ORF522 – Linear and Nonlinear Optimization

17. Operator theory

Bartolomeo Stellato — Fall 2020

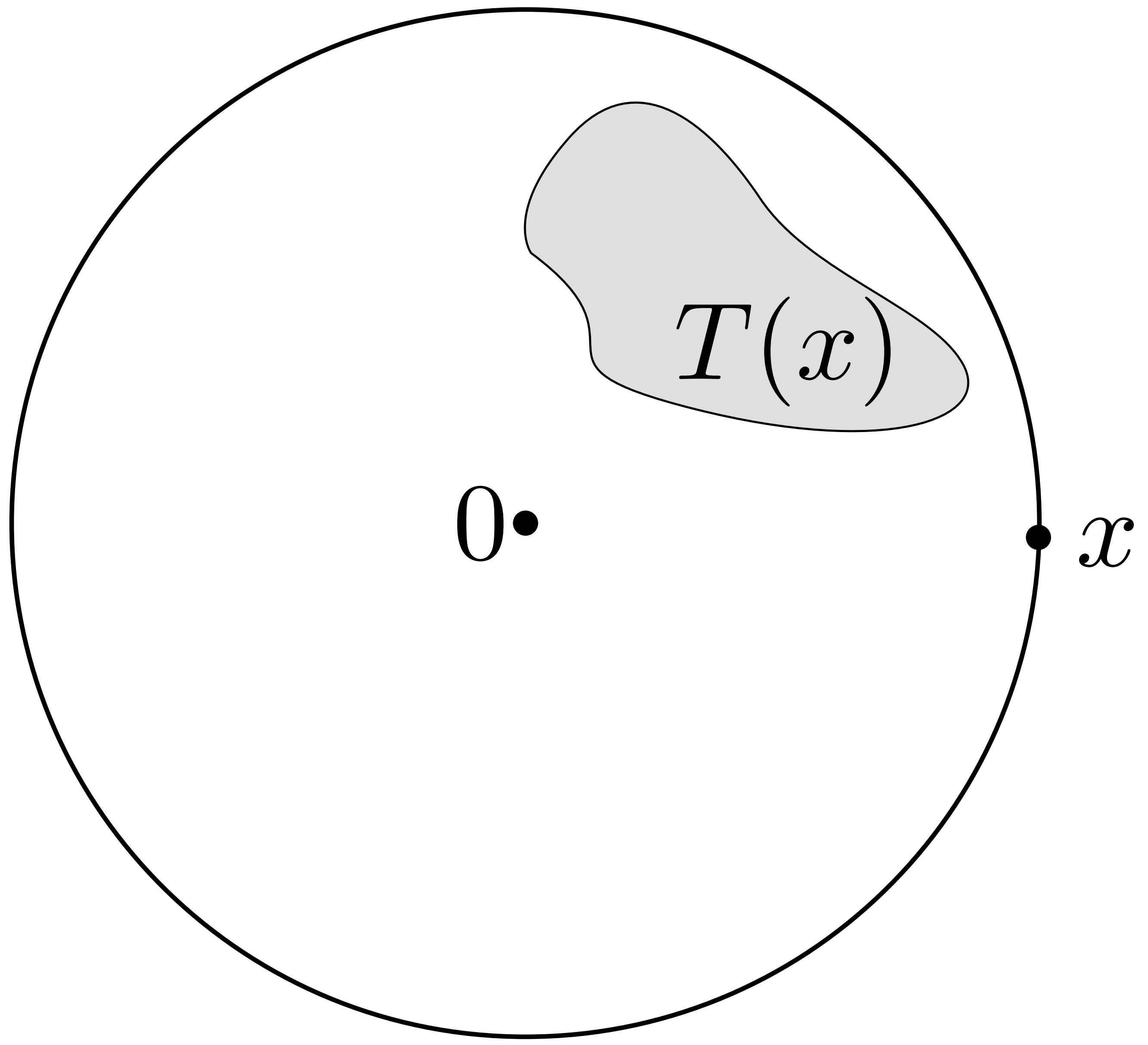
Recap

Operators

An operator T maps each point in $\mathbf{R}^n$ to a subset of $\mathbf{R}^n$

- **set valued** $T(x)$ returns a set
- **single-valued** $T(x)$ (function) returns a singleton

The **domain** of T is the set $\mathbf{dom}\, T = \{x \mid T(x) \neq \emptyset\}$



Example

- The subdifferential ∂f is a set-valued operator
- The gradient ∇f is a single-valued operator

Zeros

Zero

x is a **zero** of T if $\quad 0 \in T(x)$

Zero set

The set of all the zeros $\quad T^{-1}(0) = \{x \mid 0 \in T(x)\}$

Example
If $T = \partial f$ and $f : \mathbf{R}^n \to \mathbf{R}$, then
$0 \in T(x)$ means that x minimizes f

Many problems
can be posed as finding zeros
of an operator

Fixed points

$\bar{x}$ is a **fixed-point** of a single-valued operator T if

$$\bar{x} = T(\bar{x})$$

Set of fixed points $\quad \mathbf{fix}\, T = \{x \in \mathbf{dom}\, T \mid x = T(x)\} = (I - T)^{-1}(0)$

Examples

- **Identity** $T(x) = x$. Any point is a fixed point
- **Zero operator** $T(x) = 0$. Only 0 is a fixed point

Lipschitz operators

An operator T is L-Lipschitz if

$$\|T(x) - T(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbf{dom}\, T$$

Fact If T is Lipschitz, then it is single-valued

Proof If $y = T(x), z = T(x)$, then $\|y - z\| \leq L\|x - x\| = 0 \implies y = z$ ∎

For $L = 1$ we say T is **nonexpansive**
For $L < 1$ we say T is **contractive** (with contraction factor L)

Lipschitz operators examples

Lipschitz affine functions

$$T(x) = Ax + b$$

$\longleftrightarrow$

maximum singular value

$$L = \|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

Lipschitz operators examples

Lipschitz affine functions

$$T(x) = Ax + b$$

↔

maximum singular value

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Lipschitz differentiable functions

