All (in)equalities are elementwise

Optimization terminology

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \\ & Dx = f \end{array}$$

x is **feasible** if it satisfies the constraints $Ax \le b$ and $Dx = f$

The **feasible set** is the set of all feasible points

$x^\star$ is **optimal** if it is feasible and $c^T x^\star \le c^T x$ for all feasible x

The **optimal value** is $p^\star = c^T x^\star$

Unbounded problem: $c^T x$ is unbounded below on the feasible set ($p^\star = -\infty$)

Infeasible problem: feasible set is empty ($p^\star = +\infty$)

Standard form

Definition

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Minimization
- Equality constraints
- Nonnegative variables

- Matrix notation for **theory**
- Standard form for **algorithms**

Standard form

Transformation tricks

Change objective

If "maximize", use $-c$ instead of c and change to "minimize".

Eliminate inequality constraints

If $Ax \leq b$, define s and write $Ax + s = b, \ s \geq 0$.

If $Ax \geq b$, define s and write $Ax - s = b, \ s \geq 0$.

s are the **slack variables**

Change variable signs

If $x_i \leq 0$, define $y_i = -x_i$.

Eliminate "free" variables

If x_i unconstrained, define $x_i = x_i^+ - x_i^-$, with $x_i^+ \geq 0$ and $x_i^- \geq 0$.

Standard form

Transformation example

$$\begin{array}{lrcrl} \text{minimize} & 2x_1 & + & 4x_2 & \\ \text{subject to} & x_1 & + & x_2 & \geq 3 \\ & 3x_1 & + & 2x_2 & = 14 \\ & x_1 & & & \geq 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{lrcrcrcrl} \text{minimize} & 2x_1 & + & 4x_2^+ & - & 4x_2^- & & & \\ \text{subject to} & x_1 & + & x_2^+ & - & x_2^- & - & x_3 & = 3 \\ & 3x_1 & + & 2x_2^+ & - & 2x_2^- & & & = 14 \\ & x_1, & & x_2^+, & & x_2^-, & & x_3 & \geq 0. \end{array}$$

Linear, affine and convex functions

Linear function: $f(x) = a^T x$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \quad \forall x, y \in \mathbf{R}^n,\ \alpha, \beta \in \mathbf{R}$$

Affine function: $f(x) = a^T x + b$

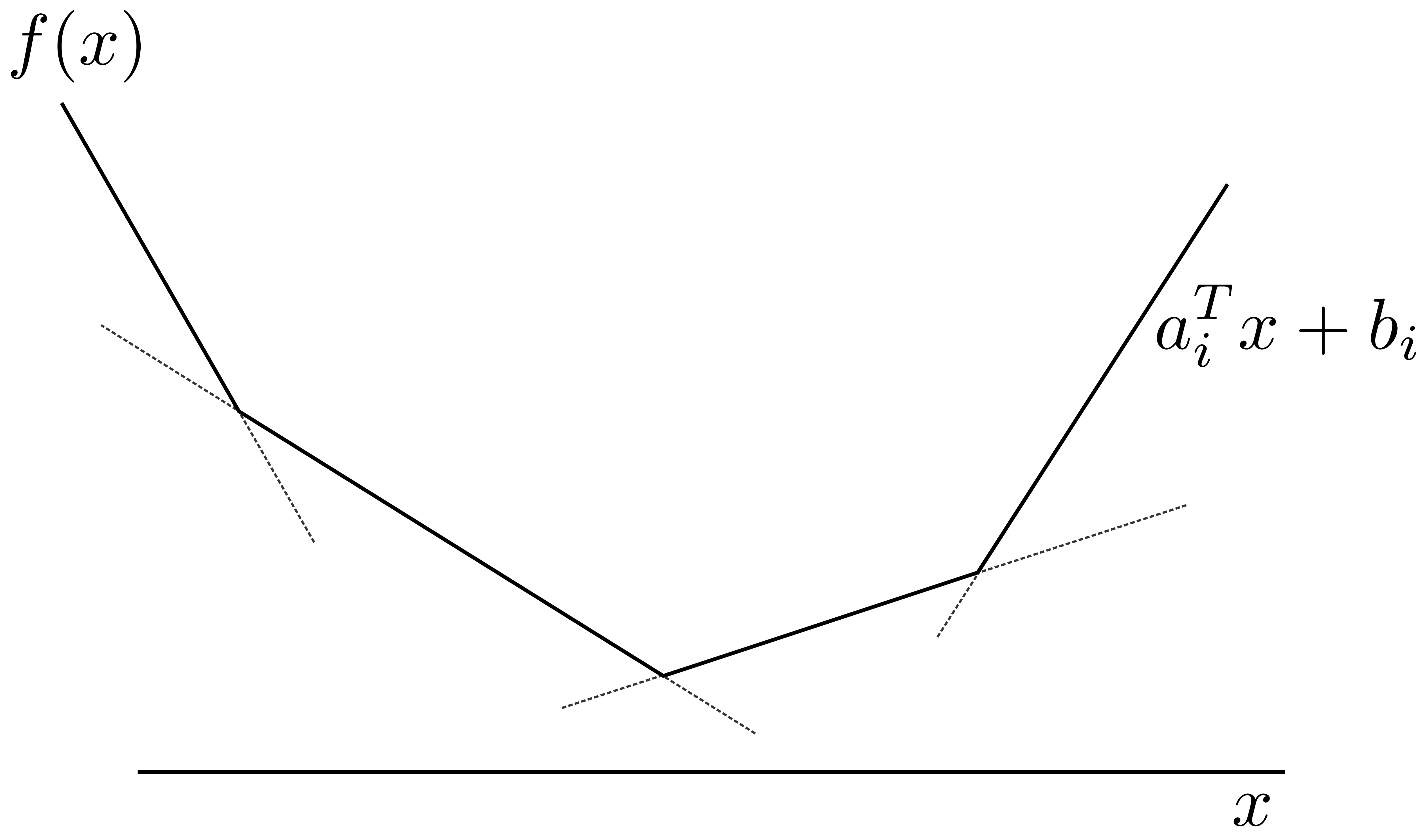
$$f(\alpha x + (1-\alpha) y) = \alpha f(x) + (1-\alpha) f(y), \quad \forall x, y \in \mathbf{R}^n,\ \alpha \in \mathbf{R}$$

