ORF522 – Linear and Nonlinear Optimization

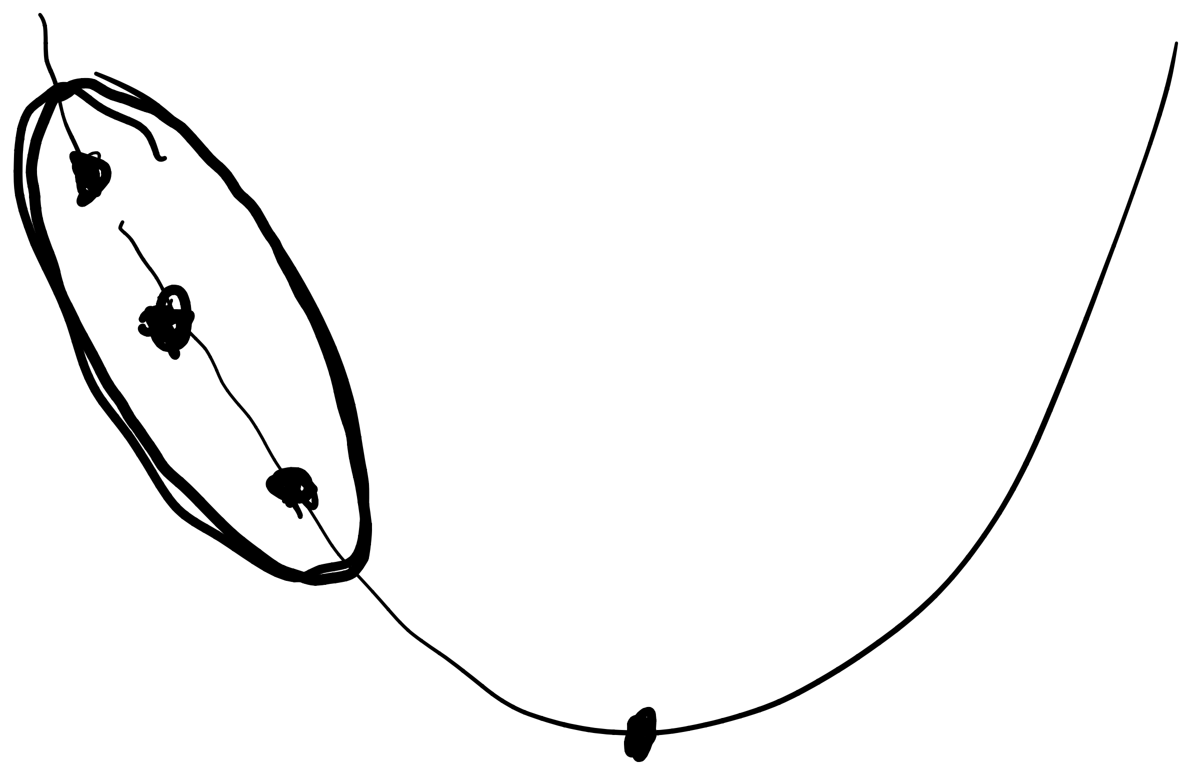
20. Sequential Convex Programming

Bartolomeo Stellato — Fall 2020

Ed forum

- Theorem of lower bounds (Nesterov'83). The theorem declares the existence of a fuction f, and gives its lower bound for first order methods; but how does it give lower bounds for all convex L-smooth functions?

- Can we include more previous directions instead of just rewinding 1 step in the momentum acceleration scheme? Yes! Anderson Acceleration



Today's lecture

[Chapter 4 and 17, Numerical Optimization, Nocedal and Wright]
[Stanford ee364b Lecture Notes, Boyd]

Convex algorithms to solve nonconvex optimization problems

- Sequential convex programming
- Trust region methods
- Building convex approximations
- Difference of convex programming

Methods for nonconvex optimization

Convex optimization algorithms: global and typically **fast**

Nonconvex optimization algorithms: must give up one, global or fast

Methods for nonconvex optimization

Convex optimization algorithms: global and typically **fast**

Nonconvex optimization algorithms: must give up one, global or fast

- **Local methods: fast** but **not global** → **heuristics**
  Need not find a global (or even feasible) solution.
  They cannot certify global optimality because
  KKT conditions are not sufficient.

- **Global methods: global** but often **slow**
  They find a global solution and certify it.

Sequential Convex Programming

Sequential convex programming (SCP)

Local optimization method that leverages convex optimization

Subproblems are convex → we can solve them efficiently

Sequential convex programming (SCP)

Local optimization method that leverages convex optimization

Subprogrems are convex ⟶ we can solve them efficiently

It is a **heuristic**

- It **can fail** to find an optimal (or even feasible point)
- Results **depend on the starting point.**
  We can run the algorithm from many initial points and take the best result.

It often works very well
it finds a feasible point with good objective value (often optimal!)

Gradient descent as SCP

Problem

minimize $\quad f(x)$

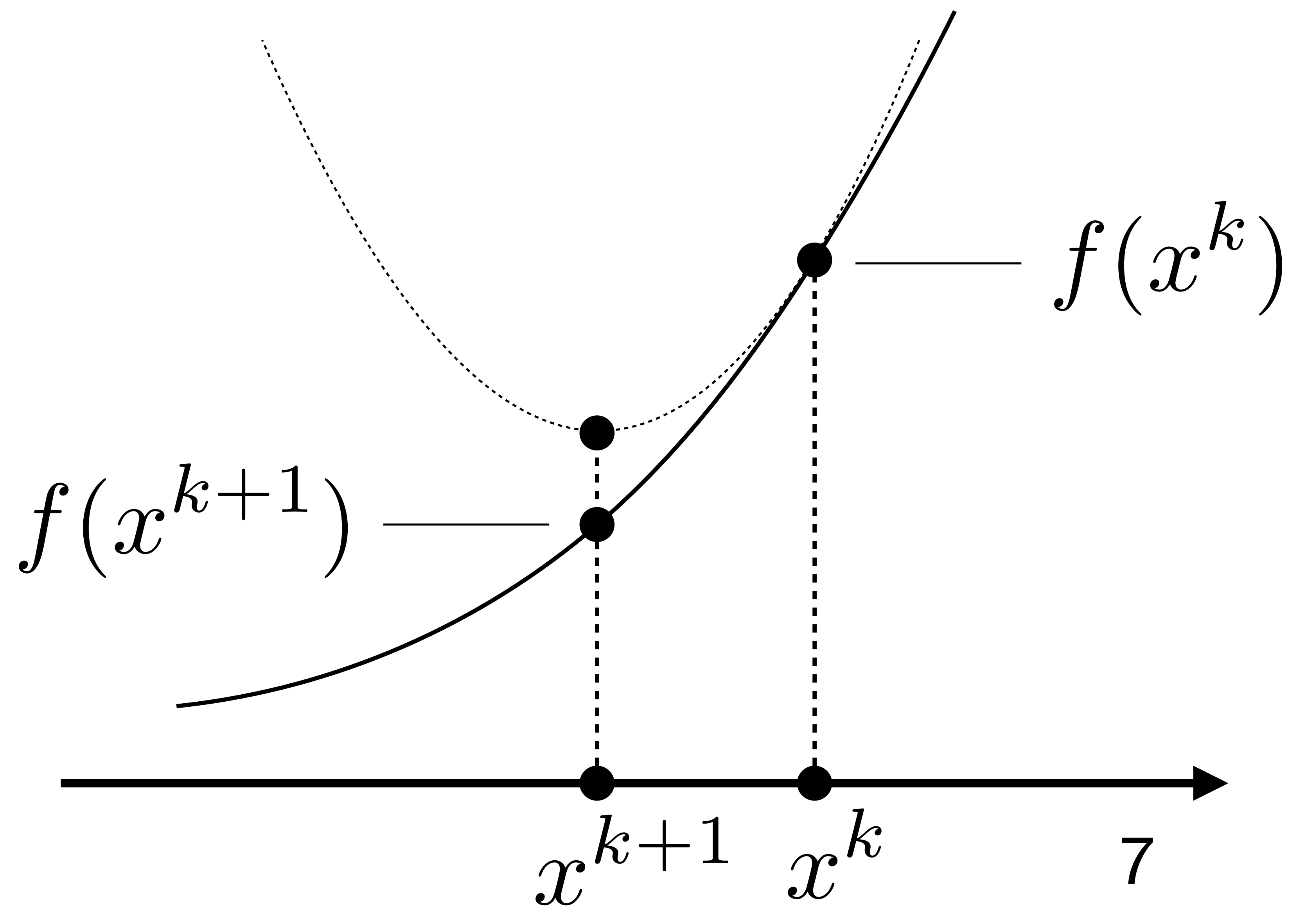
Iterates

$$x^{k+1} = x^k - t_k \nabla f(x^k)$$

Quadratic approximation, replace $\nabla^2 f(x^k)$ with $\dfrac{1}{t_k} I$

$$x^{k+1} = \underset{y}{\text{argmin}}\ f(x^k) + \nabla f(x^k)^T (y - x^k) + \frac{1}{2t_k} \|y - x^k\|_2^2$$

strongly convex problem



The problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

with $x \in \mathbf{R}^n$

- f and g_i can be nonconvex
- h_i can be nonaffine

Trust region methods

Main idea

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

Main idea

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

↓ iterate x^k
trust region $\mathcal{T}^k$

approximate convex problem

$$\begin{array}{ll} \text{minimize} & \hat{f}(x) \\ \text{subject to} & \hat{g}_i(x) \leq 0, \quad i = 1, \ldots, m \\ & \hat{h}_i(x) = 0, \quad i = 1, \ldots, p \\ & x \in \mathcal{T}^k \end{array}$$

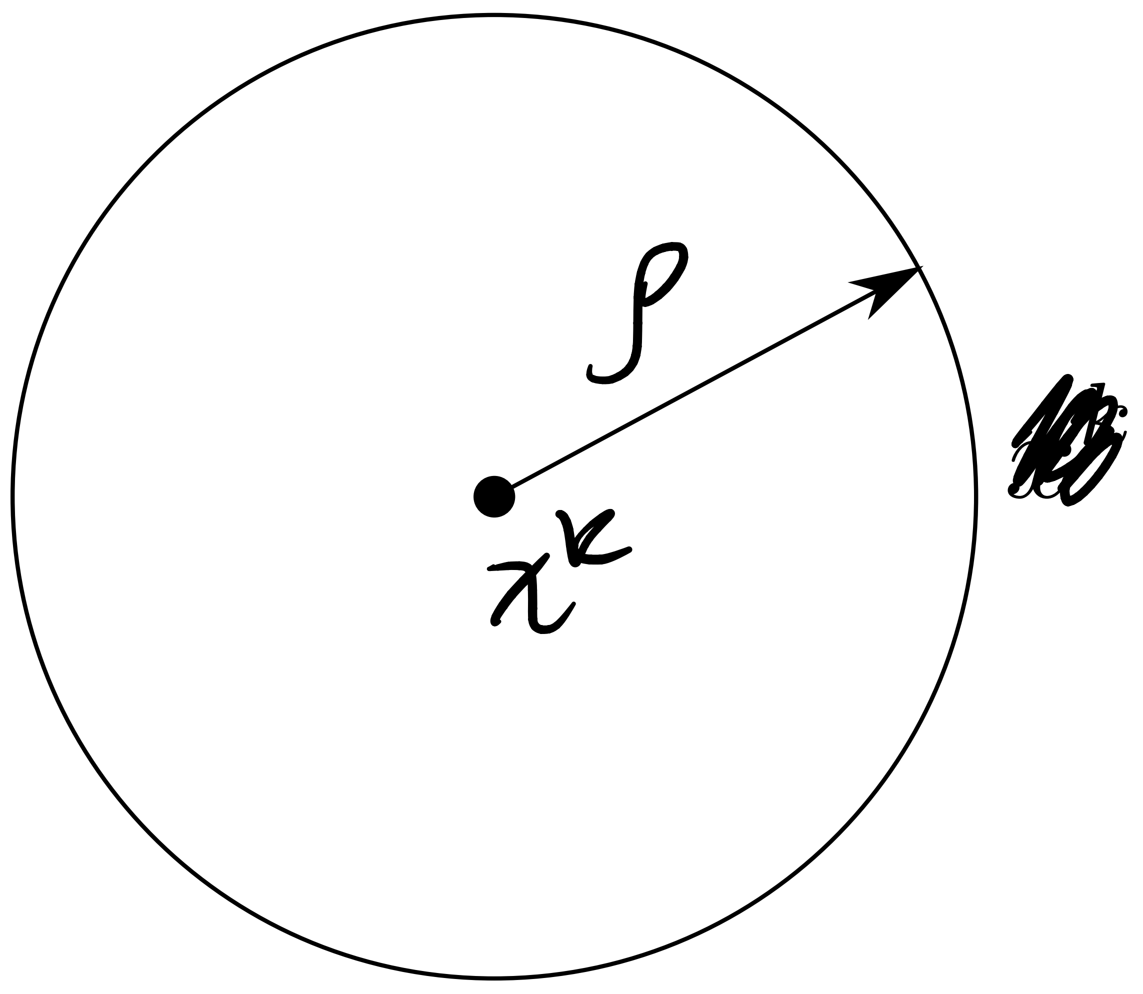
→ solve to get x^{k+1}

- $\hat{f}$ ($\hat{g}_i$) is a convex approximation of f (g_i) over $\mathcal{T}^k$
- $\hat{h}$ is an affine approximation of h over $\mathcal{T}^k$