T such that there exists derivative DT

↔

derivative is bounded

$$\|DT\|_2 \leq L$$

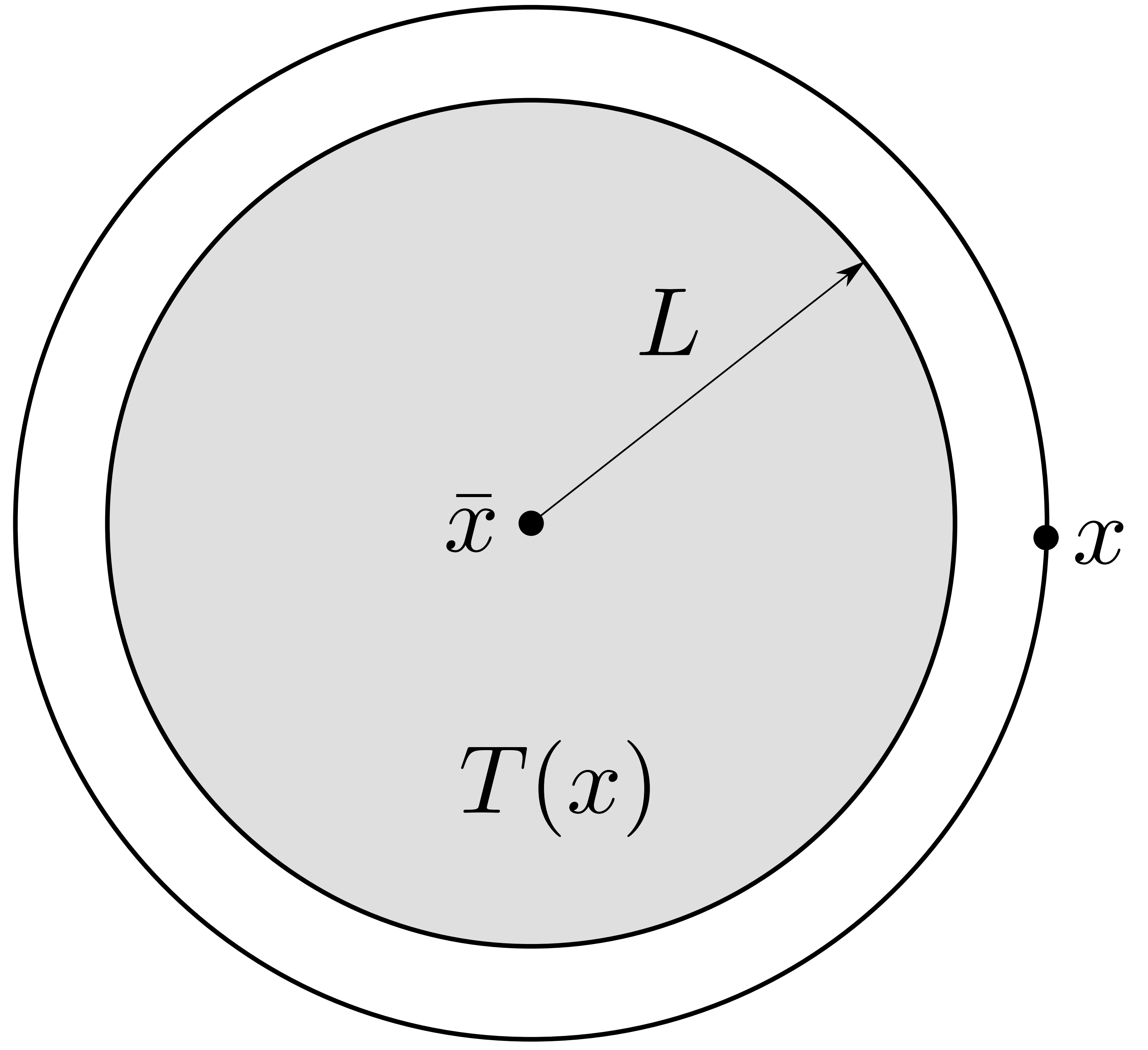
Lipschitz operators and fixed points

Given a L-Lipschitz operator T and a fixed point $\bar{x} = T\bar{x}$,

$$\|Tx - \bar{x}\| = \|Tx - T\bar{x}\| \leq L\|x - \bar{x}\|$$

A contractive operator ($L < 1$) can have at most one fixed point, i.e., $\mathbf{fix}\, T = \{\bar{x}\}$

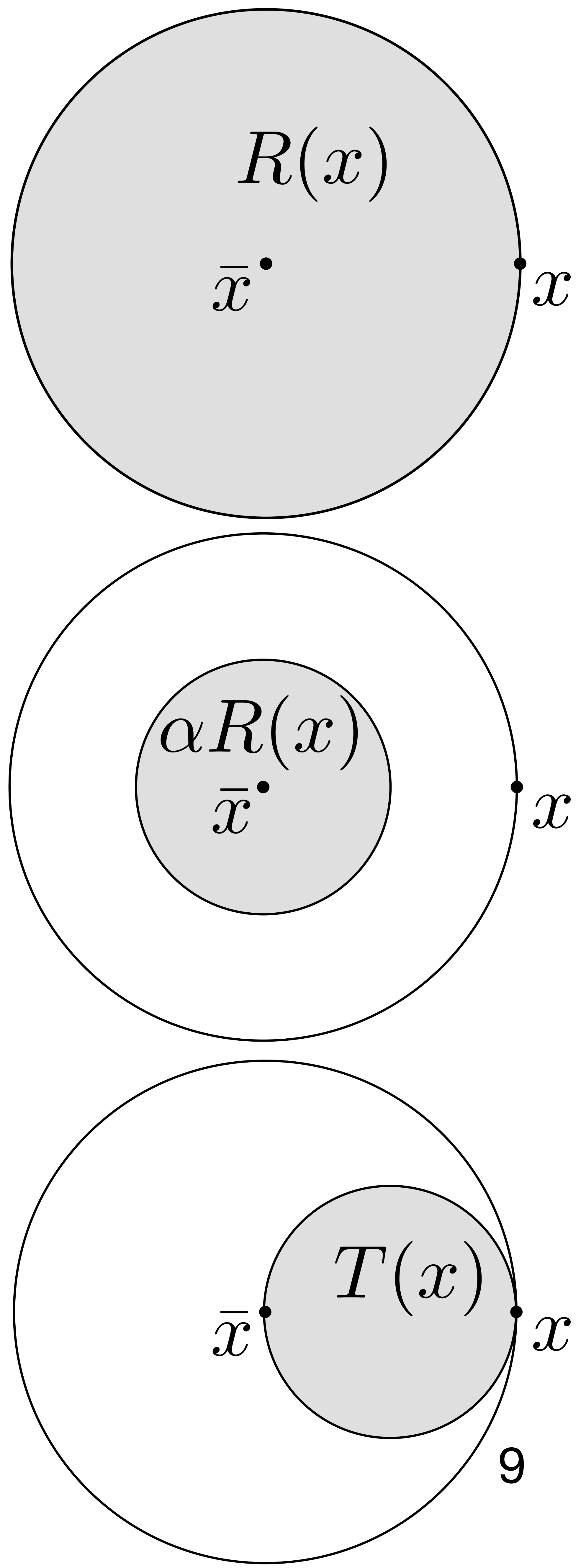
Proof
If $\bar{x}, \bar{y} \in \mathbf{fix}\, T$ and $\bar{x} \neq \bar{y}$ then
$\|\bar{x} - \bar{y}\| = \|T(\bar{x}) - T(\bar{y})\| < \|x - y\|$ (contradiction) ■



A nonexpansive operator ($L = 1$) need not have a fixed point

Example $T(x) = x + 2$

Example $\alpha = 1/2$

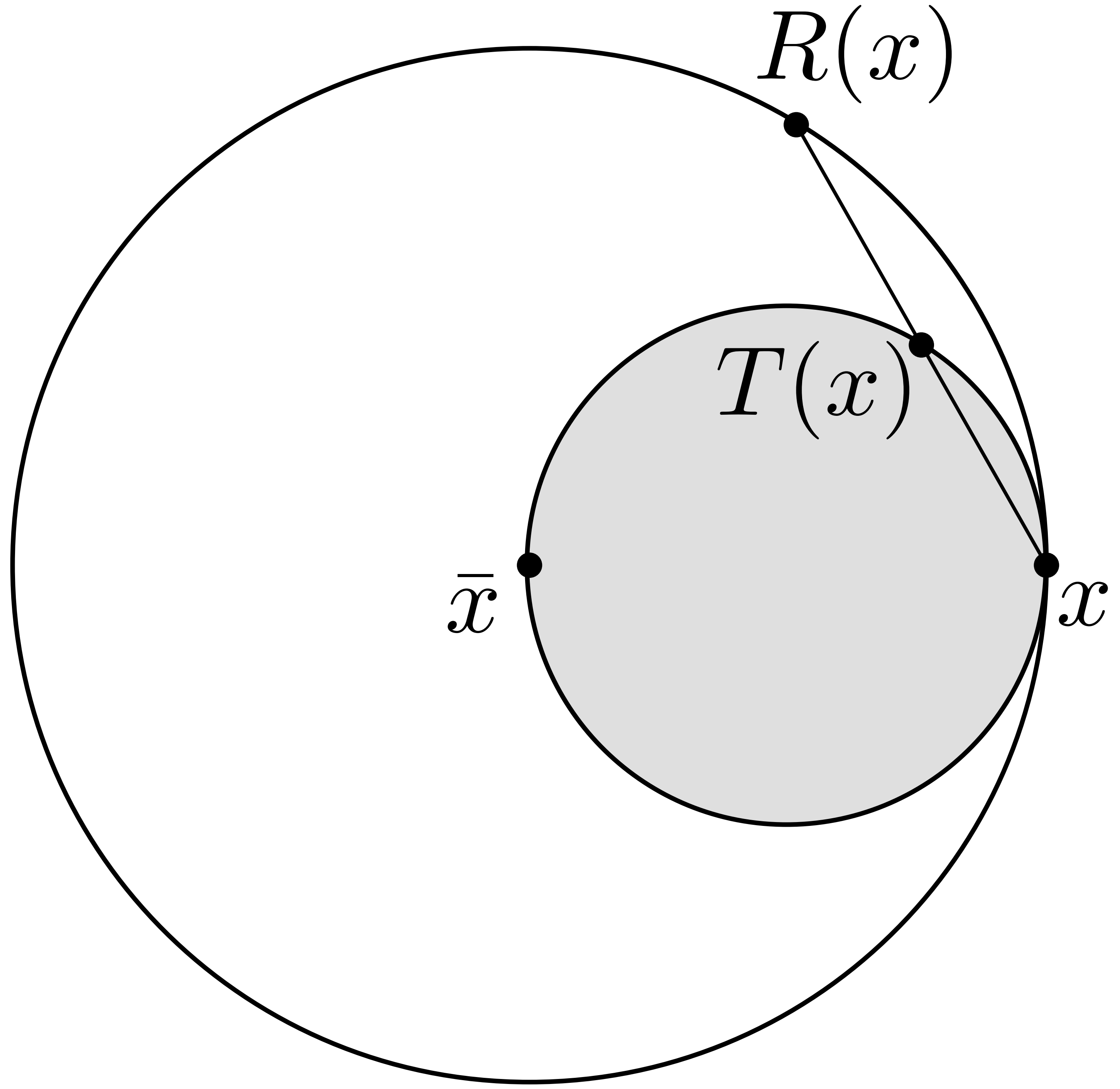


Averaged operators

We say that an operator T is $\alpha-$**averaged** with $\alpha \in (0,1)$ if

$$T = (1-\alpha)I + \alpha R$$

and R is nonexpansive.