Convex function:

$$f(\alpha x + (1-\alpha) y) \le \alpha f(x) + (1-\alpha) f(y), \quad \forall x, y \in \mathbf{R}^n,\ \alpha \in [0, 1]$$

Convex piecewise-linear functions

$$f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$



Convex piecewise-linear minimization

$$\text{minimize} \quad f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$

Equivalent linear optimization

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

Matrix notation

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \leq \tilde{b} \end{array}$$

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} a_1^T & -1 \\ \vdots & \vdots \\ a_m^T & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix}$$

Sum of piecewise-linear functions

minimize $\quad f(x) + g(x) = \max_{i=1,\dots,m} (a_i^T x + b_i) + \max_{i=1,\dots,p} (c_i^T x + d_i)$

Cost function is piecewise-linear

$$f(x) + g(x) = \max_{\substack{i=1,\dots,m \\ j=1,\dots,p}} ((a_i + c_j)^T x + (b_i + d_j))$$

Equivalent linear optimization

$$\begin{array}{ll} \text{minimize} & t_1 + t_2 \\ \text{subject to} & a_i^T x + b_i \leq t_1, \quad i = 1, \dots, m \\ & c_i^T x + d_i \leq t_2, \quad i = 1, \dots, p \end{array}$$

Matrix

notation?

Examples

Cheapest cat food problem

- Choose quantities $x_1, \ldots, x_n$ of n ingredients each with unit cost c_j.
- Each ingredient j has nutritional content a_{ij} for nutrient i.
- Require a minimum level b_i for each nutrient i.

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i = 1 \ldots m \\ & x_j \geq 0, \qquad j = 1 \ldots n \end{aligned}$$

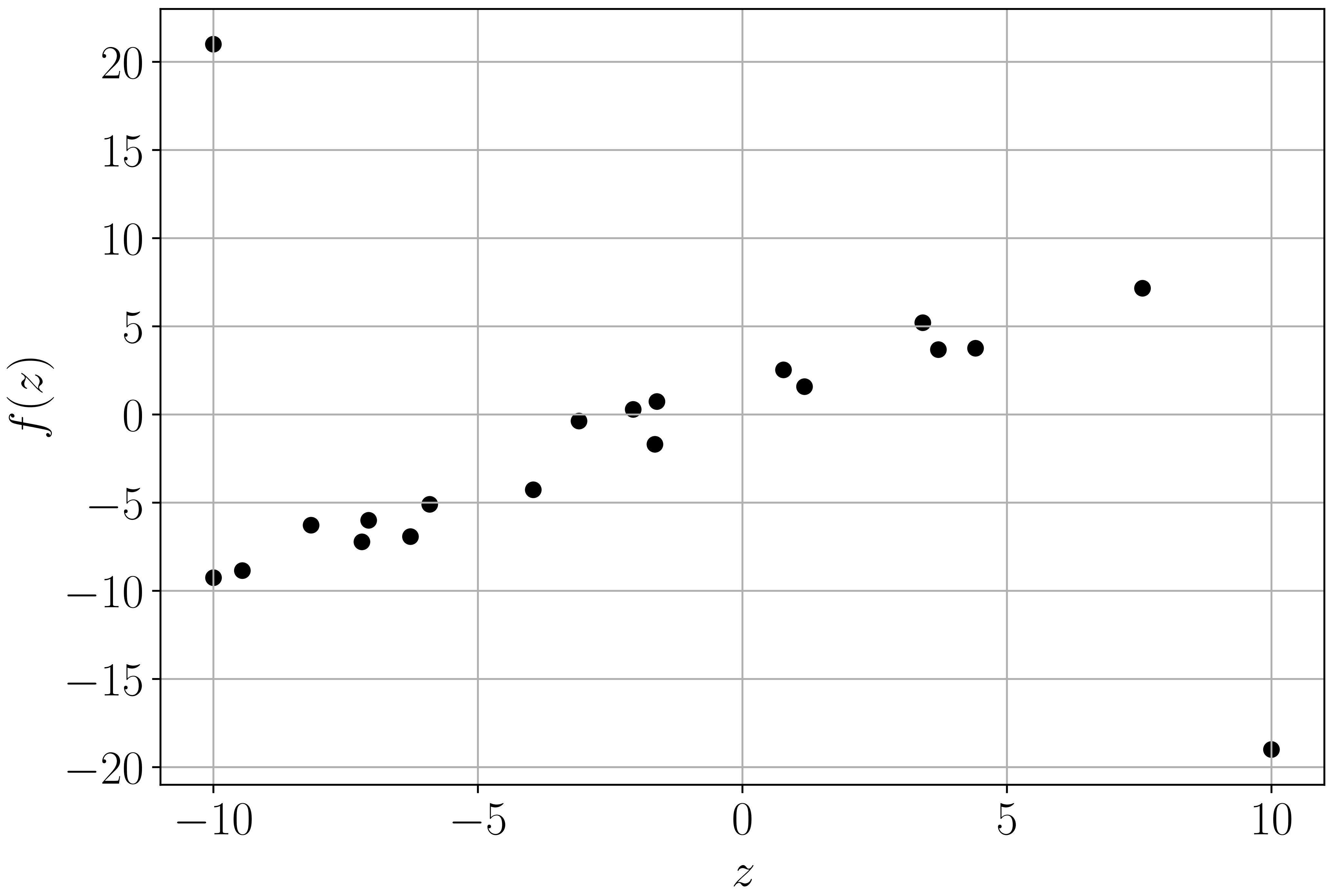
[Photo Phoebe, my cat]

Would you give her the optimal food ?