The trust region

$$\mathcal{T}^k = \{x \mid \|x - x^k\| \leq \rho\}$$

Ball $\mathcal{T}^k = \{x \mid \|x - x^k\|_2 \leq \rho\}$



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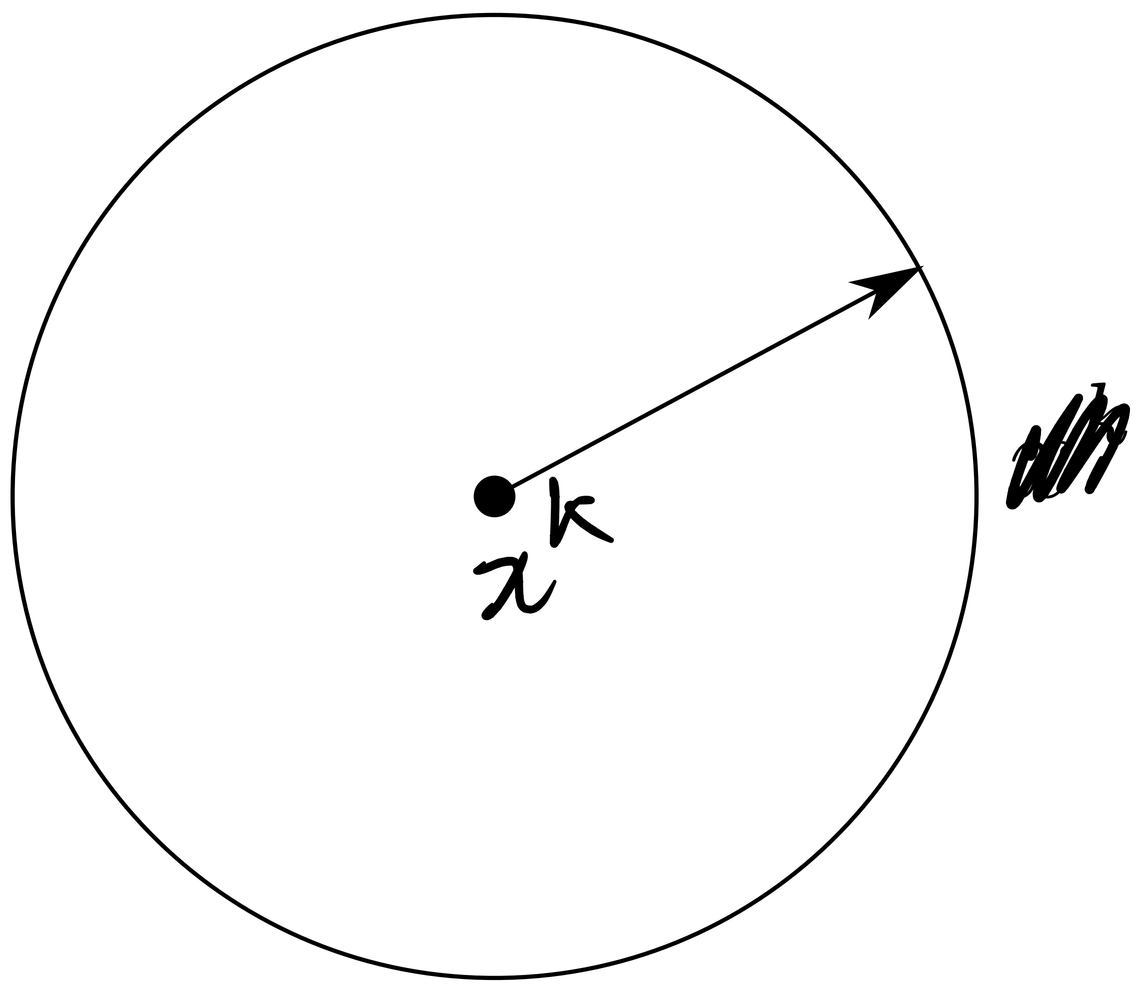
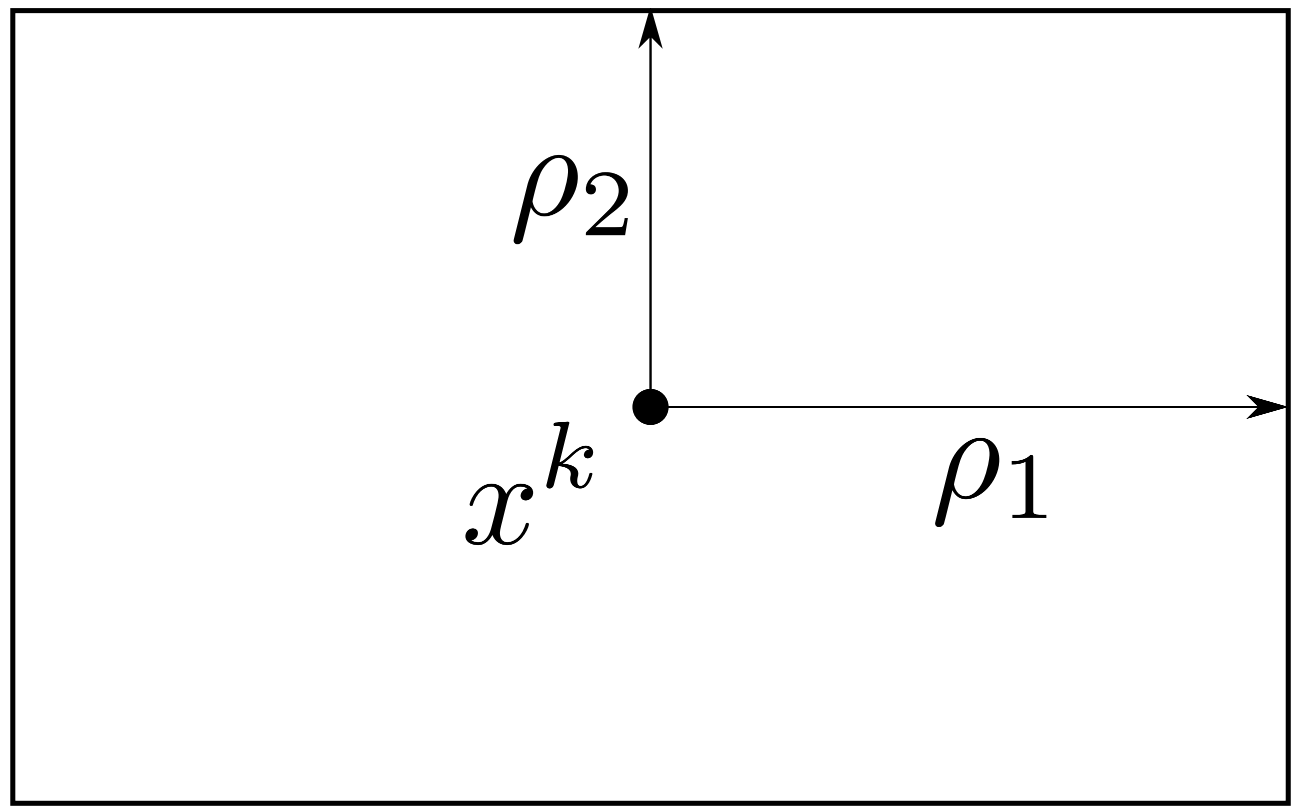


Box $\mathcal{T}^k = \{x \mid |x_i - x_i^k| \leq \rho_i\}$



Note: if f, g_i h_i are convex or affine in x_i, then we can take $\rho_i = \infty$

Proximal operator interpretation

trust region problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \|x - x^k\|_2 \leq \rho \end{array}$$

proximal problem

$$\text{minimize} \quad f(x) + \frac{1}{2\lambda}\|x - x^k\|_2^2$$

Proximal operator interpretation

trust region problem

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proximal problem

$$\text{minimize} \quad f(x) + \frac{1}{2\lambda}\|x - x^k\|_2^2$$

optimality conditions

$$0 \in \partial f(x^{\text{tr}}) + \mu \frac{x^{\text{tr}} - x^k}{\|x^{\text{tr}} - x^k\|_2},$$

$$\|x^{tr} - x^k\|_2 = \rho$$

$$\lambda = \rho/\mu$$

$$\longleftrightarrow$$

optimality conditions

$$0 \in \partial f(x^{\text{pr}}) + \frac{1}{\lambda}(x^{\text{pr}} - x^k)$$

Note: write Lagrangian and use $\partial\|x - v\|_2 = \dfrac{x - v}{\|x - v\|_2}$

Building convex approximations

Convex Taylor expansions

Given nonconvex function f

First order $\hat{f}(x) = f(x^k) + \nabla f(x^k)^T (x - x^k)$

Convex Taylor expansions

Given nonconvex function f

First order $$\hat{f}(x) = f(x^k) + \nabla f(x^k)^T (x - x^k)$$

Second order $$\hat{f}(x) = f(x^k) + \nabla f(x^k)^T (x - x^k) + (1/2)(x - x^k)^T P_+ (x - x^k)$$

where $P_+ = \Pi_{\mathbf{S}_+}(\nabla^2 f(x)) = U(\mathbf{diag}(\lambda))_+ U^T$

positive semidefinite cone projection

Convex Taylor expansions

Given nonconvex function f

First order $\hat{f}(x) = f(x^k) + \nabla f(x^k)^T(x - x^k)$

Second order $\hat{f}(x) = f(x^k) + \nabla f(x^k)^T(x - x^k) + (1/2)(x - x^k)^T P_+(x - x^k)$

where $P_+ = \Pi_{\mathbf{S}_+}(\nabla^2 f(x)) = U(\mathbf{diag}(\lambda))_+ U^T$ **positive semidefinite cone projection**

Local approximation
it does not depend on trust-region radius ρ

Quasi-linearization

Very easy and cheap method for affine approximation

write h as $h(x) = A(x)x + b(x)$

↓

use $\hat{h}(x) = A(x^k)x + b(x^k)$

Quasi-linearization

Very easy and cheap method for affine approximation

write h as $h(x) = A(x)x + b(x)$

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use $\hat{h}(x) = A(x^k)x + b(x^k)$

Example $f(x) = (1/2)x^T Px + q^T x = ((1/2)Px + q)^T x + r$

Quasi-linear: $\hat{x} = ((1/2)Px^k + q)^T x + r$

Taylor: $\hat{x} = h(x^k) + (Px^k + q)^T (x - x^k)$

Quasi-linearization

Very easy and cheap method for affine approximation

write h as $h(x) = A(x)x + b(x)$

$\downarrow$

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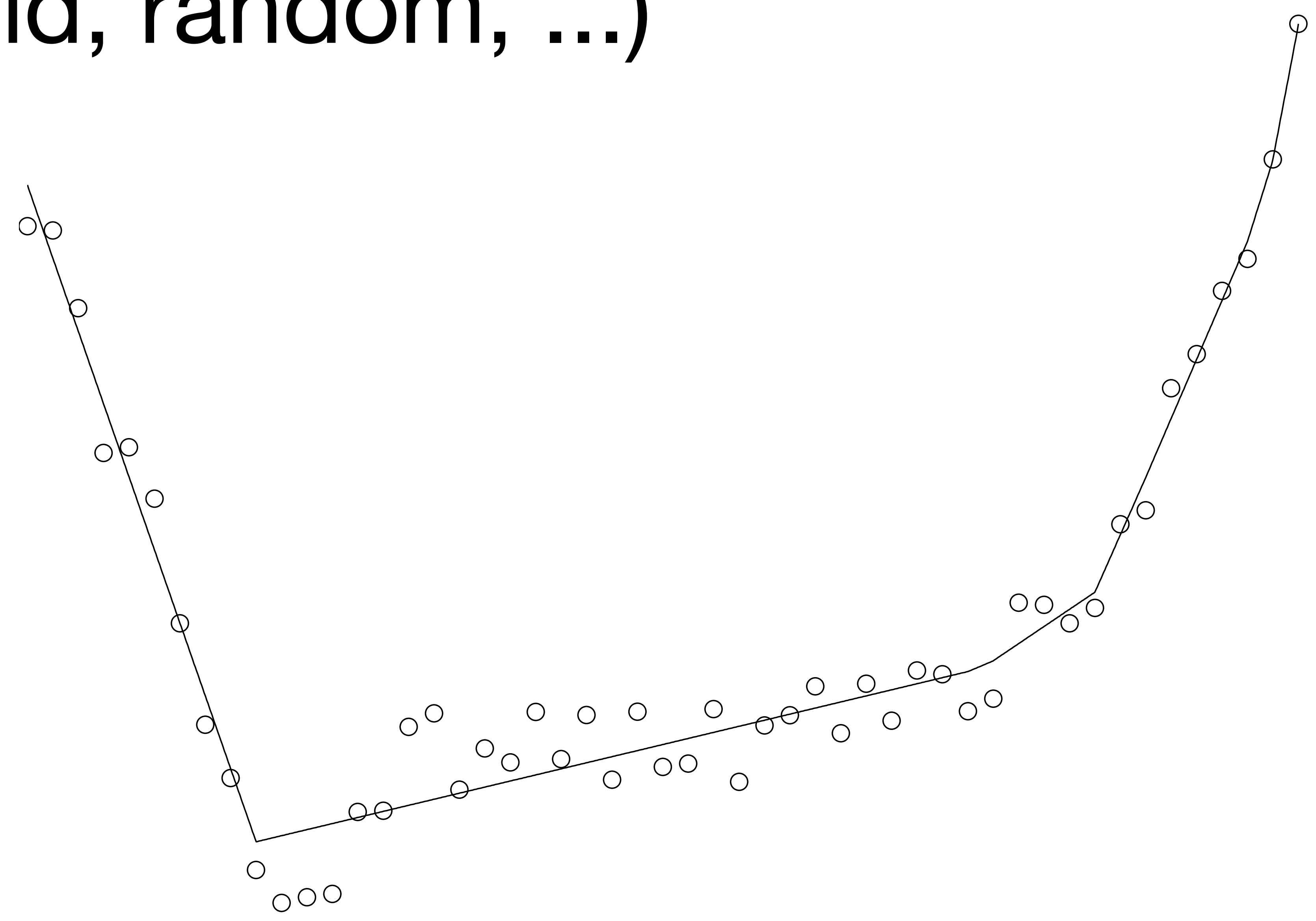
Local approximation

it does not depend on trust-region radius ρ

Particle methods

Idea

- Choose points $z_1, \ldots, z_K \in \mathcal{T}^k$ (e.g., verticles, grid, random, ...)
- Evaluate function $y_i = f(z_i)$
- Fit data (z_i, y_i) with convex functions (convex optimization)