How to design an algorithm

Problem

minimize $\quad f(x)$

Algorithm (operator) construction

1. Find a suitable T such that $\bar{x} \in \mathbf{fix}\, T$ solve your problem
2. Show that the fixed point iteration converges

If T is contractive $\implies$ **linear convergence**
If T is averaged $\implies$ **sublinear convergence**

Most first order algorithms can be constructed in this way

Today’s lecture

[Chapter 4, First-order methods in optimization, Beck]
[Proximal Algorithms, Parikh and Boyd]
[A premier on monotone operator methods, Parikh and Boyd]

Monotone operators

- Conjugate functions and duality
- Monotone and cocoercive operators
- Subdifferential operator and monotonicity
- Operators in optimization problems
- Operators in algorithms
- Building contractions

Conjugate functions and duality

Convex closed proper functions

A function $f : \mathbf{R}^n \to \mathbf{R}$ is called **CCP** if it is

closed $\quad \mathbf{epi}\, f$ is a closed set

convex $\quad f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y), \quad \alpha \in [0,1]$

proper $\quad \mathbf{dom}\, f$ is nonempty

If not otherwise stated, we assume functions to be **CCP**

Conjugate function

Given a function $f : \mathbf{R}^n \to \mathbf{R}$ we define its **conjugate** $f^* : \mathbf{R}^n \to \mathbf{R}$ as

$$f^*(y) = \max_x \; y^T x - f(x)$$

Note f^* is always convex (pointwise maximum of affine functions in y)

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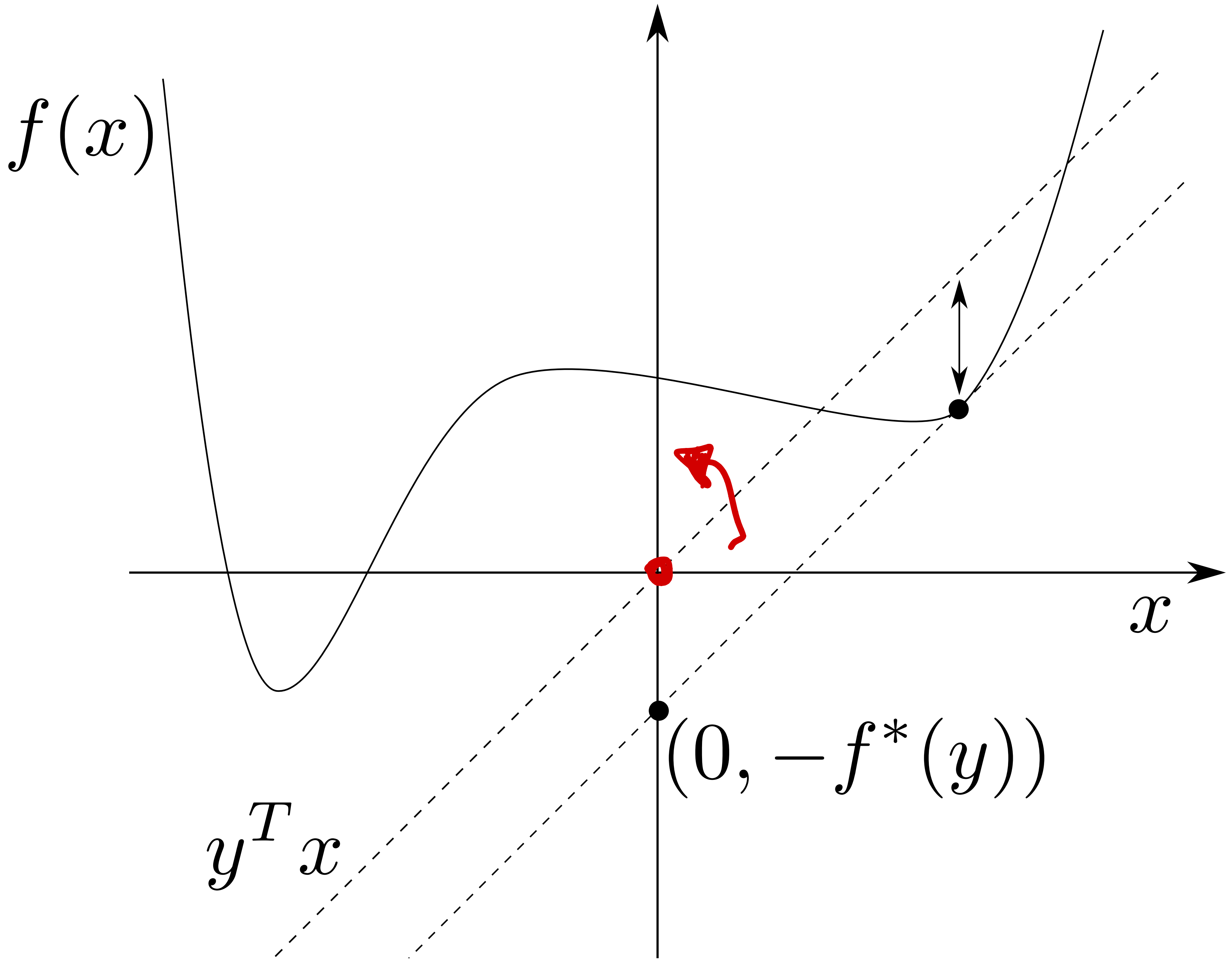

f^* is the *maximum gap* between $y^T x$ and $f(x)$

Conjugate function properties and examples

Properties

Fenchel's inequality $f(x) + f^*(y) \geq y^T x$

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Biconjugate $f^{**}(y) = \max_x \; y^T x - f^*(x) \quad \implies \quad f(x) \geq f^{**}(x)$

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Biconjugate for CCP functions If f CCP, then $f^{**} = f$

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Examples

Norm $f(x) = \|x\|$: $f^*(y) = \mathcal{I}_{\|y\|_* \leq 1}(y)$ **indicator function of dual norm set**

$1/p + 1/q = 1$

$p = 2 \quad q = 2$

$p = 1 \quad q = \infty$

Conjugate function properties and examples

Properties

Fenchel's inequality $f(x) + f^*(y) \geq y^T x$

Biconjugate $f^{**}(y) = \max_x \; y^T x - f^*(x) \quad \Longrightarrow \quad f(x) \geq f^{**}(x)$

Biconjugate for CCP functions If f CCP, then $f^{**} = f$

Examples

Norm $f(x) = \|x\|$: $f^*(y) = \mathcal{I}_{\|y\|_* \leq 1}(y)$ **indicator function of dual norm set**

Indicator function $f(x) = \mathcal{I}_C(x)$: $f^*(y) = \mathcal{I}_C^*(y) = \max_{x \in C} \; y^T x = \sigma_C(y)$ **support function**

More examples of conjugate functions [Page 101, First Order Methods in Optimization, Beck]

Fenchel dual

Dual using conjugate functions

minimize $\quad f(x) + g(x)$ $\longrightarrow$

Equivalent form (variables split)

minimize $\quad f(x) + g(z)$

subject to $\quad x = z$

Fenchel dual

Dual using conjugate functions

minimize $\quad f(x) + g(x)$ $\longrightarrow$

Equivalent form (variables split)

minimize $\quad f(x) + g(z)$

subject to $\quad x = z$

Lagrangian

$$L(x, z, y) = f(x) + g(z) + y^T(z - x) = -(y^T x - f(x)) - (-y^T z - g(z))$$

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Dual using conjugate functions