Data-fitting example

Fit a linear function $f(z) = a + bz$ to m data points (z_i, f_i):

Approximation problem $Ax \approx b$ where $\underbrace{\begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{b}$



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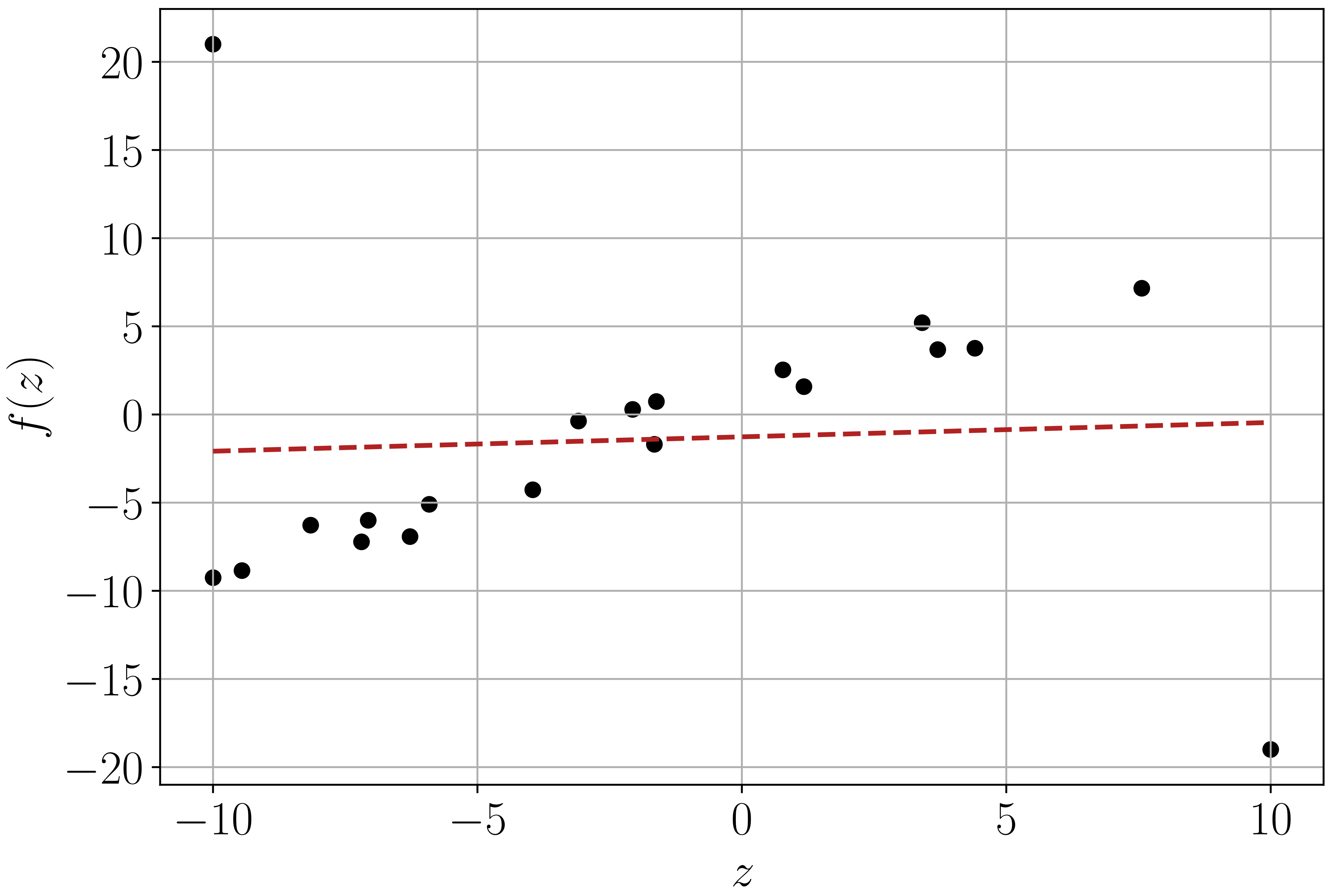


Least squares way:

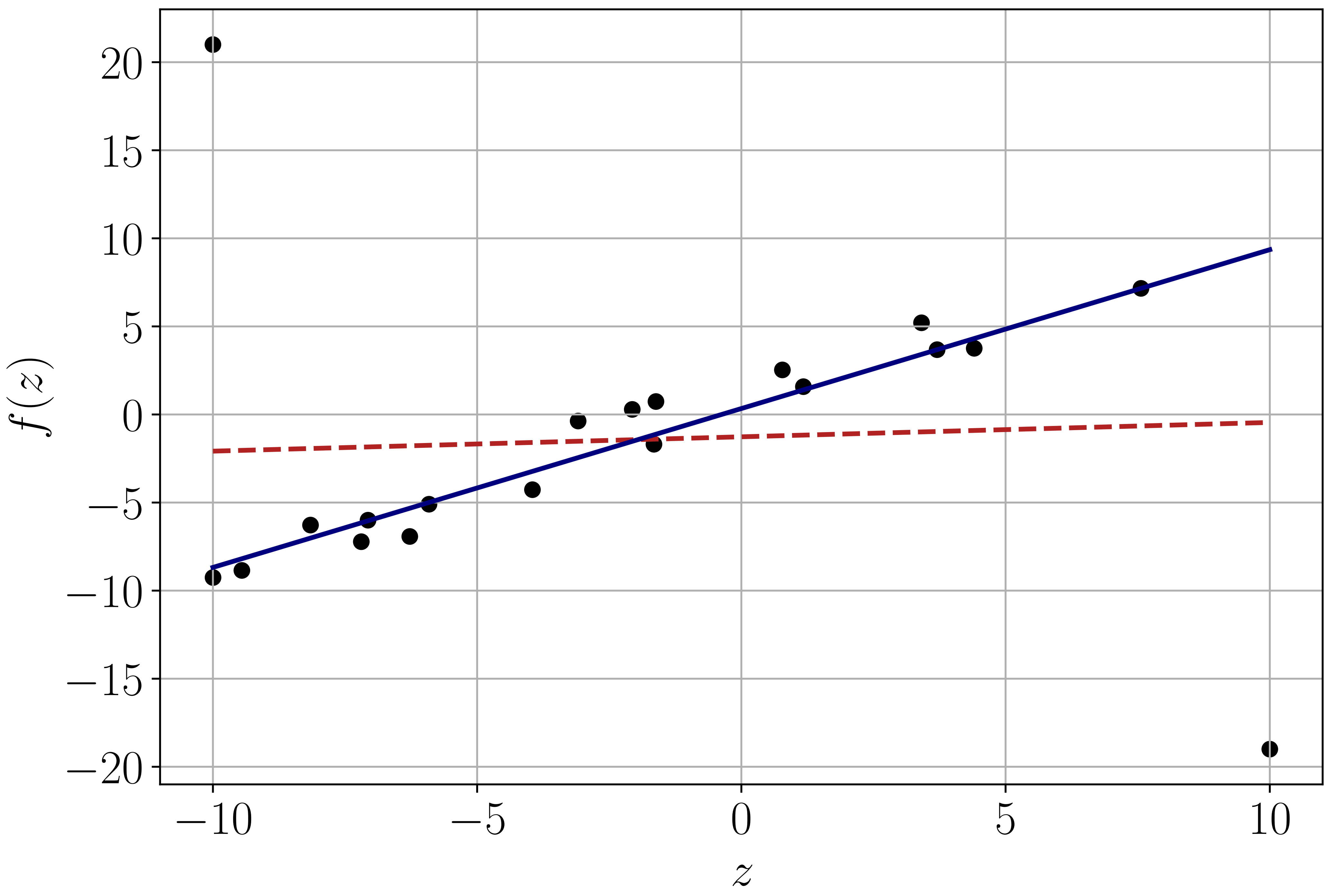
minimize $\displaystyle\sum_{i=1}^{m} (Ax - b)_i^2 = \|Ax - b\|_2^2$

Good news: solution is in closed form $x^\star = (A^T A)^{-1} A^T b$
Bad news: solution is very sensitive to outliers!

Data-fitting example

Fit a linear function $f(z) = a + bz$ to m data points (z_i, f_i):

Approximation problem $Ax \approx b$ where $\underbrace{\begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{b}$



A different way:

minimize $\displaystyle\sum_{i=1}^{m} |Ax - b|_i = \|Ax - b\|_1$

Good news: solution is much more robust to outliers.
Bad news: there is no closed form solution.

1-norm approximation

$$\text{minimize} \quad \|Ax - b\|_1$$

The 1-**norm** of m-vector y is

$$\|y\|_1 = \sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} u_i \\ \text{subject to} & -u \leq Ax - b \leq u \end{array}$$