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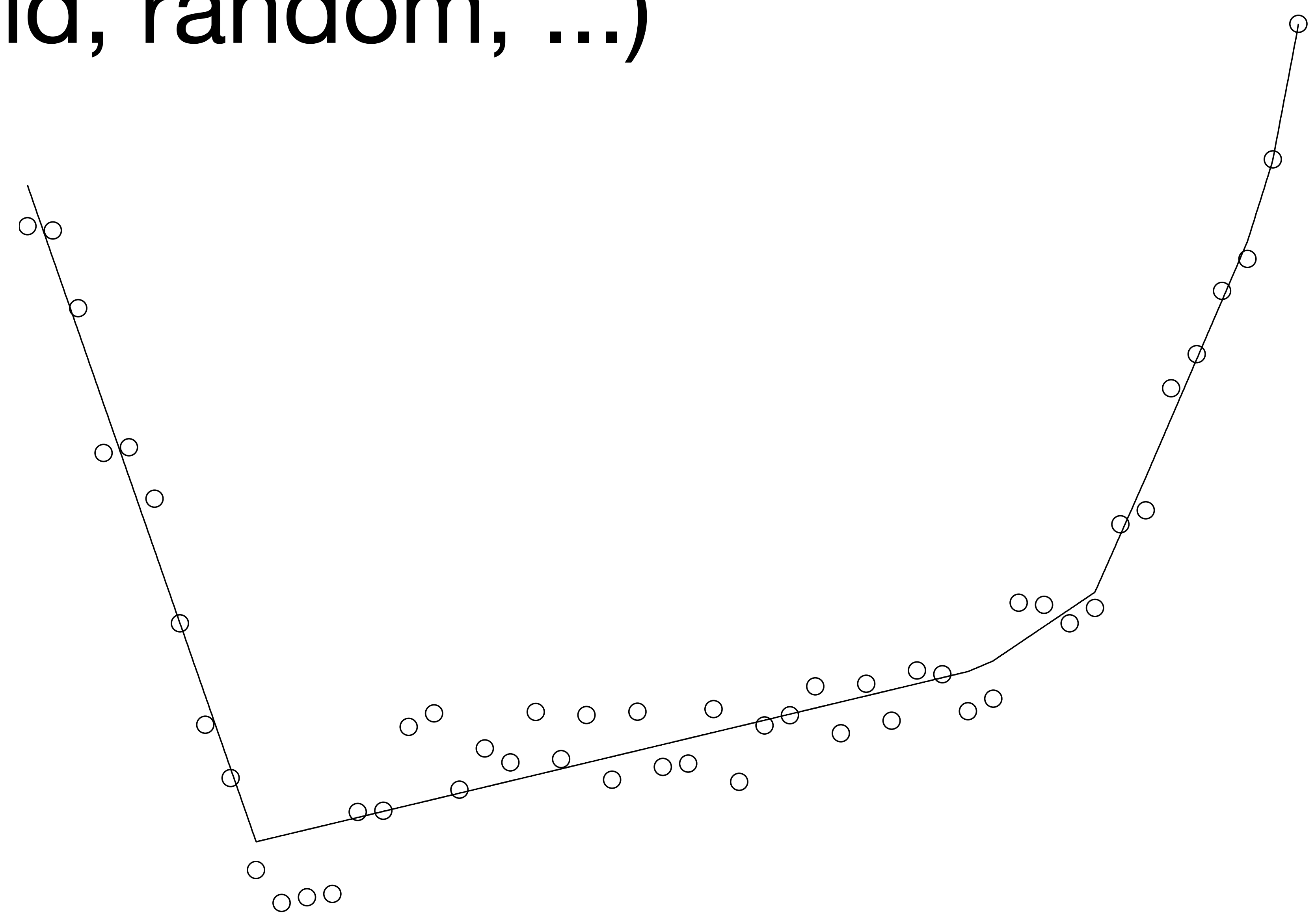

Advantages

- Nondifferentiable functions
- **regional models**: they depend on current x^k and radii ρ_i

Particle methods

Fit piecewise linear functions to data

$$\hat{f}(x) = \max_i \{\hat{y}_i + g_i^T(x - z_i)\}$$

$\hat{y}_i$ act as function values $\hat{f}(z_i)$

g_i act as subgradients $\partial \hat{f}(z_i)$

Particle methods

Fit piecewise linear functions to data

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Fitting problem

minimize	$\sum_{i=1}^K (\hat{y}_i - y_i)^2$	
subject to	$\hat{y}_j \geq \hat{y}_i + g_i^T(z_j - z_i), \quad i, j = 1, \ldots, K$	**convexity**
	$\hat{y}_i \leq y_i, \quad i = 1, \ldots, K$	**lower bound**

Particle methods

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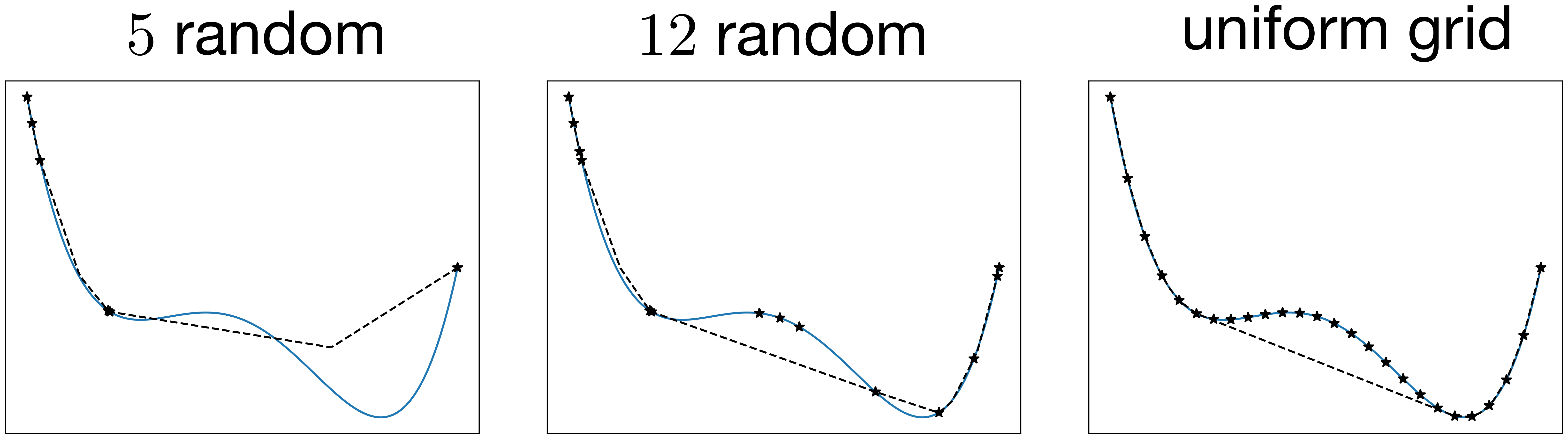
convexity

lower bound

$$f(x) = x^4 - 2x^3 + 0.3x$$



Particle methods

Fit quadratic functions to data

$$\hat{f}(x) = (1/2)(x - x^k)^T P(x - x^k) + q^T(x - x^k) + r$$

Fitting problem

$\hat{f}(z_i) = \hat{y}_i$

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{K} \left((1/2)(z_i - x^k)^T P(z_i - x^k) + q^T(z_i - z^k) + r - y_i\right)^2 \\ \text{subject to} & P \succeq 0 \end{array}$$

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Remarks

- No necessarily upper/lower bound
- We can add other objectives, convex constraints and norm penalties
- Can be more sample efficient than piecewise linear
- Need to solve a **convex problem for every function at every SCP iteration**

Trust region example

Example: nonconvex quadratic program

$$\begin{array}{ll} \text{minimize} & f(x) = (1/2)x^T P x + q^T x \\ \text{subject to} & \|x\|_\infty \leq 1 \end{array}$$

P is symmetric but not positive semidefinite

Taylor approximation

$$\hat{f}(x) = f(x^k) + (Px^k + q)^T (x - x^k) + (1/2)(x - x^k)^T P_+ (x - x^k)$$

Example: nonconvex quadratic program

Lower bound via convex duality

$$\begin{array}{ll} \text{minimize} & f(x) = (1/2)x^T P x + q^T x \\ \text{subject to} & \|x\|_\infty \leq 1 \end{array}$$

Lagrangian

$$\begin{aligned} L(x, \lambda) &= (1/2)x^T P x + q^T x + \textstyle\sum_{i=1}^n \lambda_i (x_i^2 - 1) \\ &= (1/2)x^T (P + 2\mathbf{diag}(\lambda))x + q^T x - \mathbf{1}^T \lambda \end{aligned}$$

Example: nonconvex quadratic program

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Dual problem (always convex)

$$\begin{array}{ll} \text{minimize} & -(1/2)q^T (P + 2\mathbf{diag}(\lambda))^{-1} q - \mathbf{1}^T \lambda \\ \text{subject to} & P + 2\mathbf{diag}(\lambda) \succ 0 \\ & \lambda \geq 0 \end{array}$$

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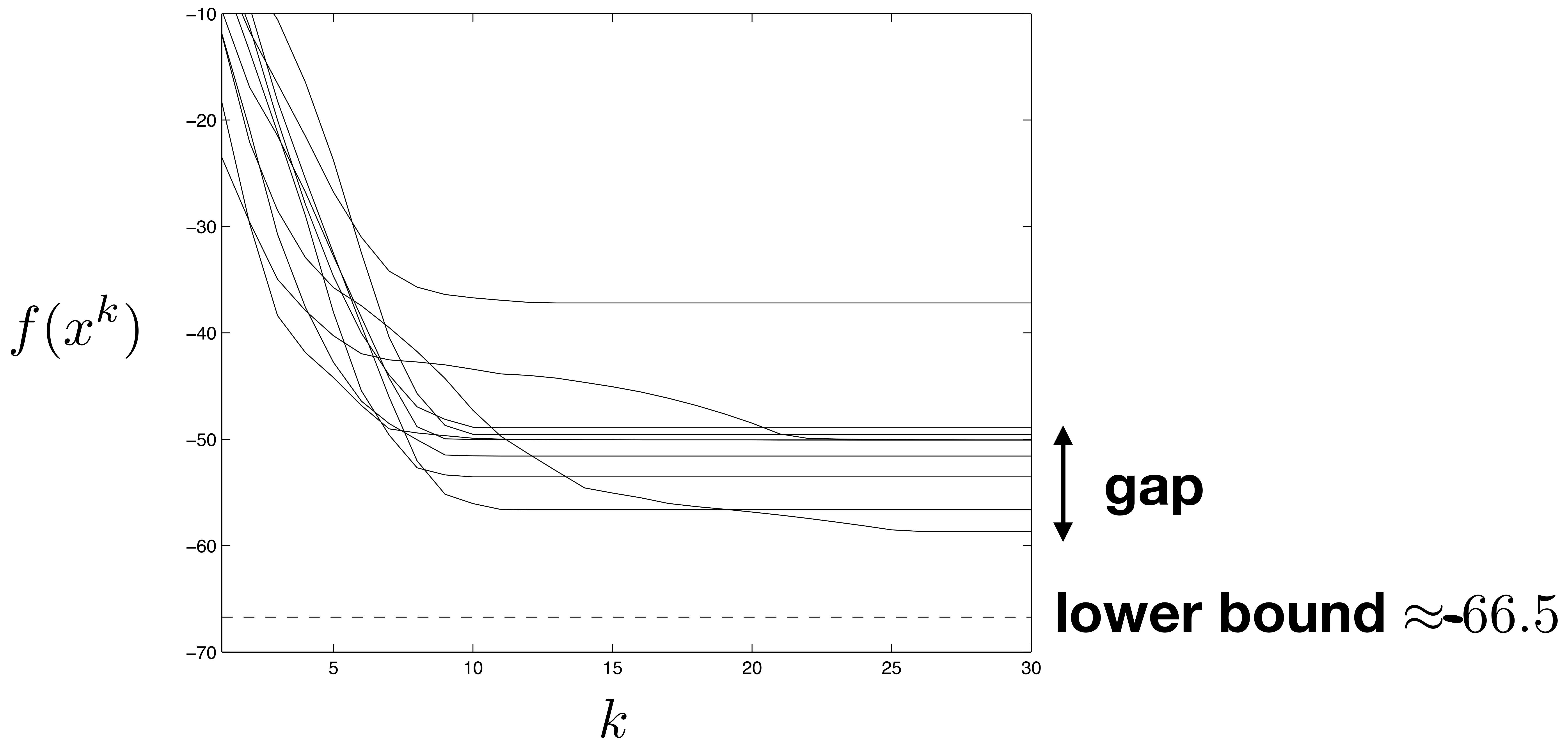
$g(\lambda)$

Example: nonconvex quadratic program

SCP with $\rho = 0.2$ with 10 different random $x_0 \in \mathbf{R}^n$



Regularized trust region methods

Issues with vanilla sequential ~~quadratic~~ CONVEX programming

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \hat{f}(x) \\ \text{subject to} & \hat{g}_i(x) \leq 0, \quad i = 1, \dots, m \\ & \hat{h}_i(x) = 0, \quad i = 1, \dots, p \\ & x \in \mathcal{T}^k \end{array}$$

Infeasibility

Approximate problem can be infeasible (e.g. too small ρ)