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Dual function

$$\min_{x,z} L(x, z, y) = -f^*(y) - g^*(-y)$$

Fenchel dual

Dual using conjugate functions

minimize $f(x) + g(x)$ $\longrightarrow$

Equivalent form (variables split)

$$\begin{array}{ll} \text{minimize} & f(x) + g(z) \\ \text{subject to} & x = z \end{array}$$

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Dual function

$$\min_{x,z} L(x, z, y) = -f^*(y) - g^*(-y)$$

Dual problem

$$\text{maximize} \ -f^*(y) - g^*(-y)$$

Fenchel dual example

Constrained optimization

minimize $f(x) + \mathcal{I}_C(x)$

$\longrightarrow$

Dual problem

maximize $-f^*(y) - \sigma_C(-y)$

Fenchel dual example

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Norm penalization

minimize $\quad f(x) + \|x\|$

$\longrightarrow$

Dual problem

maximize $\quad -f^*(y)$

subject to $\quad \|y\|_* \leq 1$

Fenchel dual example

Constrained optimization

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maximize $\; -f^*(y) - \sigma_C(-y)$

Norm penalization

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Dual problem

maximize $\quad -f^*(y)$

subject to $\quad \|y\|_* \leq 1$

Remarks

- Fenchel duality can simplify derivations
- Useful when conjugates are known
- Very common in operator splitting algorithms

Monotone cocoercive operators

Monotone operators

An operator T on $\mathbf{R}^n$ is **monotone** if

$$(u - v)^T (x - y) \geq 0, \quad \forall (x, u), (y, v) \in \mathbf{gph} T$$

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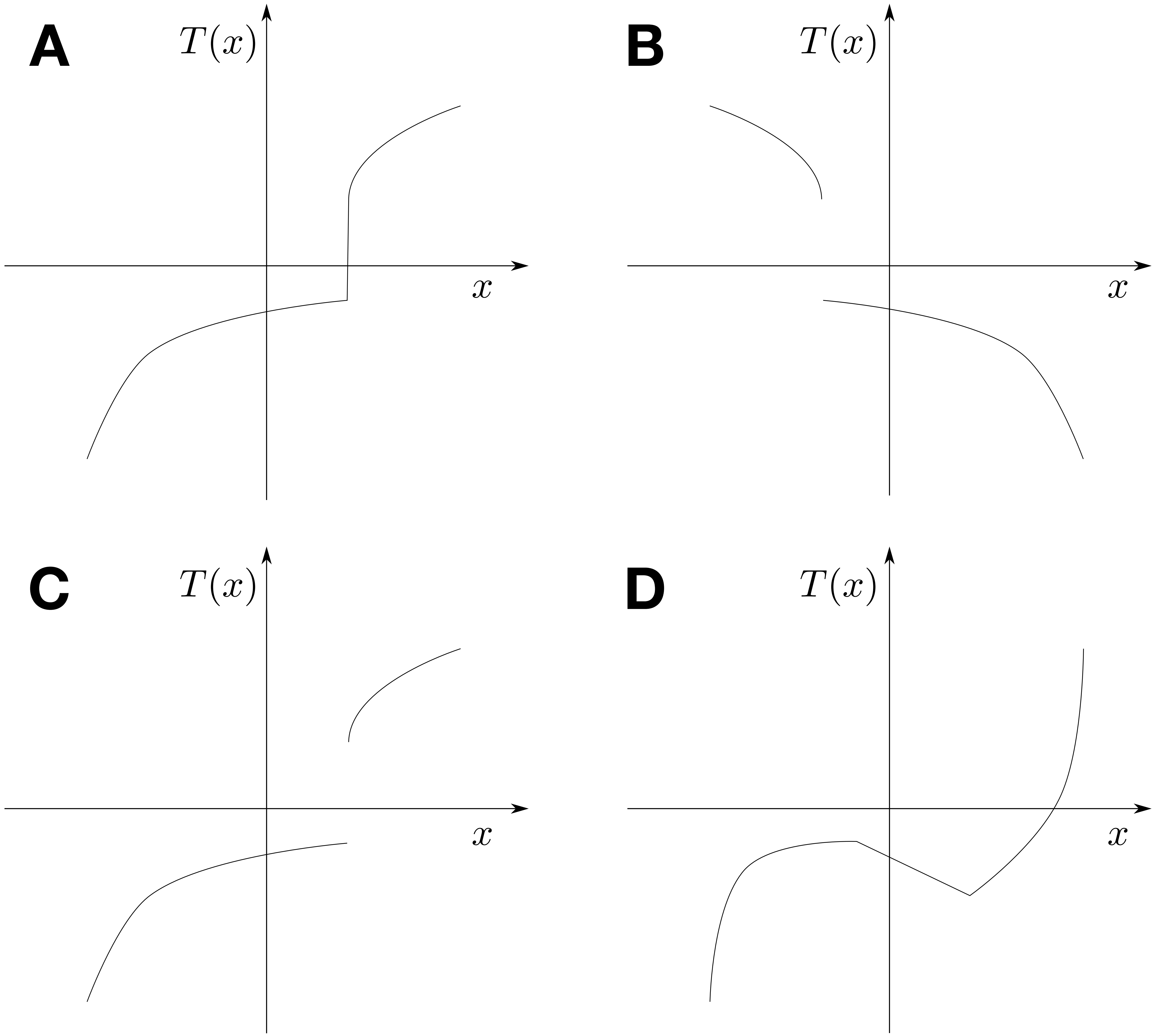
T is **maximal monotone** if
$\nexists (\bar{x},\bar{u}) \notin \mathbf{gph} T$ such that

$$(\bar{u}-u)^T(\bar{x}-x) \geq 0$$

Equivalently: $\nexists$ monotone R
such that $\mathbf{gph} T \subset \mathbf{gph} R$

Monotone operators in 1D



Let's fill the table

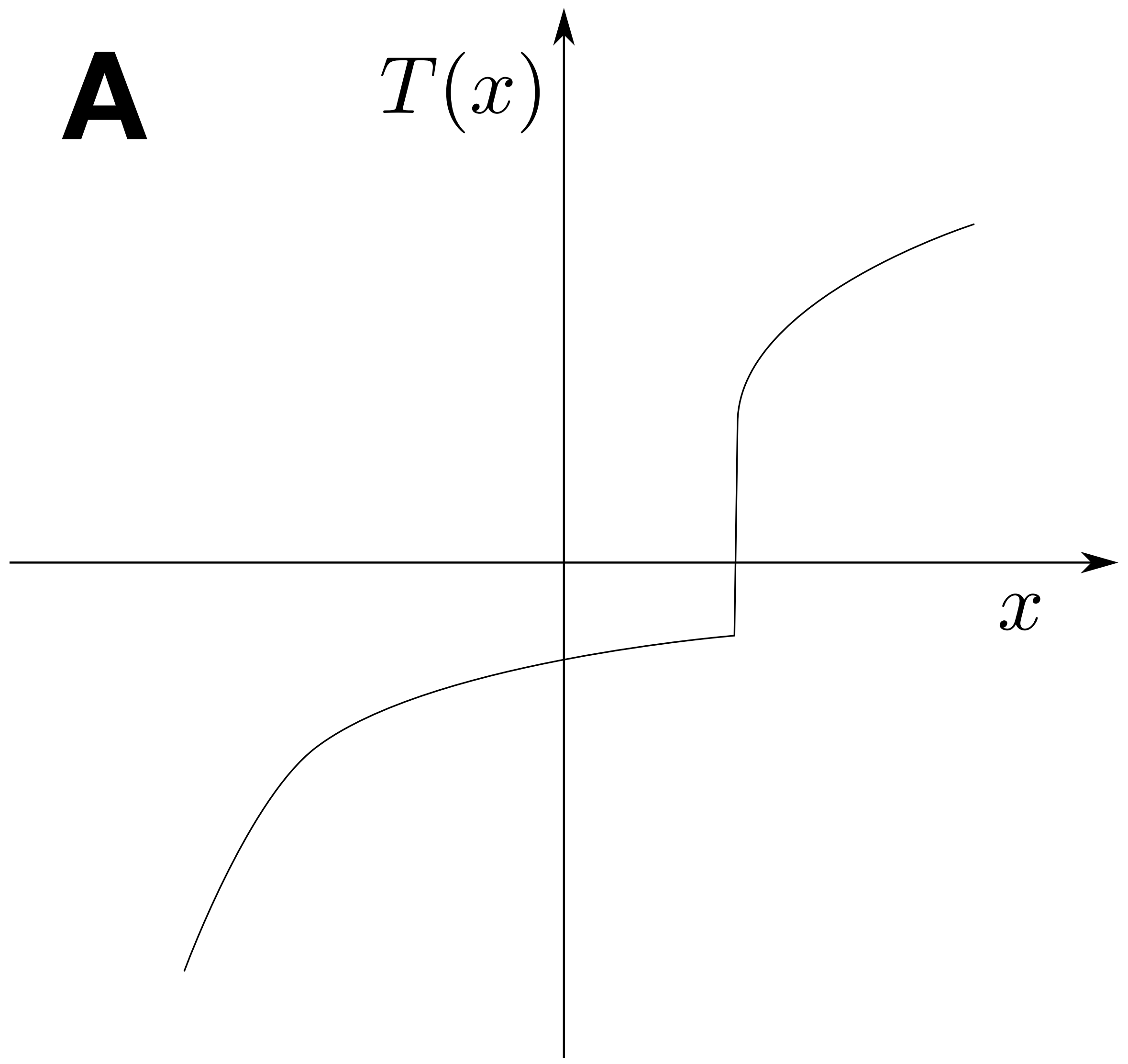
	Monotone	Max Monotone
A	✓	✓
B	✗	✗
C	✓	✗
D	✗	✗