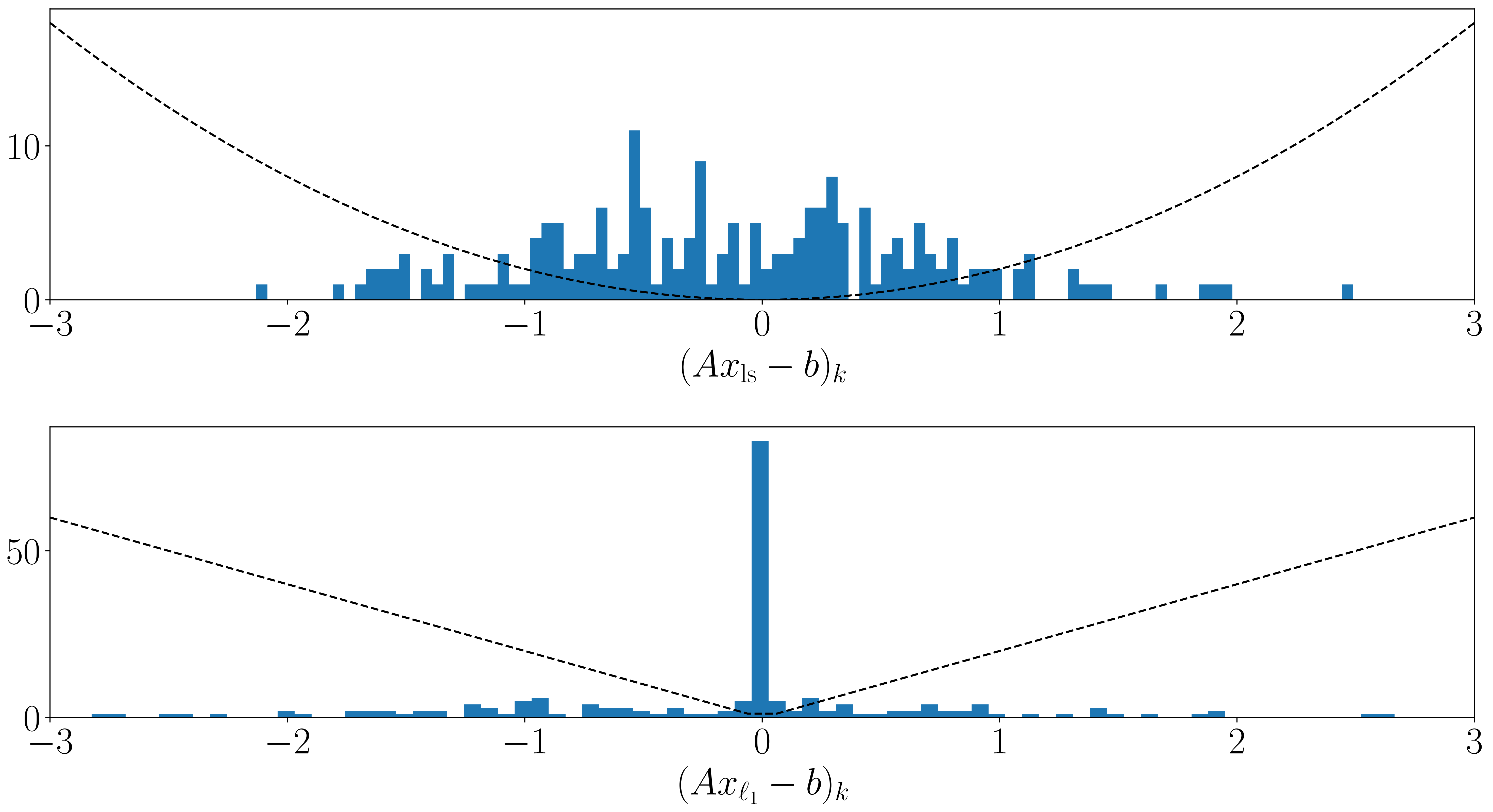
Matrix notation

$$\begin{array}{ll} \text{minimize} & \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}^T \begin{bmatrix} x \\ u \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

Comparison with least-squares

Histogram of residuals $Ax - b$ with randomly generated $A \in \mathbf{R}^{200 \times 80}$

$$x_{\text{ls}} = \text{argmin } \|Ax - b\|, \qquad x_{\ell_1} = \text{argmin } \|Ax - b\|_1$$



ℓ_1-norm distribution is **wider** with a **high peak at zero**

ℓ_∞-norm (Chebyshev) approximation

$$\text{minimize} \quad \|Ax - b\|_\infty$$

The ∞**-norm** of m-vector y is

$$\|y\|_\infty = \max_{i=1,\dots,m} |y_i| = \max_{i=1,\dots,m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -t\mathbf{1} \le Ax - b \le t\mathbf{1} \end{array}$$

Matrix notation

$$\begin{array}{ll} \text{minimize} & \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

Sparse signal recovery via ℓ_1-norm minimization

$\hat{x} \in \mathbf{R}^n$ is unknown signal, known to be sparse

We make linear measurements $y = A\hat{x}$ with $A \in \mathbf{R}^{m \times n}, m < n$

Estimate signal with smallest ℓ_1-norm, consistent with measurements

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & Ax = y \end{array}$$

Equivalent linear optimization

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & -u \leq x \leq u \\ & Ax = y \end{array}$$

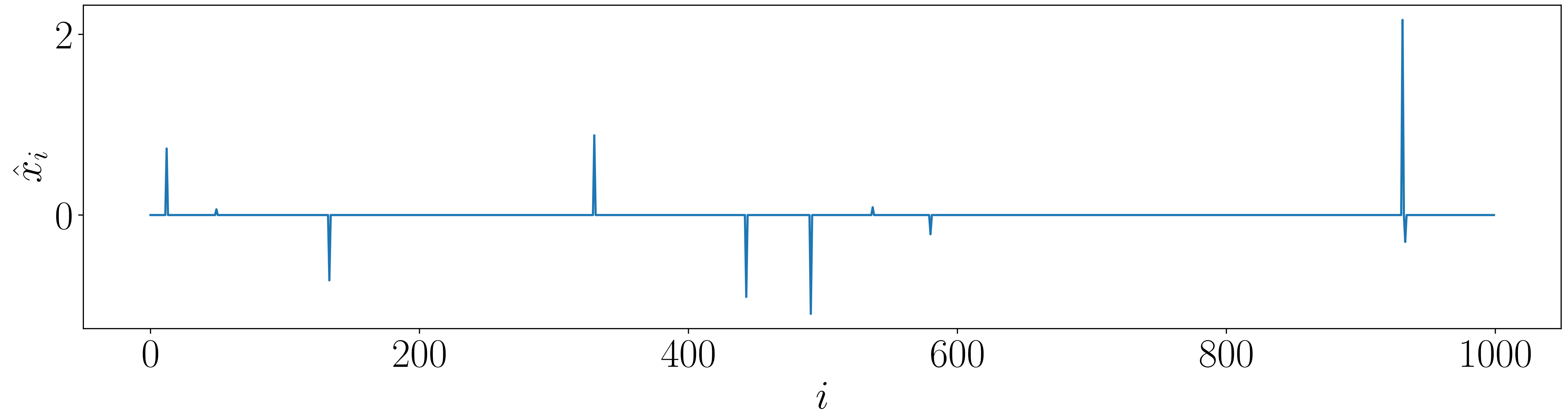
Sparse signal recovery via ℓ_1-norm minimization