Issues with vanilla sequential ~~quadratic~~ CVX programming

minimize $f(x)$

subject to $g_i(x) \leq 0, \quad i = 1, \ldots, m$

$h_i(x) = 0, \quad i = 1, \ldots, p$

$\longrightarrow$

minimize $\hat{f}(x)$

subject to $\hat{g}_i(x) \leq 0, \quad i = 1, \ldots, m$

$\hat{h}_i(x) = 0, \quad i = 1, \ldots, p$

$x \in \mathcal{T}^k$

Infeasibility

Approximate problem can be infeasible (e.g. too small ρ)

Evaluate progress

when x^k infeasible

- Objective: $f(x^k)$
- Inequality violations: $g_i(x^k)_+$
- Equality violations: $|h_i(x^k)|$

Issues with vanilla sequential quadratic programming

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \hat{f}(x) \\ \text{subject to} & \hat{g}_i(x) \leq 0, \quad i = 1, \dots, m \\ & \hat{h}_i(x) = 0, \quad i = 1, \dots, p \\ & x \in \mathcal{T}^k \end{array}$$

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Controlling trust region size

- ρ **too large**
  poor approximations $\rightarrow$ bad x^{k+1}
- ρ **too small**
  good approximations $\rightarrow$ slow progress

Exact penalty formulation

Solve unconstrained problem instead of the original problem

$$\text{minimize} \quad \phi(x) = f(x) + \lambda \left(\sum_{i=1}^{m} (f_i(x))_+ + \sum_{i=1}^{p} |h_i(x)| \right), \qquad \lambda > 0$$

For λ large enough $\longrightarrow$ $x^\star = \operatorname{argmin} \phi(x)$ solves the original problem

($\lambda > \|y^\star\|_\infty$ where $y^\star$ is the dual variable satisfying the KKT conditions)

Exact penalty formulation

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For λ large enough $\longrightarrow$ $x^\star = \operatorname{argmin} \phi(x)$ solves the original problem

($\lambda > \|y^\star\|_\infty$ where $y^\star$ is the dual variable satisfying the KKT conditions)

SCP solves the convex approximation (always feasible)

$$\hat{\phi}(x) = \hat{f}(x) + \lambda \left(\sum_{i=1}^{m} (\hat{f}_i(x))_+ + \sum_{i=1}^{p} |\hat{h}_i(x)| \right)$$

If λ not large enough, we have **sparse violations**

Trust region update

Idea judge progress in ϕ using $\hat{x} = \mathrm{argmin}\, \hat{\phi}(x)$

Exact decrease

$$\delta = \phi(x^k) - \phi(\hat{x})$$

Approximate decrease

$$\hat{\delta} = \phi(x^k) - \hat{\phi}(\hat{x})$$

Trust region update

Idea judge progress in ϕ using $\hat{x} = \operatorname{argmin} \hat{\phi}(x)$

Exact decrease

$$\delta = \phi(x^k) - \phi(\hat{x})$$

Approximate decrease

$$\hat{\delta} = \phi(x^k) - \hat{\phi}(\hat{x})$$

Updates

$\delta \geq \alpha \hat{\delta} \longrightarrow$
- accept: $x^{k+1} = \hat{x}$
- increase region $\rho = \beta^{\text{acc}} \rho$

$\delta < \alpha \hat{\delta} \longrightarrow$
- reject: $x^{k+1} = x^k$
- decrease region $\rho = \beta^{\text{rej}} \rho$

Parameters

tolerance α (e.g., $= 0.1$)
accept multiplier $\beta^{\text{acc}} \geq 1$ (e.g., $= 1.1$)
reject multiplier $\beta^{\text{rej}} \in (0, 1)$ (e.g., 0.5)

Trust region update

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accept multiplier $\beta^{\mathrm{acc}} \geq 1$ (e.g., $= 1.1$)
reject multiplier $\beta^{\mathrm{rej}} \in (0, 1)$ (e.g., 0.5)

Interpretation

If actual decrease δ is more than α fraction of predicted decrease $\hat{\delta}$ then increase trust region size (longer steps). Otherwise decrease it.

Regularized trust region example

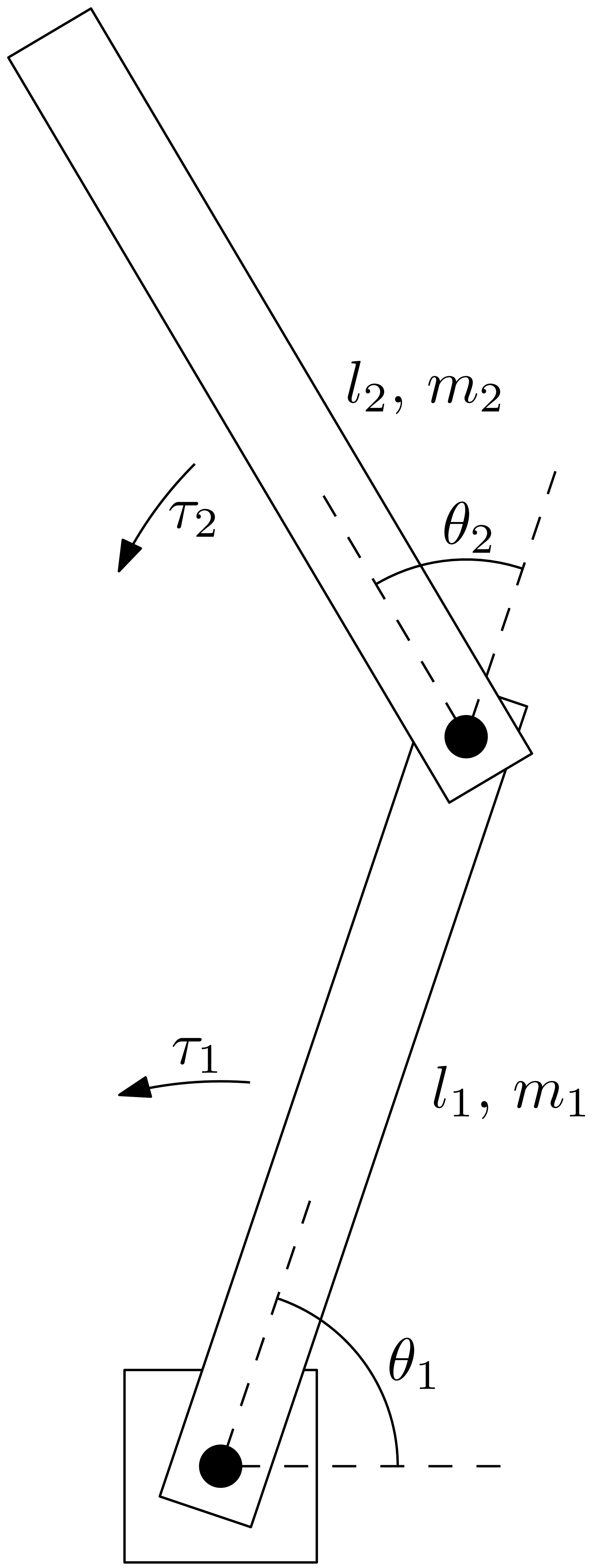
Nonlinear optimal control

Robotic arm

2-dimensional system

no gravity (horizontal)

controlled torques τ_1, τ_2



Nonlinear optimal control

The problem

$$
\begin{aligned}
\text{minimize} \quad & J = \int_0^T \|\tau(t)\|_2^2 \mathrm{d}t \\
\text{subject to} \quad & \theta(0) = \theta_{\text{init}}, \quad \theta(T) = \theta_{\text{final}} \\
& \dot{\theta}(0) = 0, \quad \dot{\theta}(T) = 0 \\
& \|\tau(t)\|_\infty \leq \tau_{\max}, \quad 0 \leq t \leq T
\end{aligned}
$$

Nonlinear optimal control

The problem

minimize $$J = \int_0^T \|\tau(t)\|_2^2 \mathrm{d}t$$

subject to $$\theta(0) = \theta_{\text{init}}, \quad \theta(T) = \theta_{\text{final}}$$

$$\dot{\theta}(0) = 0, \quad \dot{\theta}(T) = 0$$

$$\|\tau(t)\|_\infty \leq \tau_{\max}, \quad 0 \leq t \leq T$$

minimum torque

Nonlinear optimal control

The problem

minimize $J = \int_0^T \|\tau(t)\|_2^2 \mathrm{d}t$ **minimum torque**

subject to $\theta(0) = \theta_{\text{init}}, \quad \theta(T) = \theta_{\text{final}}$ **position**

$\dot{\theta}(0) = 0, \quad \dot{\theta}(T) = 0$

$\|\tau(t)\|_\infty \leq \tau_{\max}, \quad 0 \leq t \leq T$

Nonlinear optimal control

The problem

minimize	$J = \int_0^T \|\tau(t)\|_2^2 \mathrm{d}t$	**minimum torque**
subject to	$\theta(0) = \theta_{\text{init}}, \quad \theta(T) = \theta_{\text{final}}$	**position**
	$\dot{\theta}(0) = 0, \quad \dot{\theta}(T) = 0$	**velocity**
	$\|\tau(t)\|_\infty \leq \tau_{\max}, \quad 0 \leq t \leq T$	

Nonlinear optimal control

The problem

minimize $J = \int_0^T \|\tau(t)\|_2^2 \mathrm{d}t$ **minimum torque**

subject to $\theta(0) = \theta_{\text{init}}, \quad \theta(T) = \theta_{\text{final}}$ **position**

$\dot{\theta}(0) = 0, \quad \dot{\theta}(T) = 0$ **velocity**

$\|\tau(t)\|_\infty \leq \tau_{\max}, \quad 0 \leq t \leq T$

Dynamics

$$M(\theta)\ddot{\theta} + W(\theta, \dot{\theta})\dot{\theta} = \tau$$

$$M(\theta) = \begin{bmatrix} (m_1 + m_2)l_1^2 & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{bmatrix}$$

$$W(\theta, \dot{\theta}) = \begin{bmatrix} 0 & m_2 l_1 l_2 (s_1 c_2 - c_1 s_2)\dot{\theta}_2 \\ m_2 l_1 l_2 (s_1 c_2 - c_1 s_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

where $s_i = \sin(\theta_i)$ and $c_i = \cos(\theta_i)$

Nonlinear optimal control

The problem

minimize $$J = \int_0^T \|\tau(t)\|_2^2 \mathrm{d}t$$ **minimum torque**

subject to $$\theta(0) = \theta_{\text{init}}, \quad \theta(T) = \theta_{\text{final}}$$ **position**

$$\dot{\theta}(0) = 0, \quad \dot{\theta}(T) = 0$$ **velocity**

$$\|\tau(t)\|_\infty \leq \tau_{\max}, \quad 0 \leq t \leq T$$

Dynamics

$$M(\theta)\ddot{\theta} + W(\theta, \dot{\theta})\dot{\theta} = \tau$$

↓

Not convex!

(Hard to optimize)

Note: cheap to simulate

$$M(\theta) = \begin{bmatrix} (m_1 + m_2)l_1^2 & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{bmatrix}$$

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where $s_i = \sin(\theta_i)$ and $c_i = \cos(\theta_i)$

Nonlinear optimal control

Discretization

Discretize with **time intervals** $h = T/N$

Objective $$J = \int_0^T \|\tau(t)\|_2^2 \mathrm{d}t \approx h \sum_{i=1}^{N} \|\tau_i\|_2^2, \quad \text{with} \quad \tau_i = \tau(ih)$$

Nonlinear optimal control

Discretization

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Dynamics: approximate derivatives

$$M(\theta)\ddot{\theta} + W(\theta, \dot{\theta})\dot{\theta} = \tau$$

$$\dot{\theta}(ih) \approx \frac{\theta_{i+1} - \theta_{i-1}}{2h} \qquad \ddot{\theta}(ih) \approx \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2}$$

$$\theta_0 = \theta_1 = \theta_{\text{init}}$$
$$\theta_N = \theta_{N+1} = \theta_{\text{final}}$$

Nonlinear optimal control

Discretization

Discretize with **time intervals** $h = T/N$

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$$\theta_0 = \theta_1 = \theta_{\text{init}}$$
$$\theta_N = \theta_{N+1} = \theta_{\text{final}}$$

nonlinear equality constraints

$$M(\theta_i)\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + W\left(\theta_i, \frac{\theta_{i+1} - \theta_{i-1}}{2h}\right)\frac{\theta_{i+1} - \theta_{i-1}}{2h} = \tau_i$$

Nonlinear optimal control

Convexification

$$
\begin{aligned}
\text{minimize} \quad & h\sum_{i=1}^{N}\|\tau_i\|_2^2 \\
\text{subject to} \quad & \theta_0 = \theta_1 = \theta_{\text{init}}, \quad \theta_N = \theta_{N+1} = \theta_{\text{final}} \\
& \|\tau_i\|_\infty \leq \tau_{\max} \\
& M(\theta_i)\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{h^2} + W\left(\theta_i, \frac{\theta_{i+1}-\theta_{i-1}}{2h}\right)\frac{\theta_{i+1}-\theta_{i-1}}{2h} = \tau_i
\end{aligned}
$$

Nonlinear optimal control

Convexification

$$\begin{array}{ll} \text{minimize} & h \sum_{i=1}^{N} \|\tau_i\|_2^2 \\ \text{subject to} & \theta_0 = \theta_1 = \theta_{\text{init}}, \quad \theta_N = \theta_{N+1} = \theta_{\text{final}} \\ & \|\tau_i\|_\infty \leq \tau_{\max} \\ & M(\theta_i) \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + W\left(\theta_i, \frac{\theta_{i+1} - \theta_{i-1}}{2h}\right) \frac{\theta_{i+1} - \theta_{i-1}}{2h} = \tau_i \end{array}$$