Monotonicity

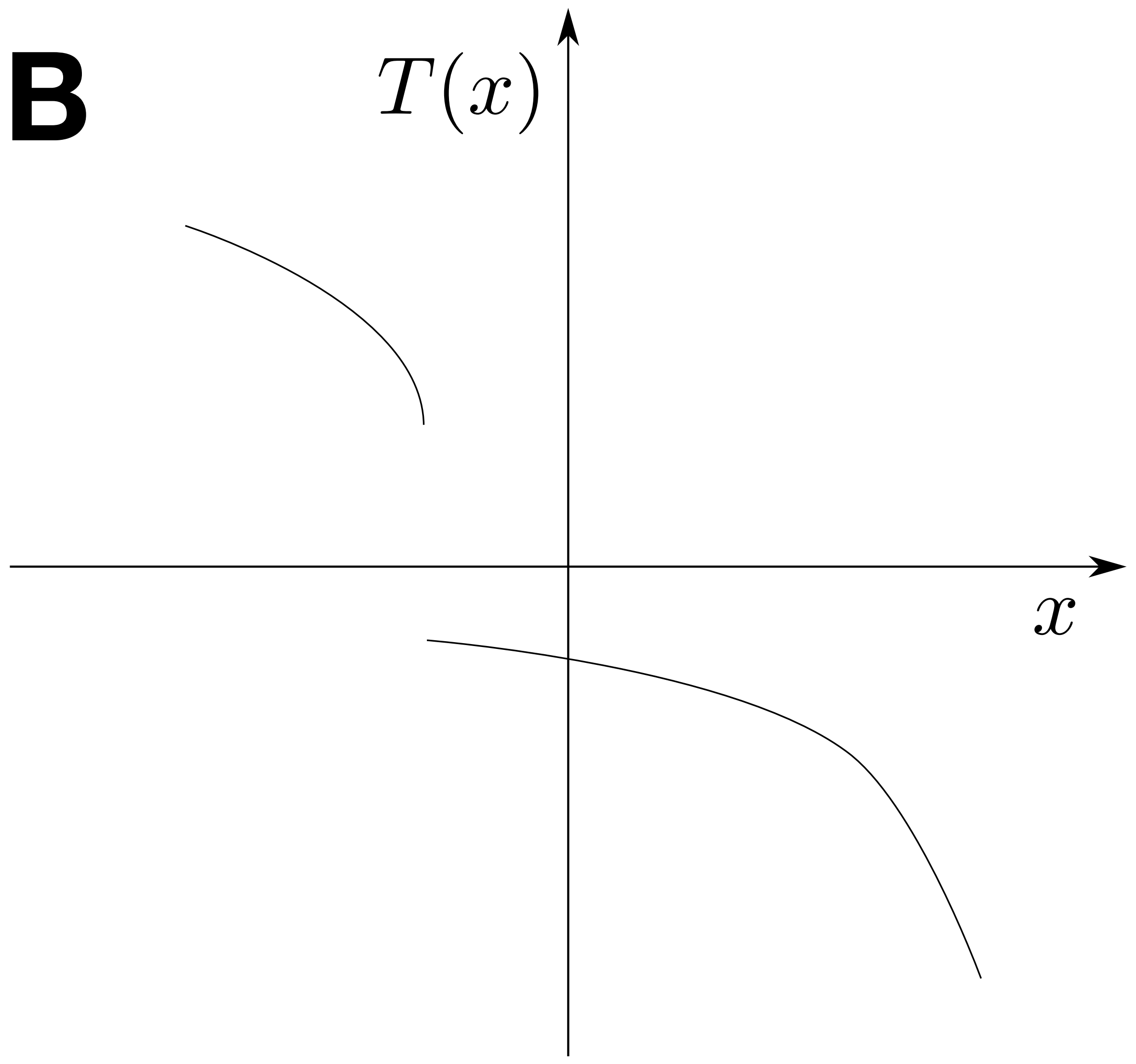
$$y > x \quad \Rightarrow \quad T(y) \geq T(x)$$