Example

Exact signal $\hat{x} \in \mathbf{R}^{1000}$

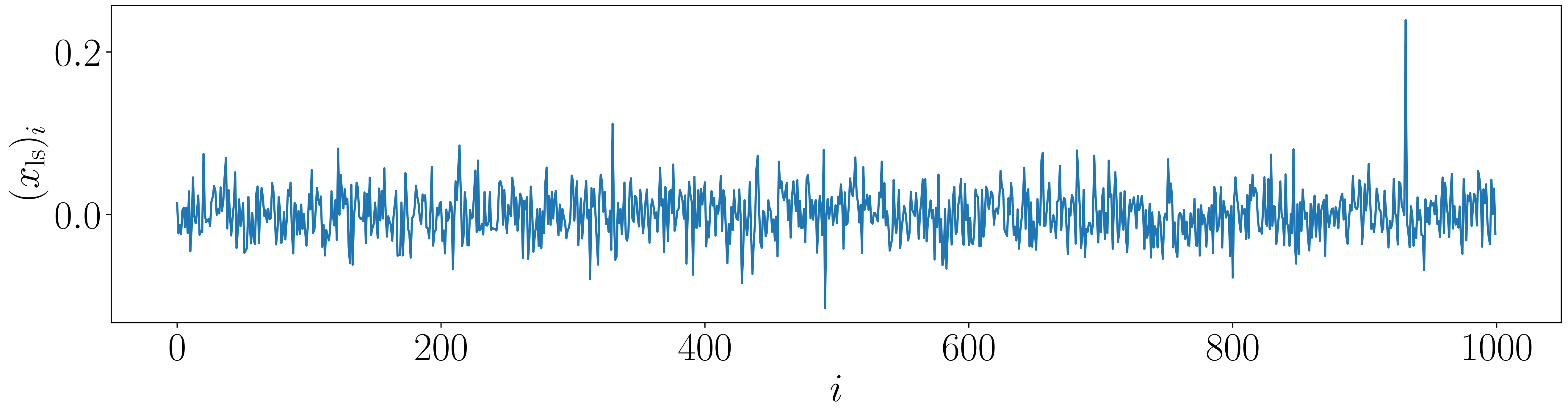
10 nonzero components

Random $A \in \mathbf{R}^{100 \times 1000}$



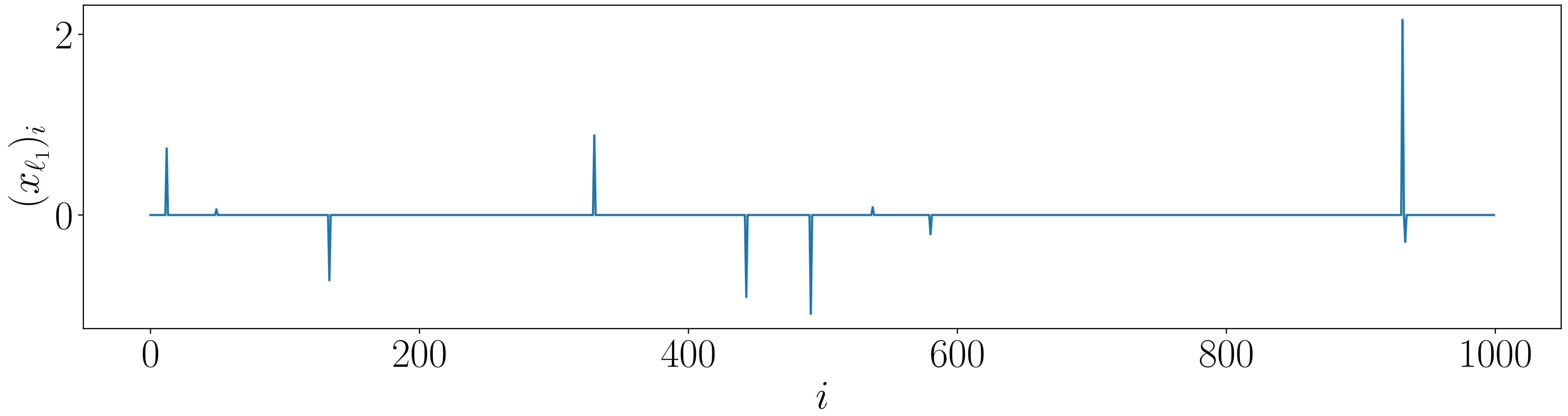
The least squares estimate cannot recover the sparse signal

$$\begin{array}{ll} \text{minimize} & \|x\|_2^2 \\ \text{subject to} & Ax = y \end{array}$$



The ℓ_1-norm estimate is **exact**

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & Ax = y \end{array}$$



Sparse signal recovery via ℓ_1-norm minimization

Exact recovery

When are these two problems equivalent?

minimize	$\mathrm{card}(x)$	minimize	$\|x\|_1$
subject to	$Ax = y$	subject to	$Ax = y$

$\mathrm{card}(x)$ is cardinality (number of nonzero components) of x

We say A allows **exact recovery** of k-sparse vectors if

$$\hat{x} = \underset{Ax=y}{\mathrm{argmin}} \|x\|_1 \quad \text{when } y = A\hat{x} \text{ and } \mathrm{card}(\hat{x}) \leq k$$