Quasi-linearization of the dynamics around previous x^k

$$M(\theta_i^k) \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + W\left(\theta_i^k, \frac{\theta_{i+1}^k - \theta_{i-1}^k}{2h}\right) \frac{\theta_{i+1} - \theta_{i-1}}{2h} = \tau_i$$

Nonlinear optimal control

Convexification

$$\begin{array}{ll} \text{minimize} & h\sum_{i=1}^{N} \|\tau_i\|_2^2 \\ \text{subject to} & \theta_0 = \theta_1 = \theta_{\text{init}}, \quad \theta_N = \theta_{N+1} = \theta_{\text{final}} \\ & \|\tau_i\|_\infty \le \tau_{\max} \\ & M(\theta_i)\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{h^2} + W\left(\theta_i, \frac{\theta_{i+1}-\theta_{i-1}}{2h}\right)\frac{\theta_{i+1}-\theta_{i-1}}{2h} = \tau_i \end{array}$$

Quasi-linearization of the dynamics around previous x^k

$$M(\theta_i^k)\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{h^2} + W\left(\theta_i^k, \frac{\theta_{i+1}^k-\theta_{i-1}^k}{2h}\right)\frac{\theta_{i+1}-\theta_{i-1}}{2h} = \tau_i$$

Remarks

- trust region only on θ_i (cost and constraints convex in τ_i)
- initialize with straight line: $\theta_i = \frac{i-1}{N-1}(\theta_{\text{final}} - \theta_{\text{init}}), \quad i = 1, \dots, N$

Nonlinear optimal control

Example

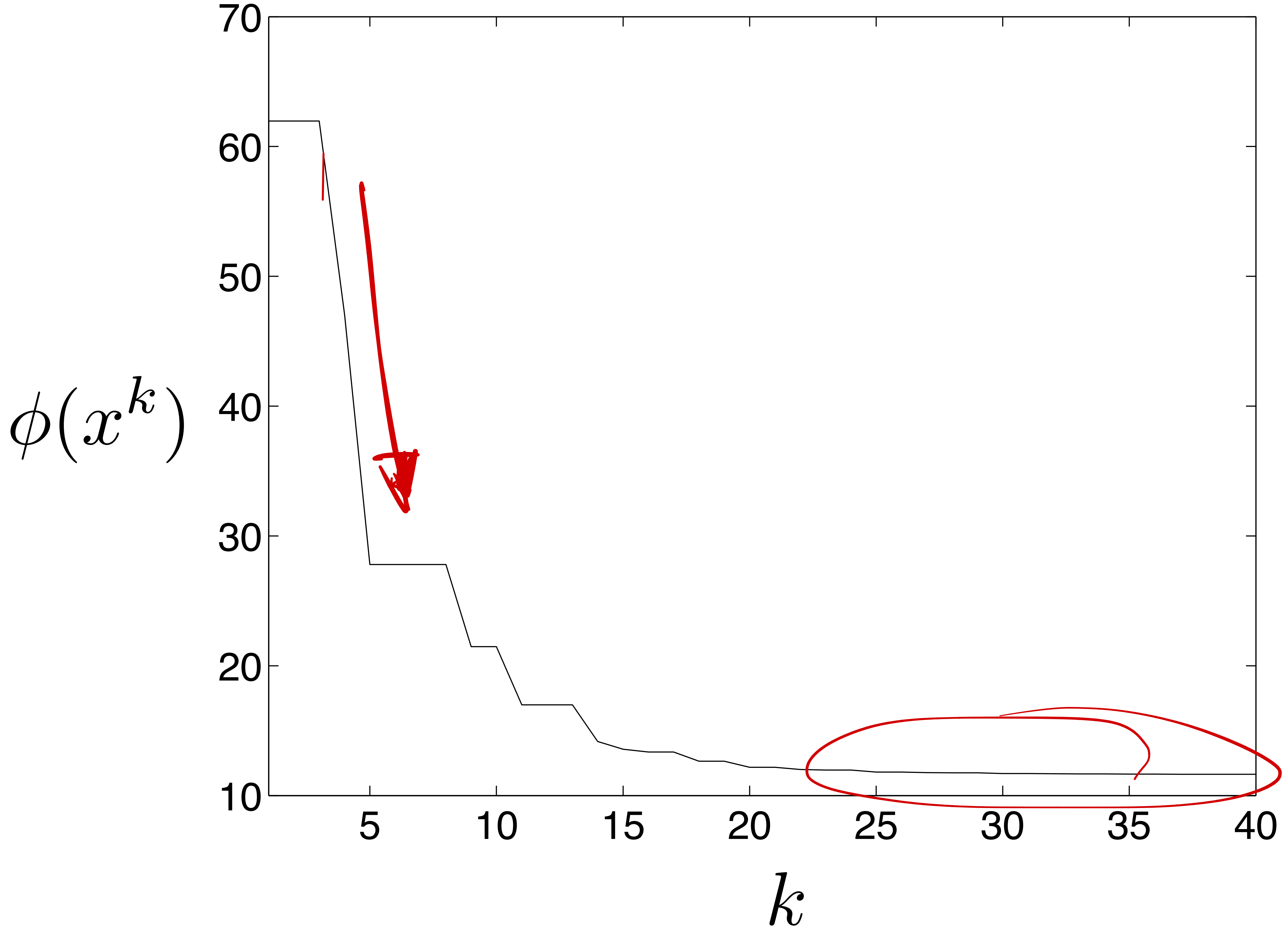
System

- $m_1 = 1$, $m_2 = 5$, $l_1 = l_2 = 1$
- $N = 40$, $T = 10$
- $\theta_{\text{init}} = (0, -2.9)$, $\quad \theta_{\text{final}} = (3, 2.9)$
- $\tau_{\max} = 1.1$

Algorithm

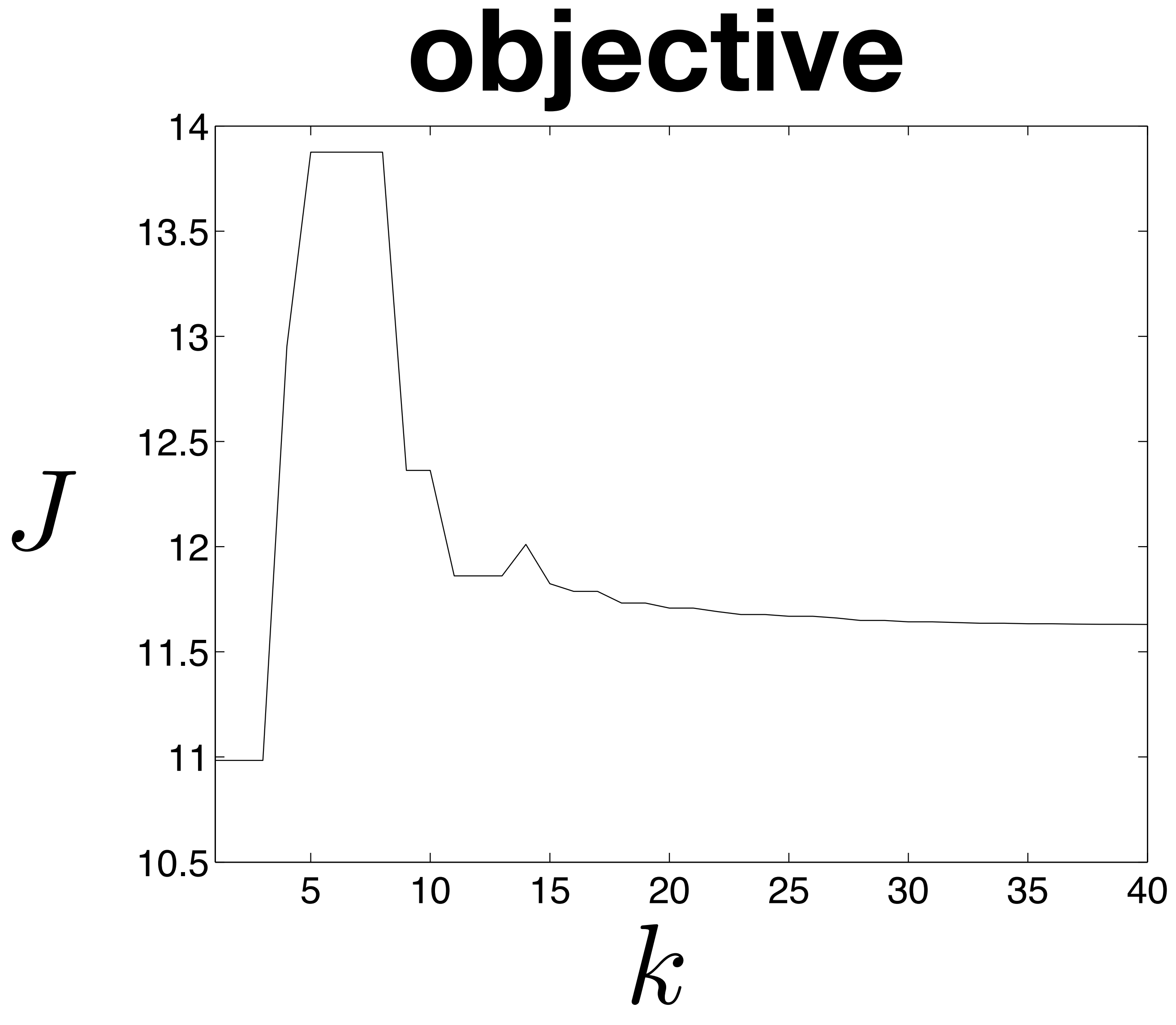
- $\lambda = 2$
- $\alpha = 0.1$, $\beta^{\text{acc}} = 1.1$, $\beta^{\text{rej}} = 0.5$
- $\rho_1 = 90°$ (very large)