Monotone operators in 1D

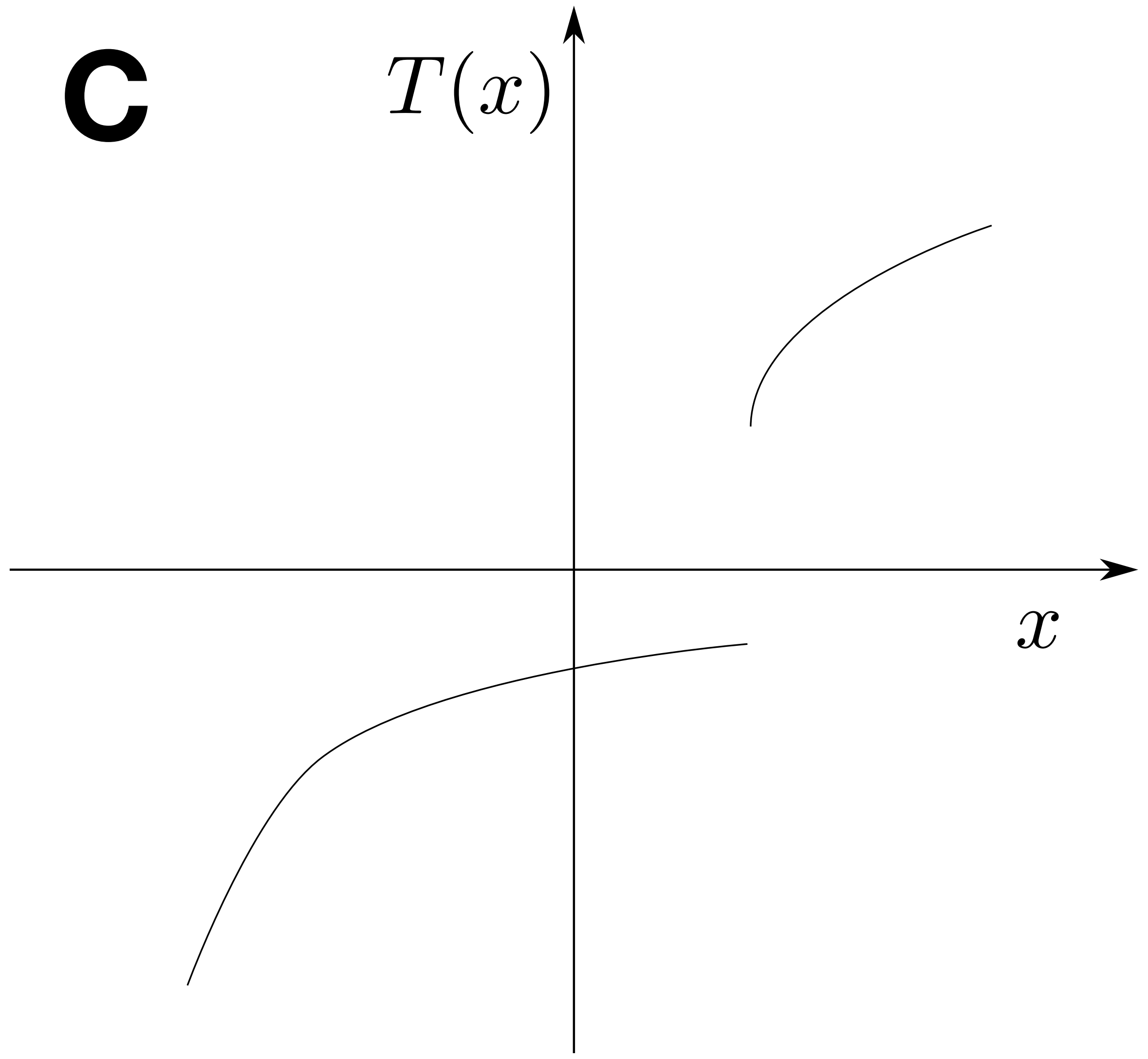
A



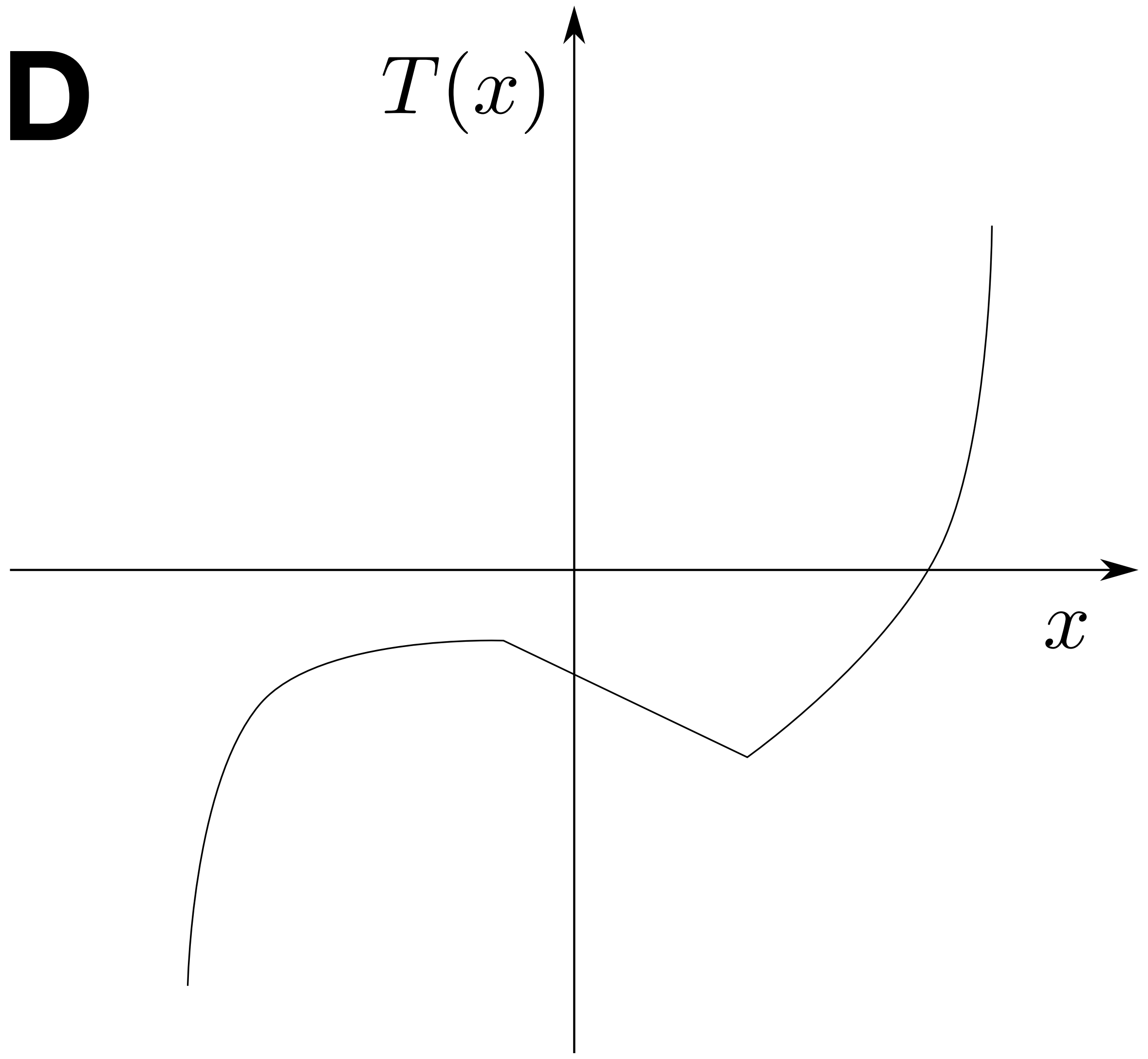
B



C



D



Let's fill the table

	Monotone	Max Monotone
A		
B		
C		
D		

Monotonicity

$$y > x \quad \Rightarrow \quad T(y) \geq T(x)$$

Continuity

If T single-valued, continuous and monotone, then it's maximal monotone

Monotone operator properties

- **sum** $T + R$ is monotone
- **nonnegative scaling** αT with $\alpha \geq 0$ is monotone
- **inverse** T^{-1} is monotone
- **congruence** for $M \in \mathbf{R}^{n \times m}$, then $M^T T(Mz)$ is monotone on $\mathbf{R}^m$

Affine function $T(x) = Ax + b$ is maximal monotone

$$\iff \quad A + A^T \succeq 0$$

$f(x) = \frac{1}{2} x^T P x + q^T x$

$\nabla f(x) = Px + q$

$P + P^T \succeq 0$

Strongly monotone operators

An operator T on $\mathbf{R}^n$ is μ-**strongly monotone** if

$$(u - v)^T(x - y) \geq \mu\|x - y\|^2, \quad \mu > 0$$

(also called μ-**coercive**)