It depends on the nullspace[1] of the "measurement matrix" A

1. Feuer & Nemirovski (IEEE Trans. On Information Technology, 2003) and several other papers on compressed sensing.

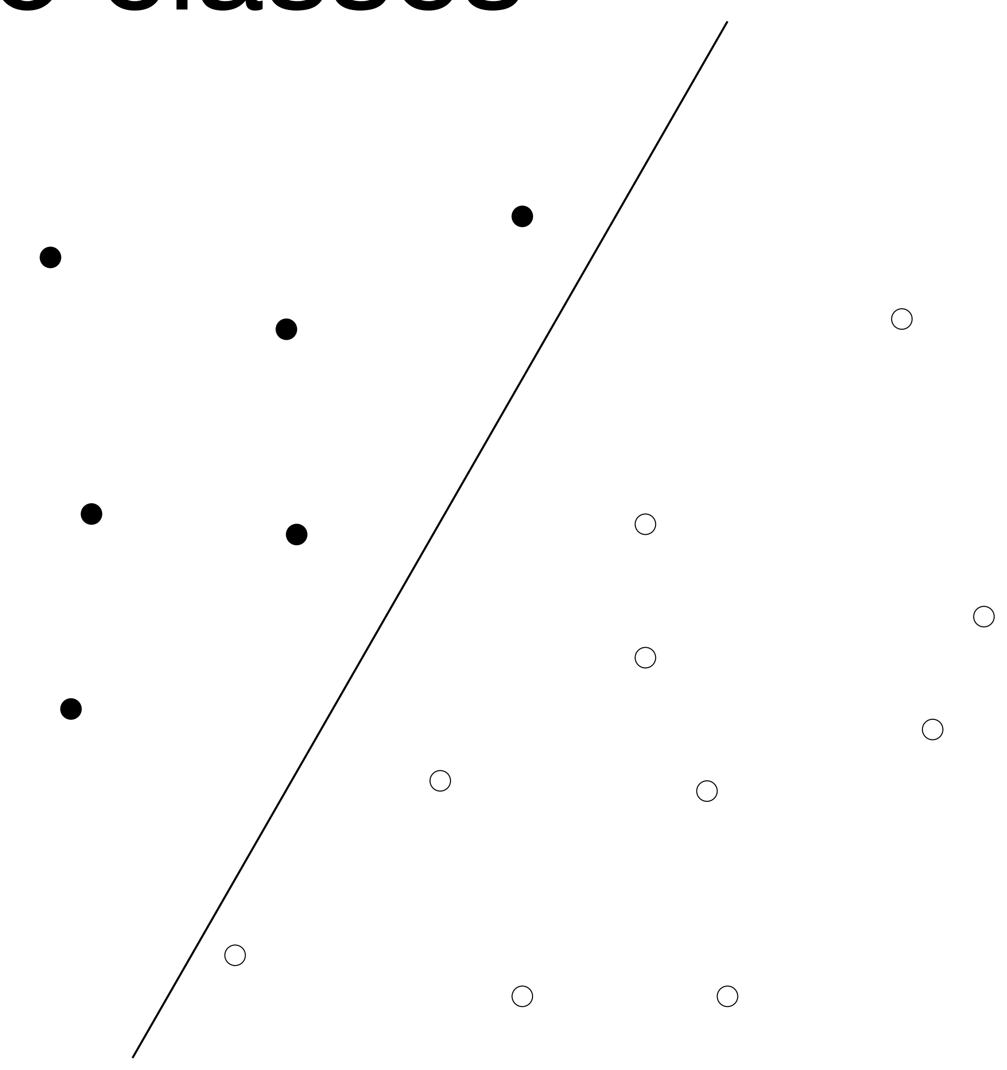
Linear classification

Support vector machine (linear separation)

Given a set of points $\{v_1, \ldots, v_N\}$ with binary labels $s_i \in \{-1, 1\}$

Find hyperplane that strictly separates the tho classes

$$a^T v_i + b > 0 \quad \text{if} \quad s_i = 1$$
$$a^T v_i + b < 0 \quad \text{if} \quad s_i = -1$$



Homogeneous in (a, b), hence equivalent to the linear inequalities (in a, b)

$$s_i(a^T v_i + b) \geq 1$$

Linear classification

Separable case

Feasibility problem

$$\begin{array}{ll} \text{find} & a, b \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

Which can be seen as a special case of LP with

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

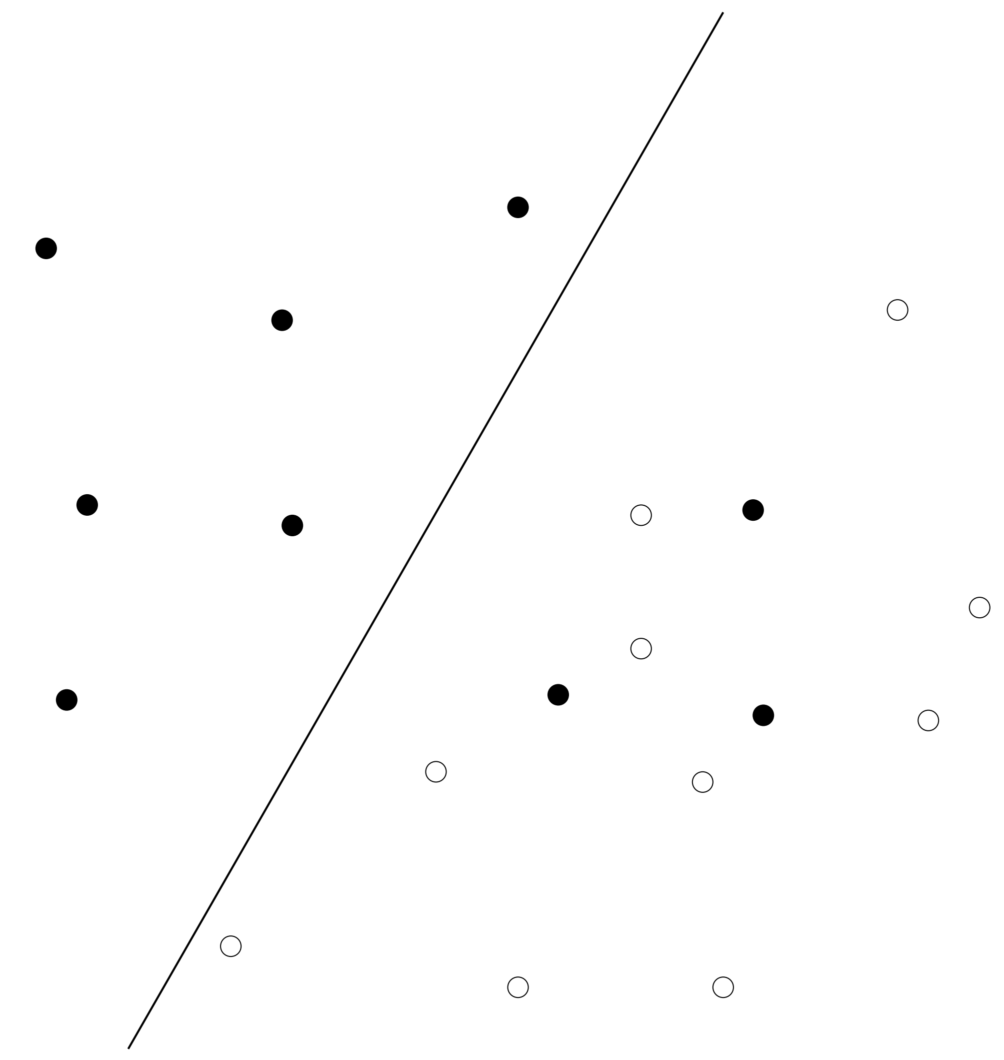
$p^\star = 0$ if problem feasible (points separable)