Progress

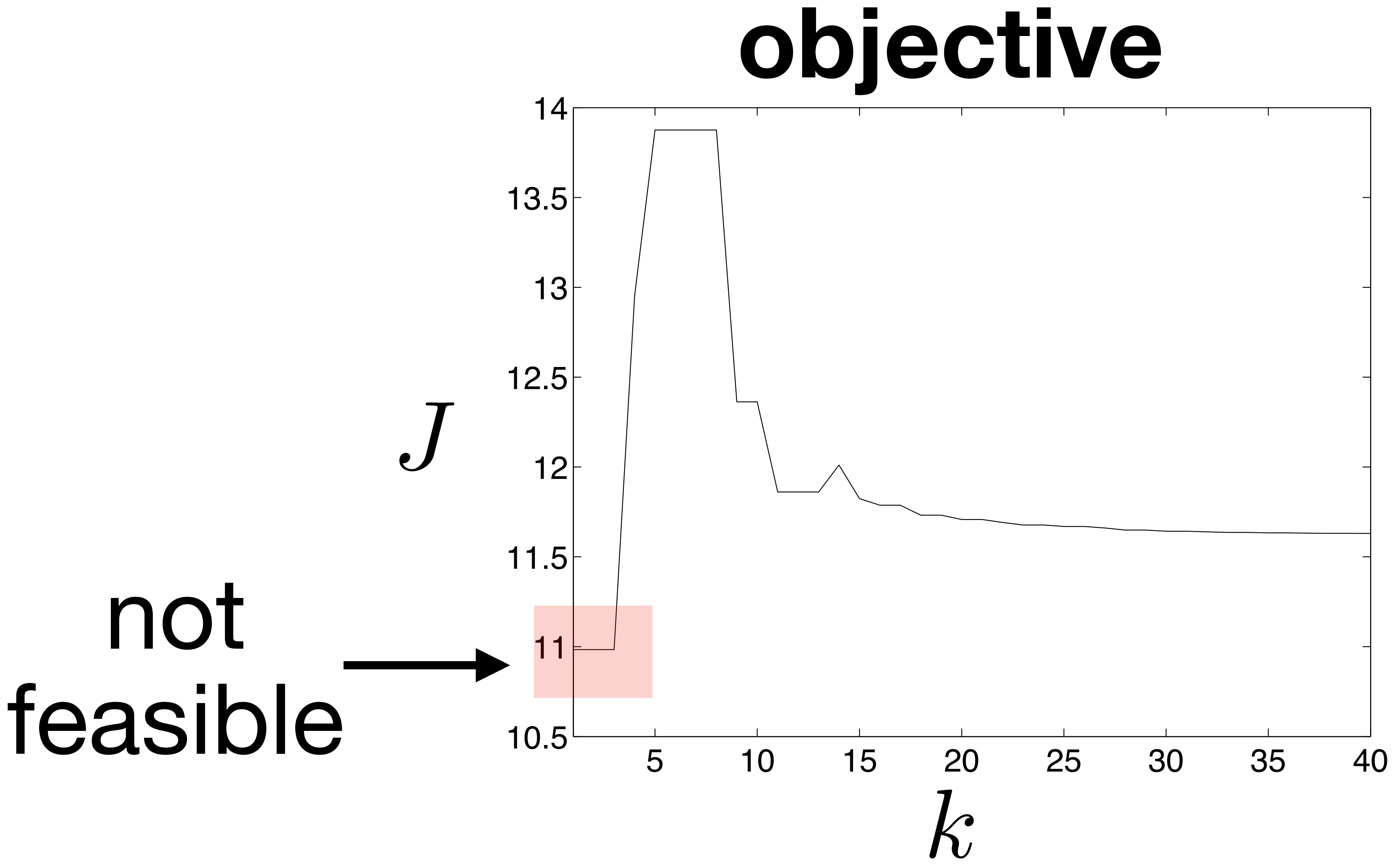


Note: does not go to 0

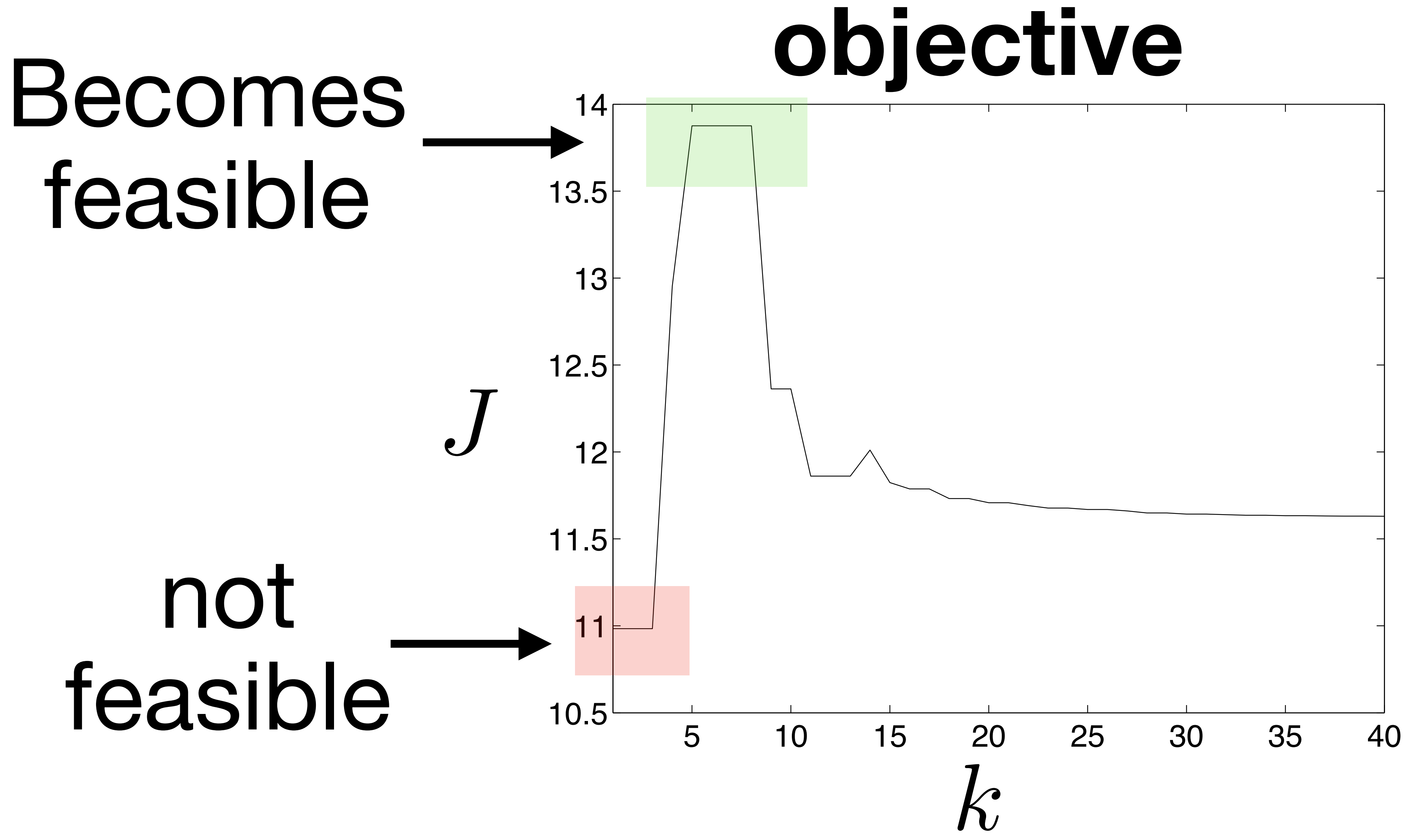
Nonlinear optimal control



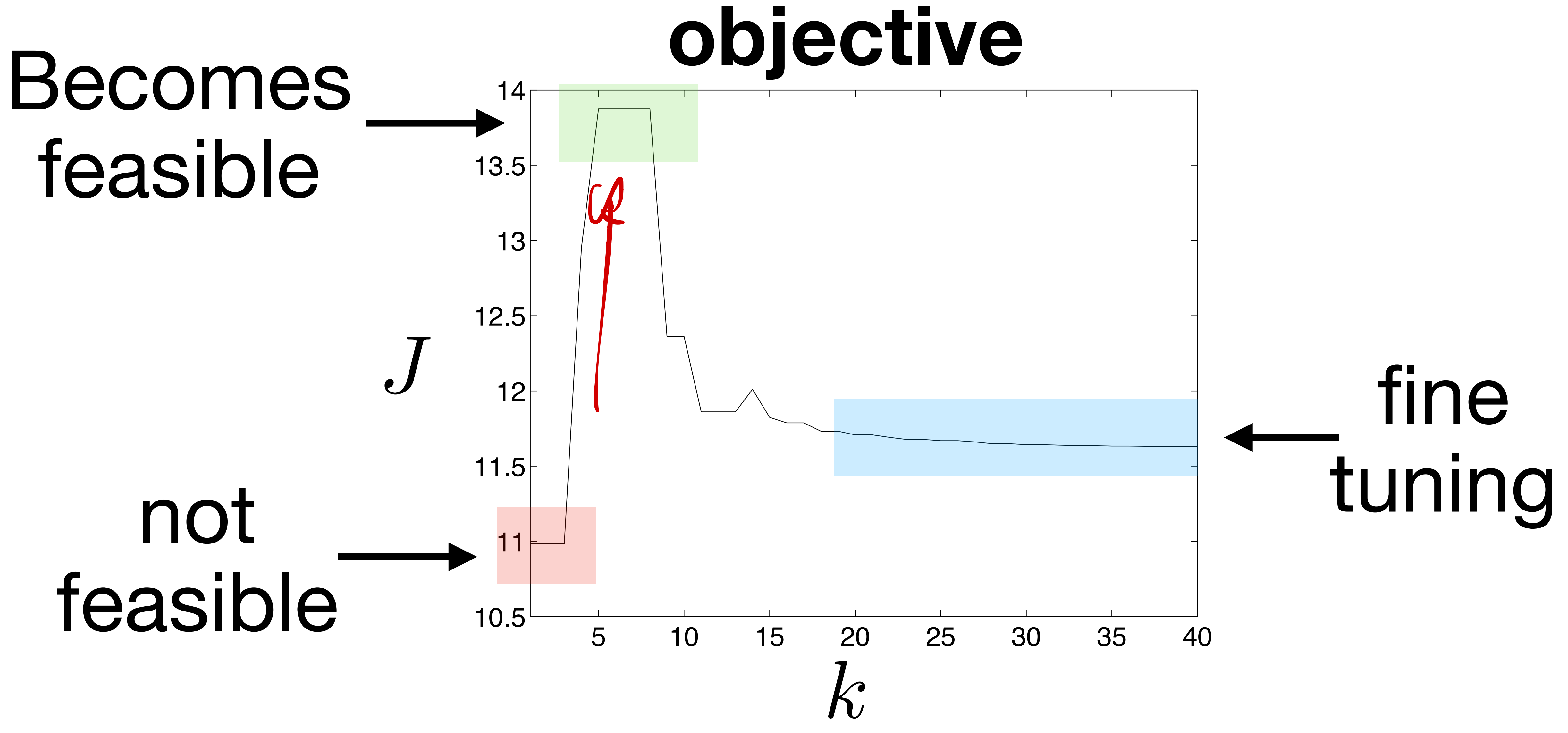
Nonlinear optimal control



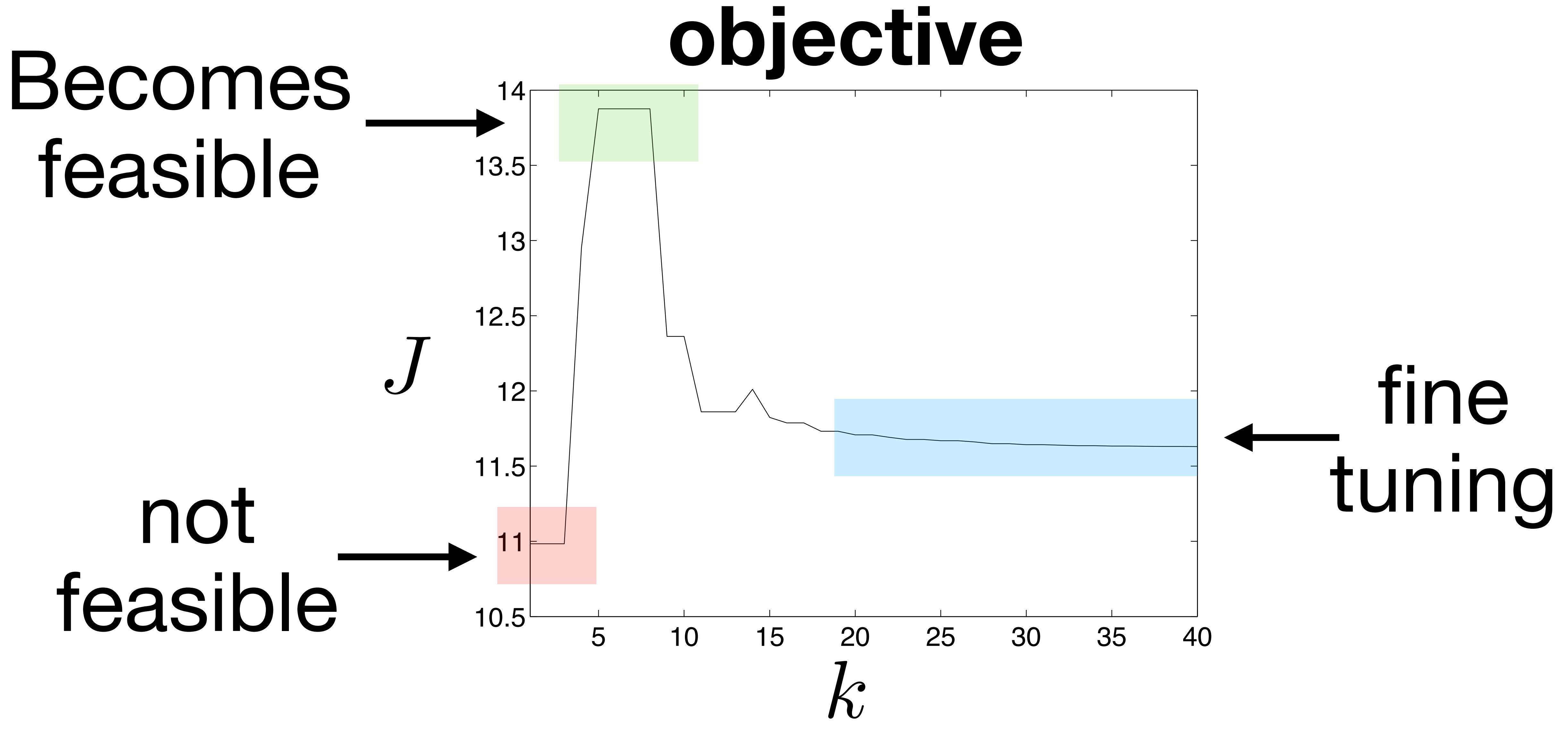
Nonlinear optimal control

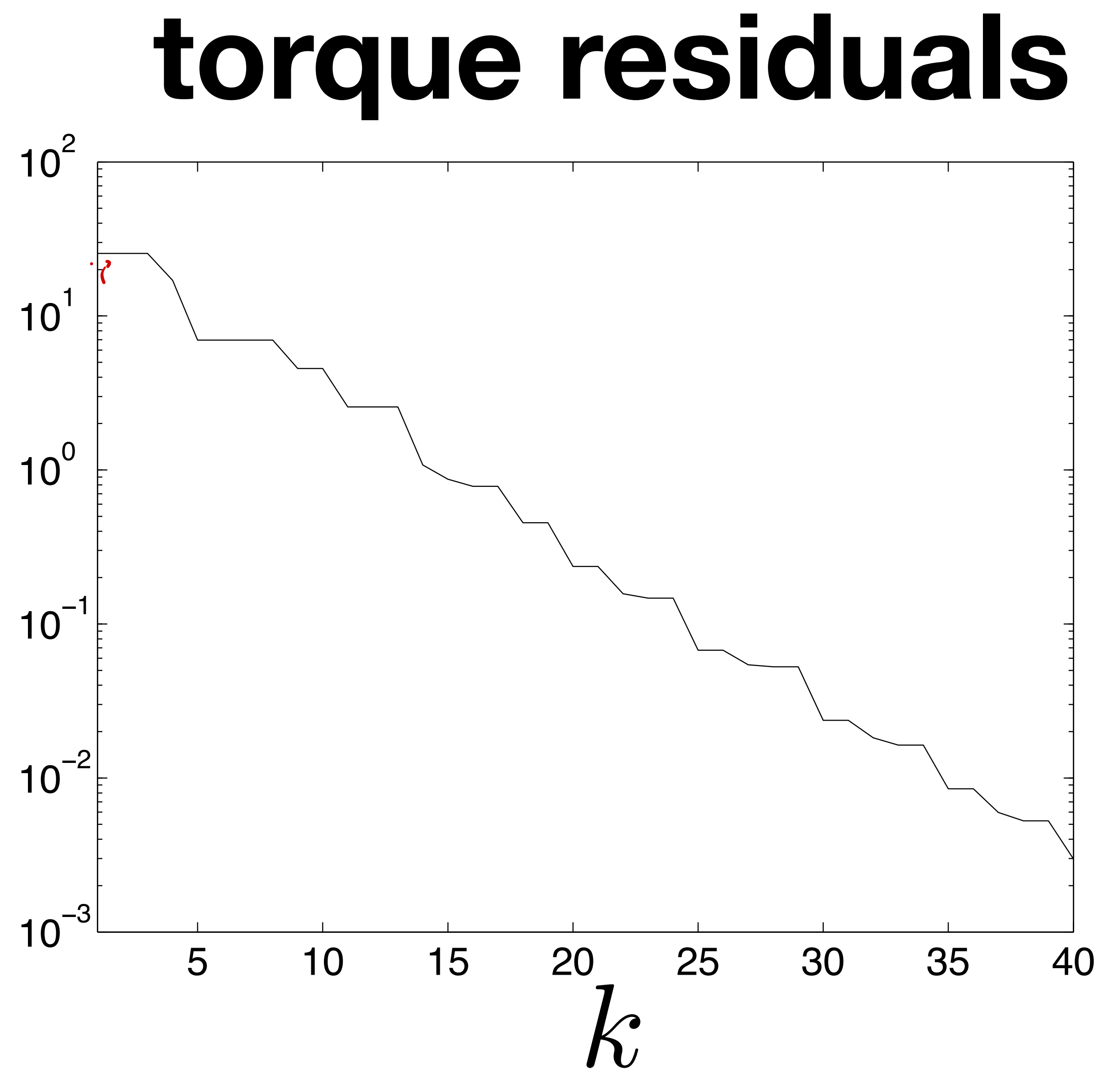


Nonlinear optimal control

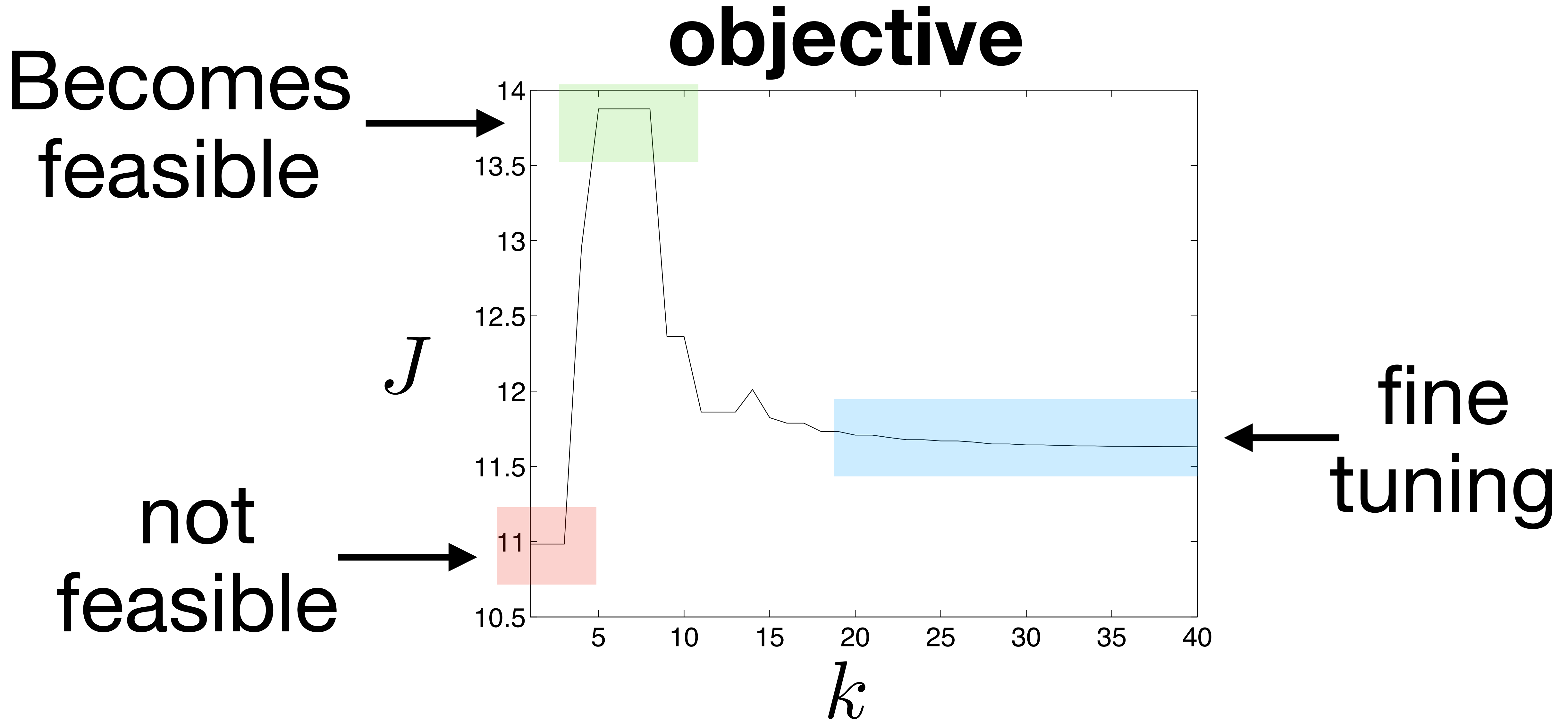


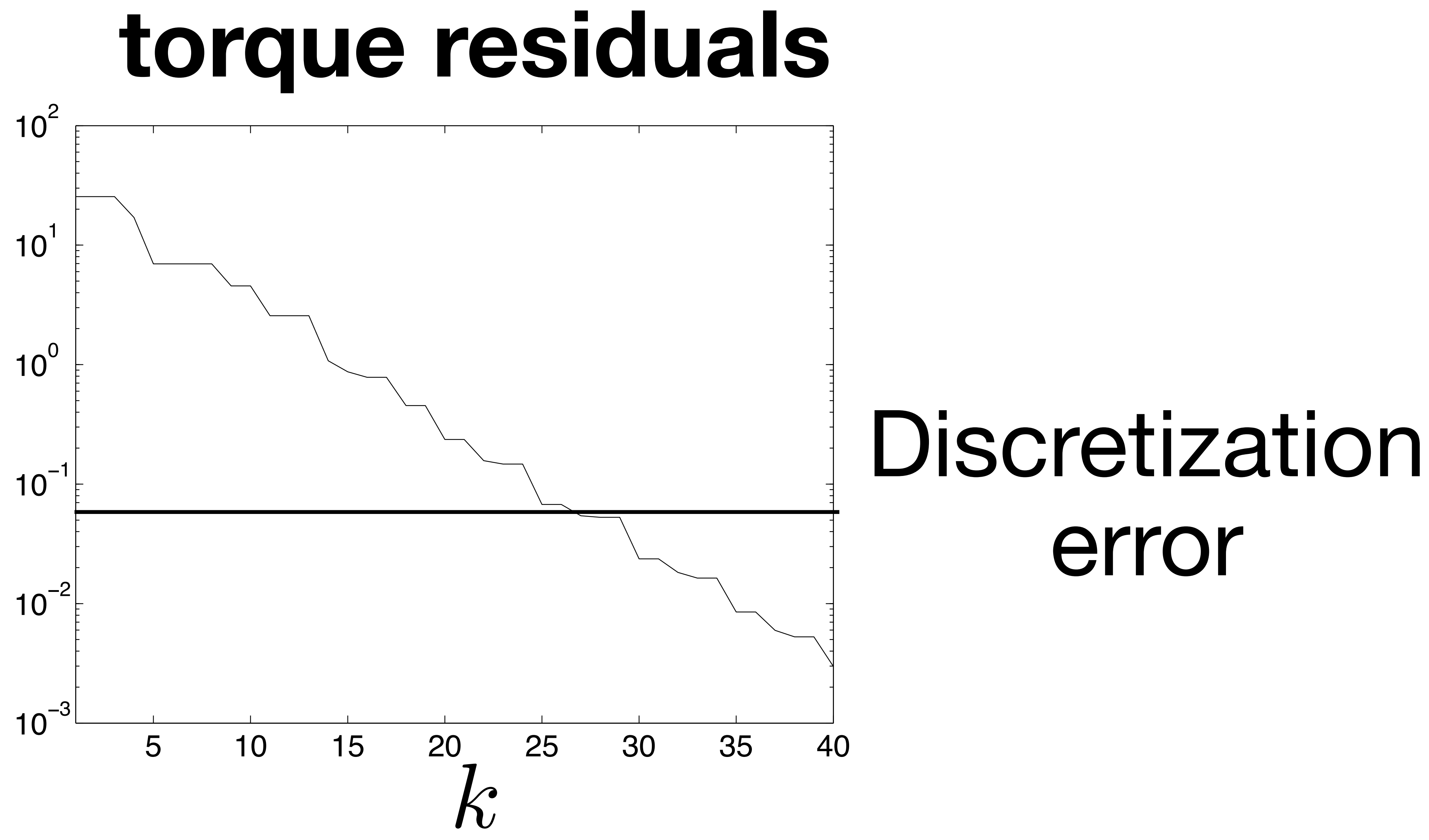