$\forall (x, u), (y, v) \in \mathbf{gph} T$

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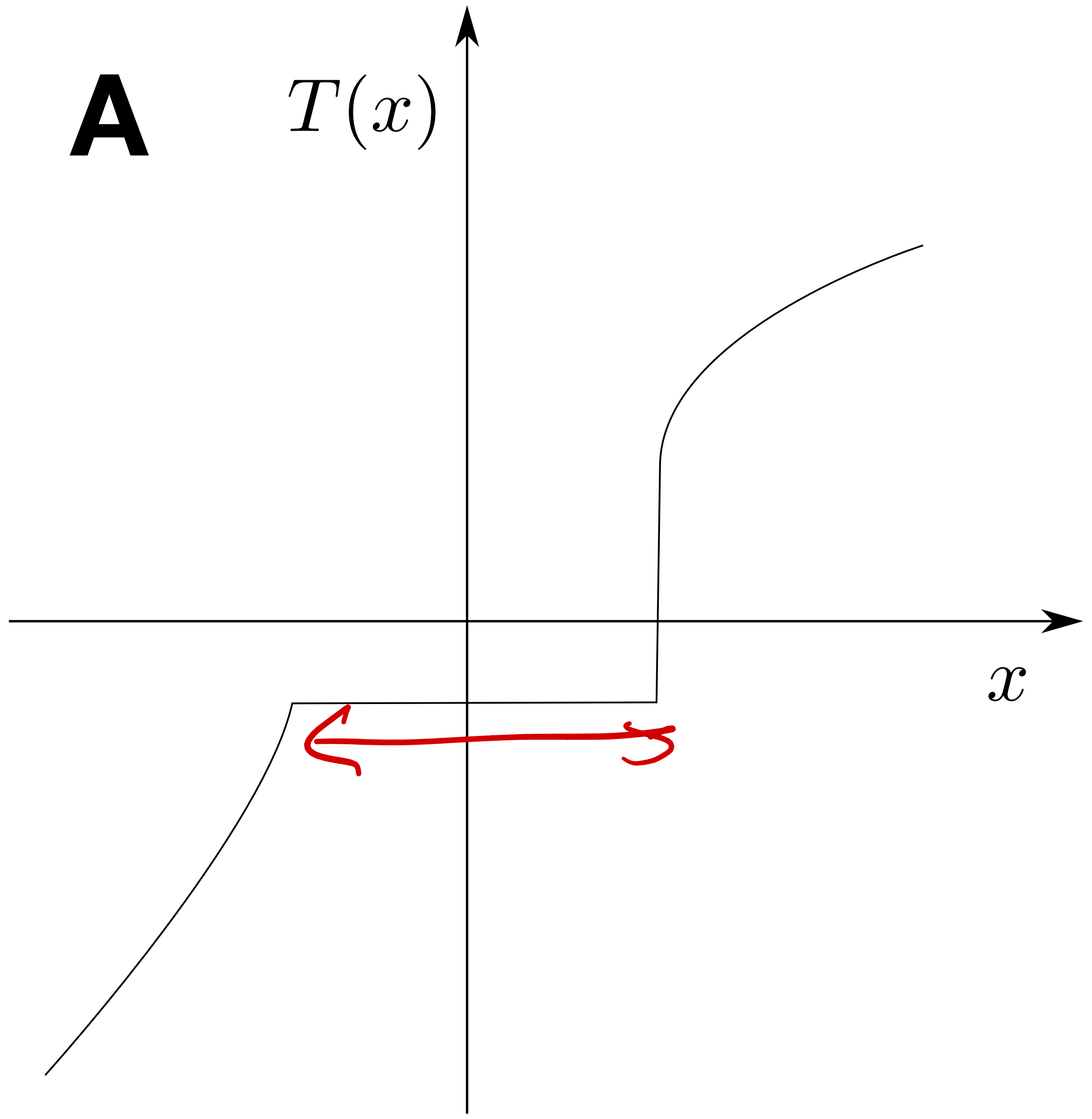
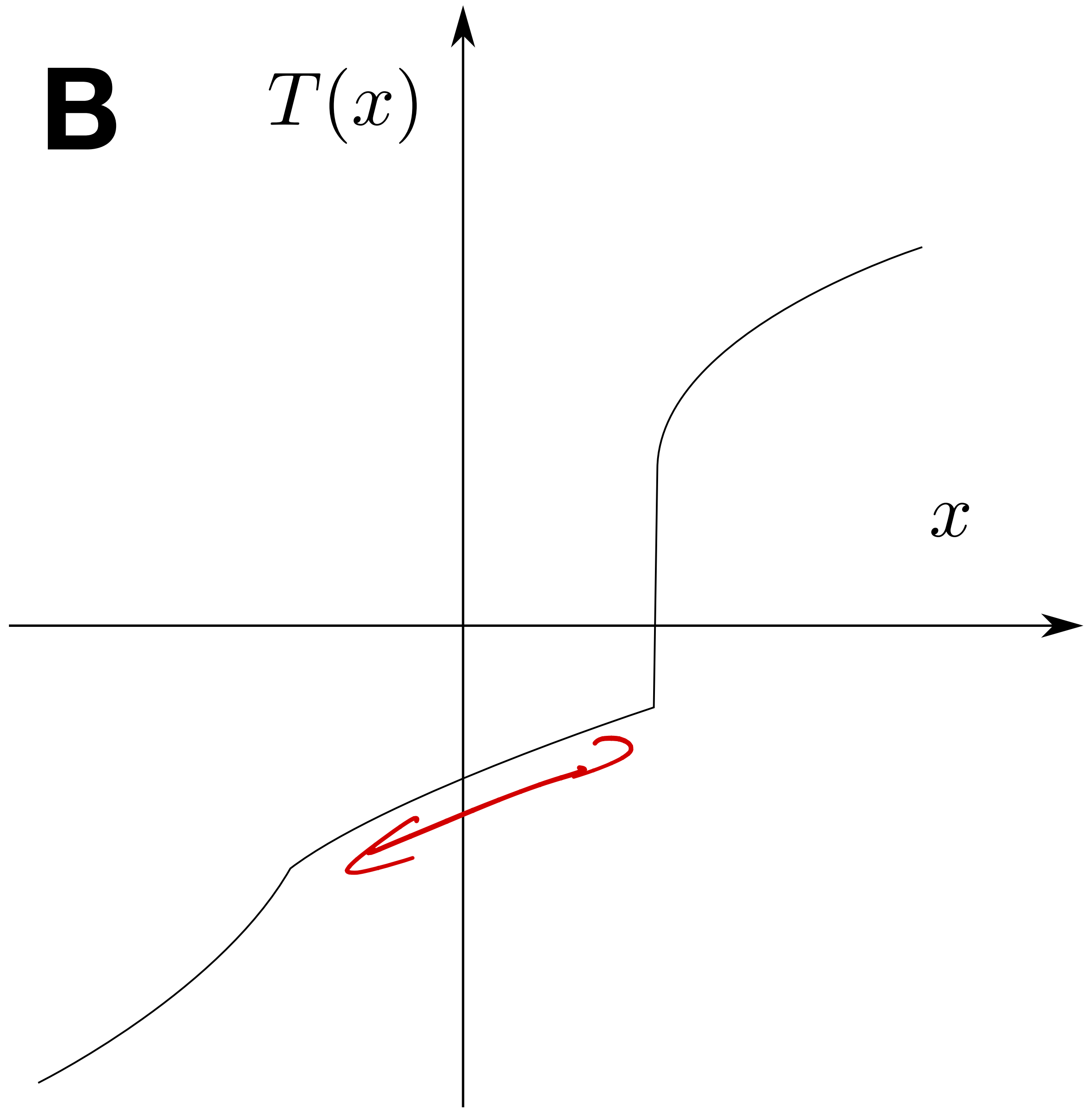