$p^\star = \infty$ if problem infeasible (points not separable) ⟶ **What then?**

Linear classification

Approximate linear separation of non-separable points

minimize $\quad \sum_{i=1}^{N}(1 - s_i(a^T v_i + b))_+ = \sum_{i=1}^{N} \max\{0, 1 - s_i(a^T v_i + b)\}$



If v_i misclassified, $1 - s_i(a_i^T v_i + b)$ is the penalty

Piecewise-linear minimization problem with variables a, b

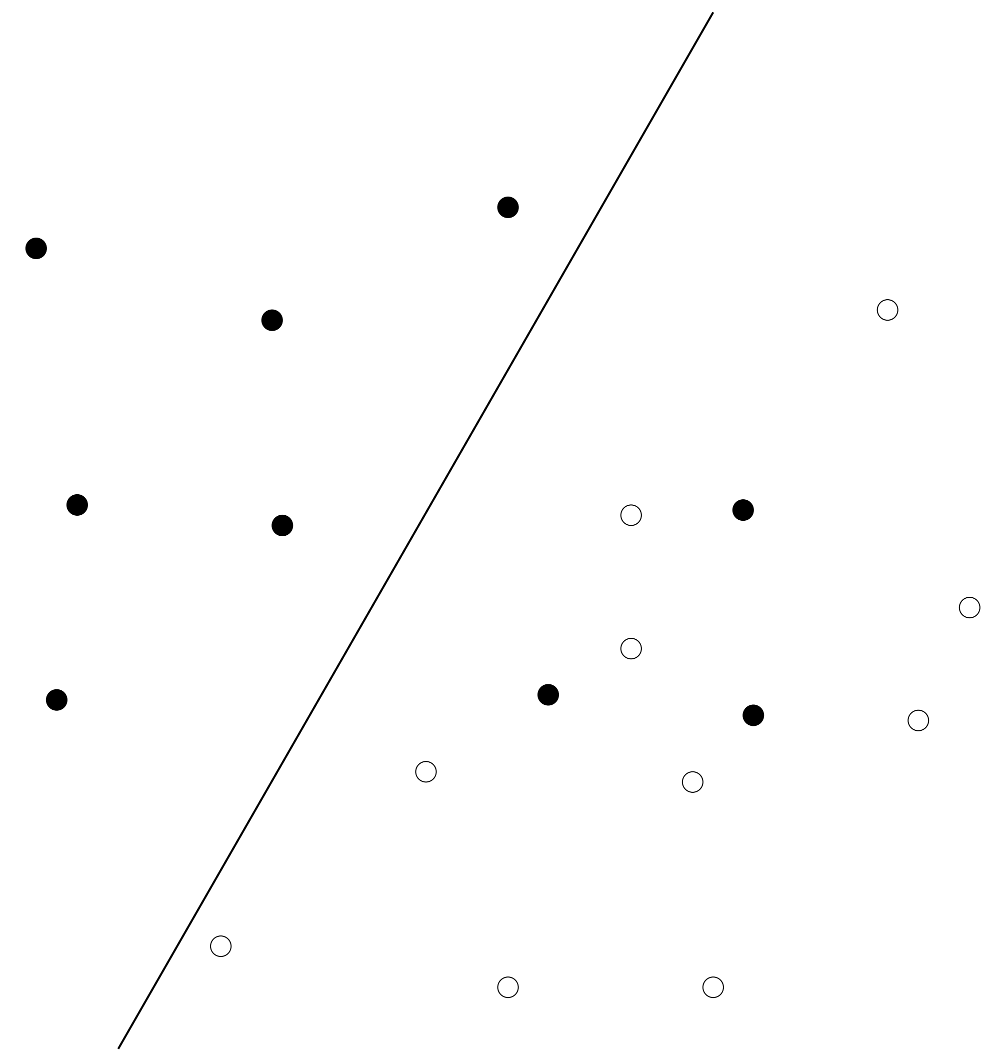
Linear classification

Approximate linear separation of non-separable points

minimize $\sum_{i=1}^N \max\{0, 1 - s_i(a^T v_i + b)\}$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^N u_i \\ \text{subject to} & 1 - s_i(v_i^T a + b) \leq u_i, \quad i = 1, \ldots, N \\ & 0 \leq u_i, \quad i = 1, \ldots, N \end{array}$$



Matrix notation?

Modelling software for linear programs

Modelling tools simplify the formulation of LPs (and other problems)

- Accept optimization problem in common notation ($\max, \|\cdot\|_1, \ldots$)
- Recognize problems that can be converted to LPs
- Express the problem in input format required by a specific LP solver

Examples

- AMPL, GAMS
- CVX, YALMIP (Matlab)
- CVXPY, Pyomo (Python)
- JuMP.jl, Convex.jl (Julia)

CVXPY example

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_1 \\ \text{subject to} & 0 \leq x \leq 1 \end{array}$$

```
x = cp.Variable(n)
objective = cp.Minimize(cp.norm(A*x - b, 1))
constraints = [0 <= x, x <= 1]
problem = cp.Problem(objective, constraints)

# The optimal objective value is returned by `problem.solve()`.
result = problem.solve()

# The optimal value for x is stored in `x.value`.
print(x.value)
```

Why linear optimization?

“Easy” to solve

- It is solvable in polynomial time, and it is tractable in practice
- State-of-the-art software can solve LPs with tens of thousands of variables. We can solve LPs with millions of variables with specific structure.

Extremely versatile

It can model many real-world problems, either exactly or approximately.

Fundamental

The theory of linear optimization lays the foundation for most optimization theories

Next lecture

Geometry of linear optimization

- Polyhedra
- Extreme points
- Basic feasible solutions