Nonlinear optimal control



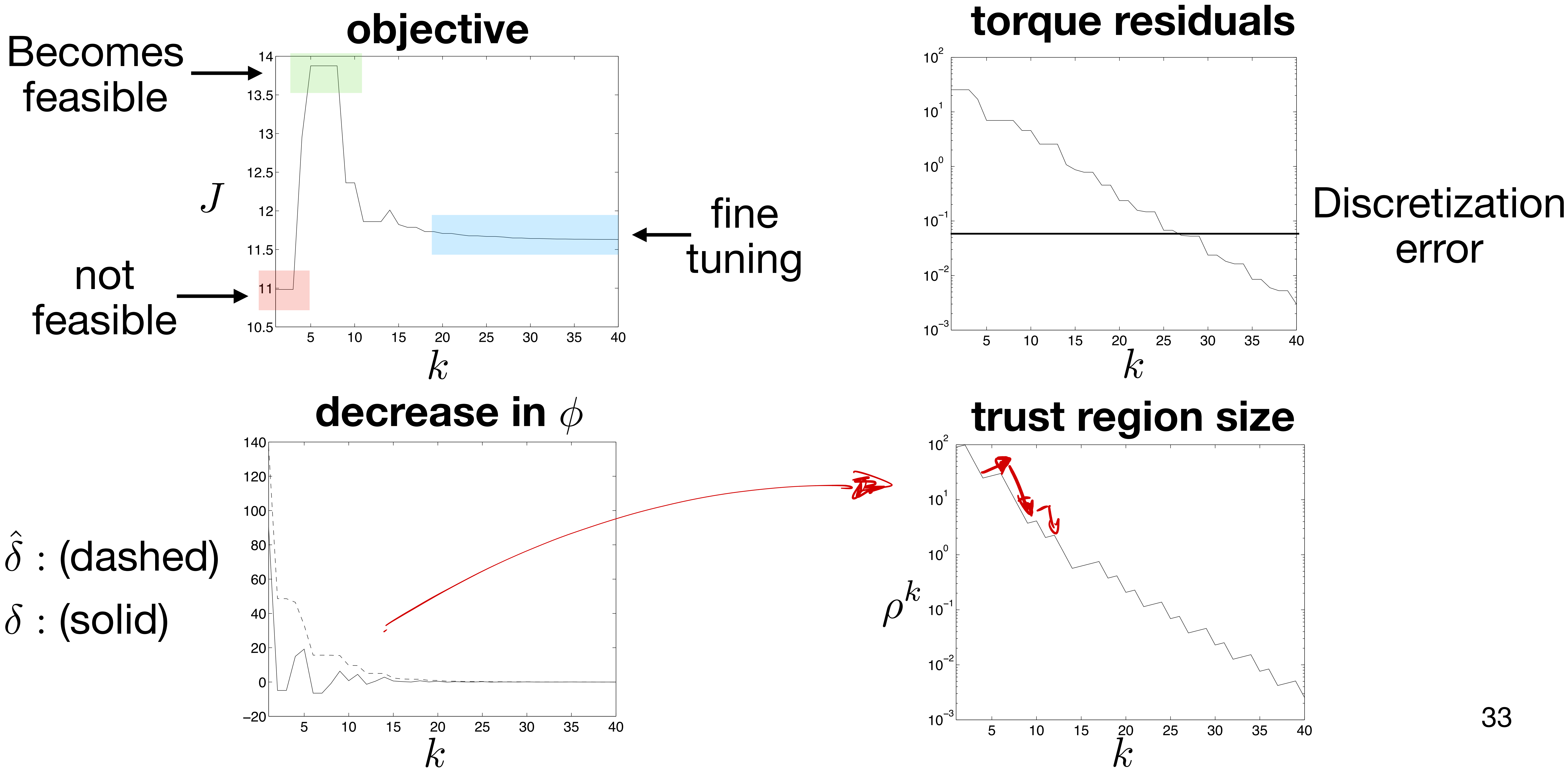


Nonlinear optimal control



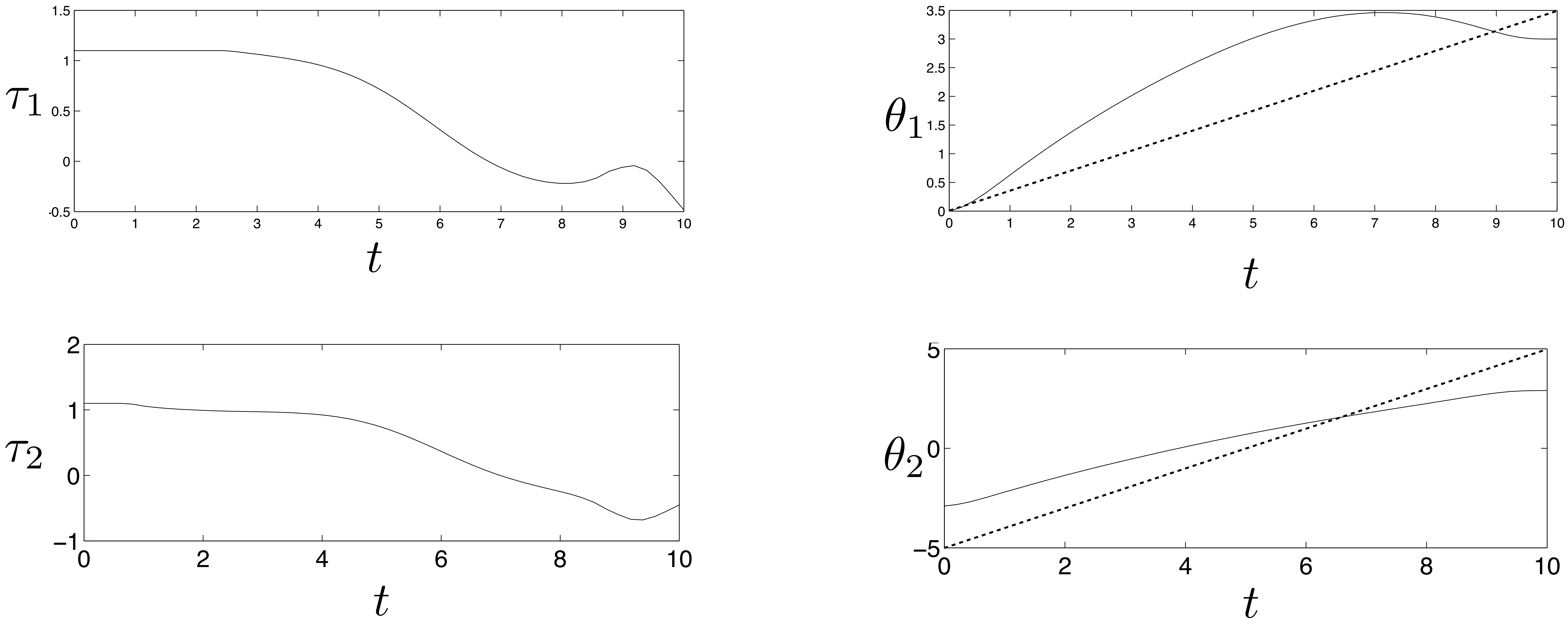


Nonlinear optimal control



Nonlinear optimal control

Trajectories



Difference of convex programming

Difference of convex programming

$$\begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & f_i(x) - g_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

where f_i and g_i are convex

Difference of convex programming

$$\begin{array}{ll} \text{minimize} & f_0(x) - g_0(x) \\ \text{subject to} & f_i(x) - g_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

Difference of convex functions

where f_i and g_i are convex

Difference of convex programming

$$
\begin{array}{ll}
\text{minimize} & f_0(x) - g_0(x) \\
\text{subject to} & f_i(x) - g_i(x) \leq 0, \quad i = 1, \dots, m
\end{array}
$$

Difference of convex functions

where f_i and g_i are convex

Very powerful
it can represent any twice differentiable function

Hard
nonconvex problem unless g_i are affine

[On Functions Representable As A Difference Of Convex Functions, Hartman]

Difference of convex programming

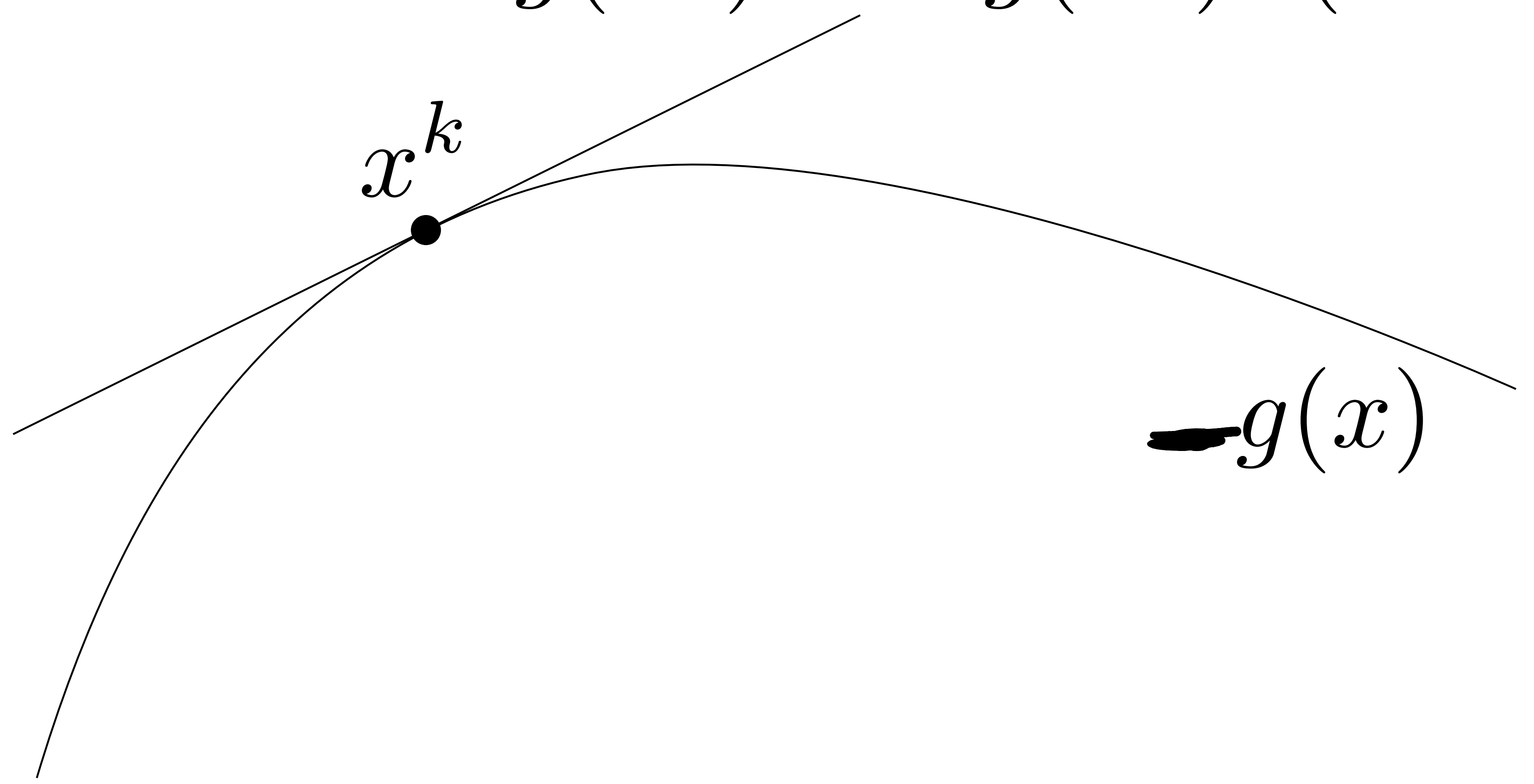
Convexification

Convexify $f(x) - g(x)$

$$f(x) - \hat{g}(x) = f(x) - g(x^k) - \nabla g(x^k)^T(x - x^k)$$

↓

$$f(x) - g(x) \leq f(x) - \hat{g}(x)$$



Remarks

- True objective better than convexified objective
- True feasible set contains convexified feasible set ⟶ **No trust region needed**

Difference of convex programming

Iterations

Convex-concave procedure

1. Convexify: form $\hat{g}_i(x) = g_i(x^k) + \nabla g_i(x^k)^T(x - x^k)$ for $i = 0, \ldots, m$
2. Solve to obtain x^{k+1}

$$\begin{array}{ll} \text{minimize} & f_0(x) - \hat{g}_0(x) \\ \text{subject to} & f_i(x) - \hat{g}_i(x) \leq 0 \end{array}$$

Remarks

It always converges to a stationary point (it might be a maximum)

[Variations and extension of the convex–concave procedure, Lipp, Boyd]

Path planning example

Find shortest path connecting a and b in $\mathbf{R}^d$

Avoid circles centered at c_j with radius r_j with $j = 1, \ldots, m$

$$
\begin{array}{ll}
\text{minimize} & L \\
\text{subject to} & x_0 = a, \quad x_n = b \\
& \|x_i - x_{i-1}\|_2 \leq L/n, \quad i = 1, \ldots, n \\
& \|x_i - c_j\|_2 \geq r_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m
\end{array}
$$

Path planning example

Find shortest path connecting a and b in $\mathbf{R}^d$

Avoid circles centered at c_j with radius r_j with $j = 1, \ldots, m$

$$
\begin{array}{ll}
\text{minimize} & L \\
\text{subject to} & x_0 = a, \quad x_n = b \\
& \|x_i - x_{i-1}\|_2 \leq L/n, \quad i = 1, \ldots, n \\
& \|x_i - p_j\|_2 \geq r_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m
\end{array}
$$

path lengths ——— $\|x_i - x_{i-1}\|_2 \leq L/n$

Path planning example

Find shortest path connecting a and b in $\mathbf{R}^d$

Avoid circles centered at c_j with radius r_j with $j = 1, \ldots, m$

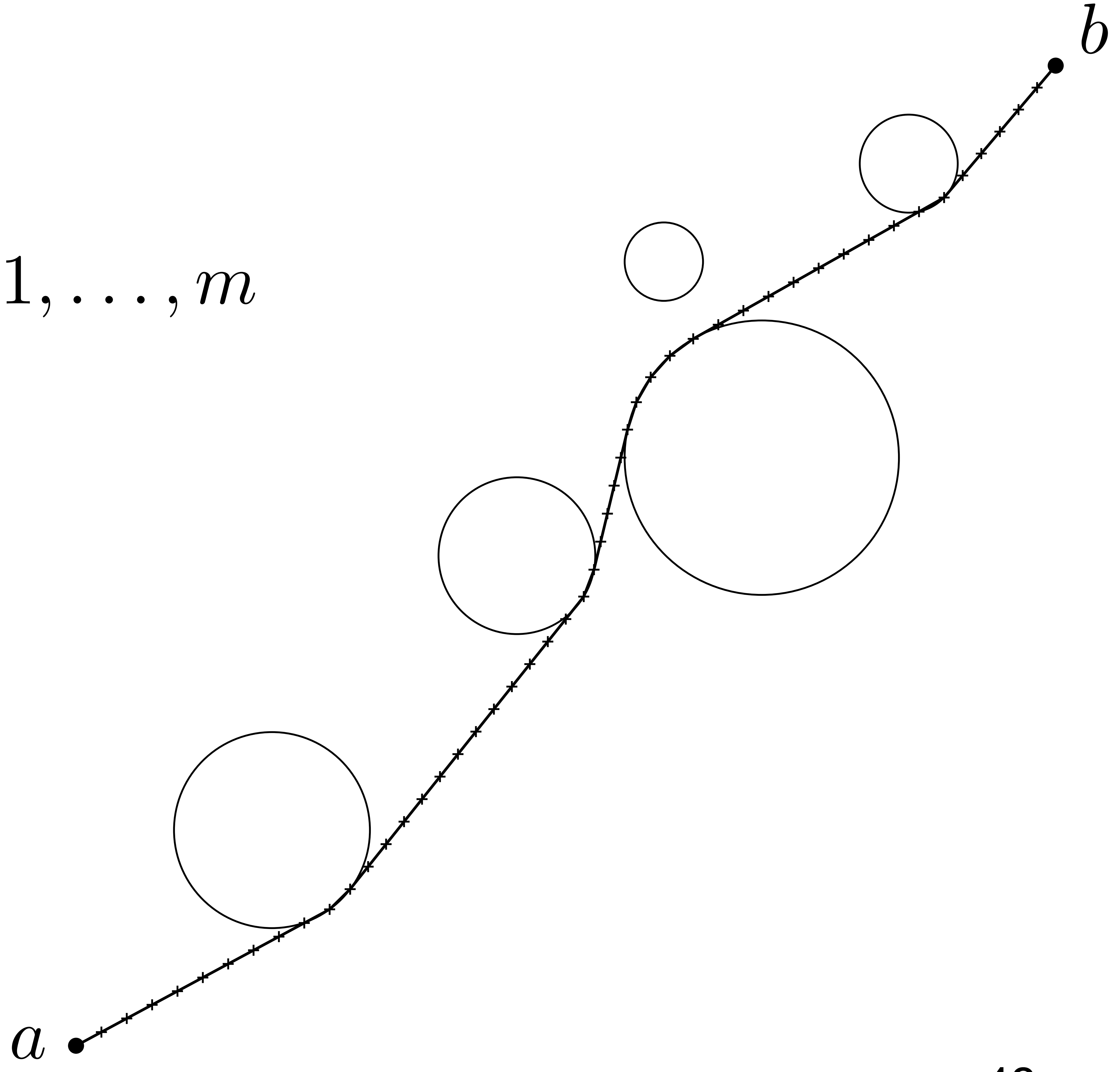
$$\begin{array}{ll} \text{minimize} & L \\ \text{subject to} & x_0 = a, \quad x_n = b \\ & \|x_i - x_{i-1}\|_2 \leq L/n, \quad i = 1, \ldots, n \\ & \|x_i - c_j\|_2 \geq r_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \end{array}$$

path lengths —— $\|x_i - x_{i-1}\|_2 \leq L/n$

obstacle constraints (not convex) —— $\|x_i - c_j\|_2 \geq r_j$

Path planning example

$$
\begin{array}{ll}
\text{minimize} & L \\
\text{subject to} & x_0 = a, \quad x_n = b \\
& \|x_i - x_{i-1}\|_2 \leq L/n, \quad i = 1, \ldots, n \\
& \|x_i - c_j\|_2 \geq r_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m
\end{array}
$$



Dimension: $d = 2$ Steps: $n = 50$

It converges in 26 iterations (convex problems)

[Disciplined Convex-Concave Programming, Shen, Diamond, Gu, Boyd]

Sequential convex programming

Today, we learned to:

- **Familiarize** with concepts of sequential convex programming
- **Develop** trust region algorithms
- **Build** convex approximations of nonlinear/nonsmooth functions
- **Develop** regularized trust region methods to account for infeasibility
- **Recognize** difference-of-convex programs and apply convex-concave procedure

Next lecture

- Branch and bound algorithms