B



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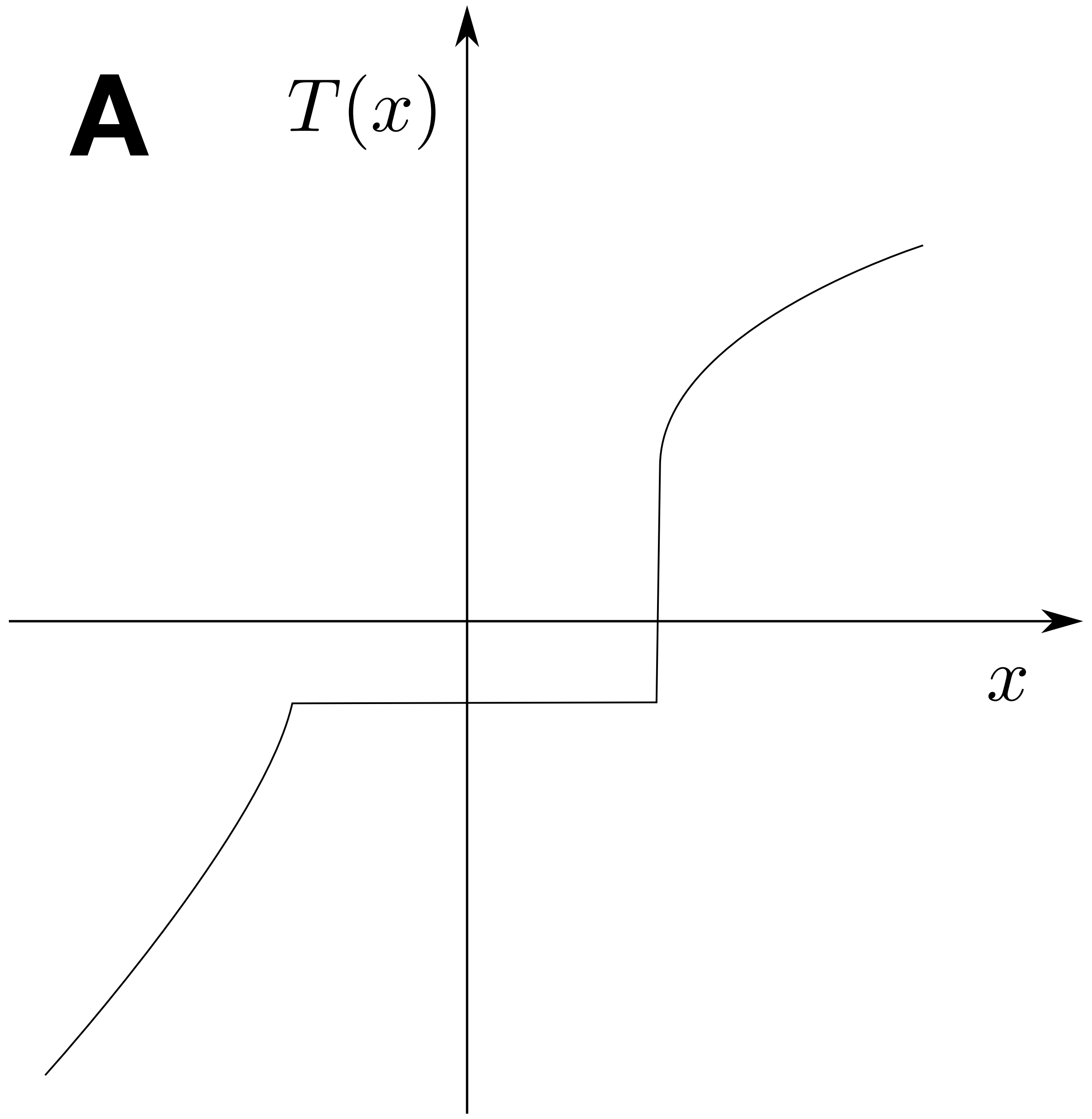
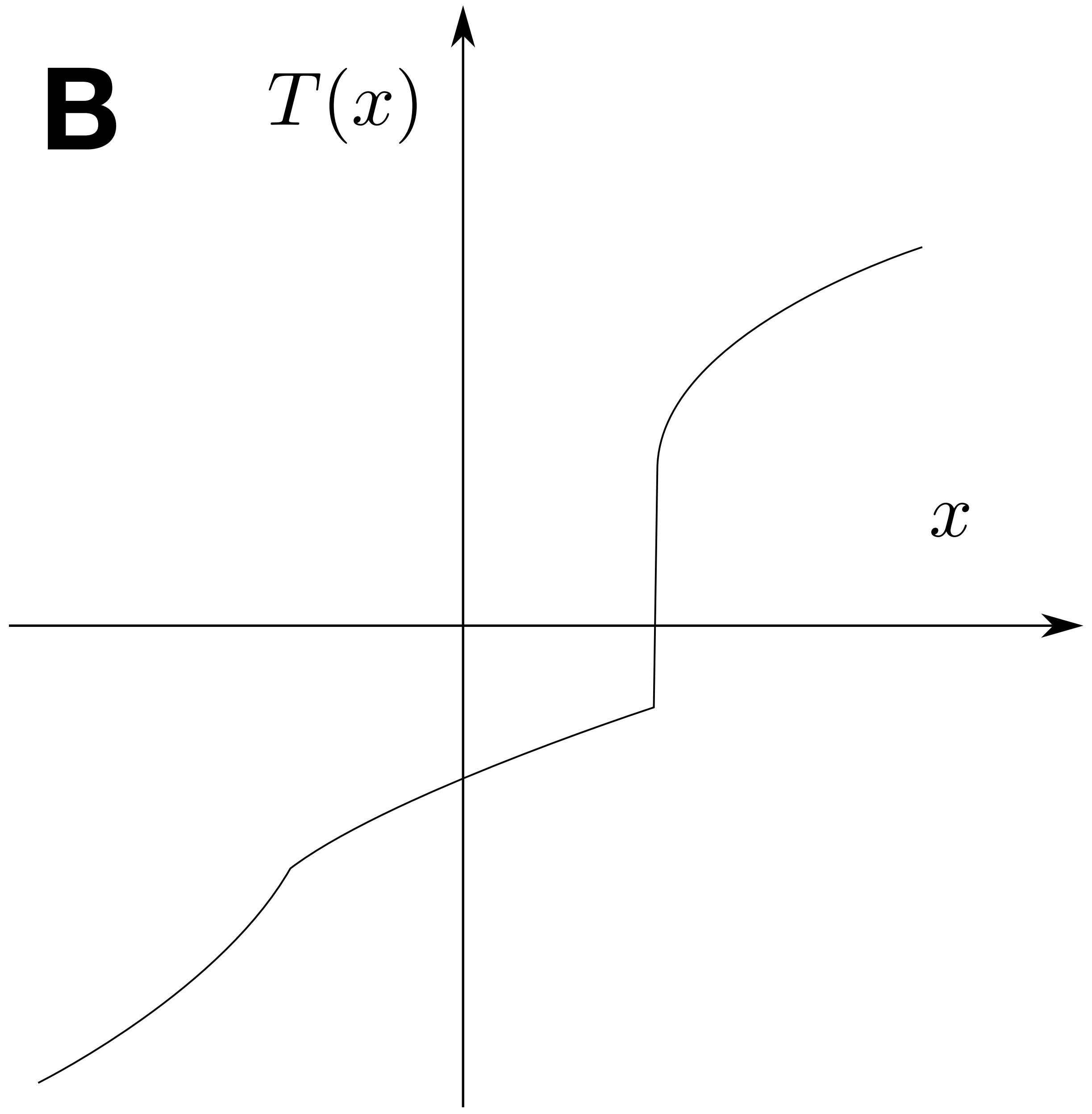


B



Let's fill the table

	Monotone	Strongly Monotone
A		
B		

The slope is at least μ

Cocoercive operators

An operator T is β**-cocoercive**, $\beta > 0$, if

$$(T(x) - T(y))^T (x - y) \geq \beta \|T(x) - T(y)\|^2$$

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If T is β**-cocoercive**, then T is $(1/\beta)$**-Lipschitz**

Proof $\beta\|T(x) - T(y)\|^2 \leq (T(x) - T(y))^T(x - y) \leq \|T(x) - T(y)\|\|x - y\|$

$$\implies \|T(x) - T(y)\| \leq (1/\beta)\|x - y\| \qquad \blacksquare$$

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If T is μ**-strongly monotone** if and only if T^{-1} is μ**-cocoercive**

Proof $(T(x) - T(x))^T(x - y) \geq \mu\|x - y\|^2$

Inverse: $u = T(x)$ and $v = T(y)$ if and only if $x \in T^{-1}(u)$ and $y \in T^{-1}(v)$

$$(u - v)^T(T^{-1}(u) - T^{-1}(v)) \geq \mu\|T^{-1}(u) - T^{-1}(v)\|^2 \qquad \blacksquare$$

Cocoercive and nonexpansive operators

If T is β-**cocoercive** if and only if $I - 2\beta T$ is **nonexpansive**

Cocoercive and nonexpansive operators

If T is β-**cocoercive** if and only if $I - 2\beta T$ is **nonexpansive**

Proof

$$\|(I{-}2\beta T)(y) - (I - 2\beta T)(x)\|^2 =$$

$$= \|y - 2\beta T(y) - x - 2\beta T(x)\|^2$$

$$= \|y - x\|^2 - 4\beta(T(y) - T(x))^T(y - x) + 4\beta^2\|T(y) - T(x)\|^2$$

$$= |y - x\|^2 - 4\beta\left((T(y) - T(x))^T(y - x) - \beta\|T(y) - T(x)\|^2\right)$$

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$$\leq \|y - x\|^2 \qquad \blacksquare$$

(cocoercive)

Summary of monotone and cocoercive operators

Monotone

$$(T(x) - T(x))^T (x - y) \geq 0$$

$\mu = 0$

Strongly monotone

$$(T(x) - T(x))^T (x - y) \geq \mu \|x - y\|^2$$

$F = T^{-1}$

Lipschitz

$$\|F(x) - F(y)\| \leq L \|x - y\|$$

$L = 1/\mu$

Cococercive

$$(F(x) - F(y))^T (x - y) \geq \mu \|F(x) - F(y)\|^2$$

$G = I - 2\mu F$

Nonexpansive

$$\|G(x) - G(y)\| \leq \|x - y\|$$

Subdifferential operator and monotonicity

Subdifferential operator monotonicity

$$\partial f(x) = \{g \mid f(y) \geq f(x) + g^T(y - x)\}$$

$\partial f(x)$ is **monotone** (also for nonconvex functions)

Subdifferential operator monotonicity

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$\partial f(x)$ is **monotone** (also for nonconvex functions)

Proof Suppose $u \in \partial f(x)$ and $v \in \partial f(y)$ then

$$f(y) \geq f(x) + u^T(y - x), \qquad f(x) \geq f(y) + v^T(x - y)$$

By adding them, we can write $(u - v)^T(x - y) \geq 0$ ■

Subdifferential operator monotonicity

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$\partial f(x)$ is **monotone** (also for nonconvex functions)

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By adding them, we can write $(u - v)^T(x - y) \geq 0$ ■

Maximal monotonicity

If f is convex, closed and proper (CCP), then $\partial f(x)$ is maximal monotone

Strongly monotone and cocoercive subdifferential

f is μ **-strongly convex** $\iff$ ∂f μ**-strongly monotone**

$$(\partial f(x) - \partial f(y))^T (x - y) \geq \mu \|x - y\|^2$$

Strongly monotone and cocoercive subdifferential

f is μ **-strongly convex** $\iff$ ∂f μ**-strongly monotone**

$$(\partial f(x) - \partial f(y))^T(x-y) \geq \mu\|x-y\|^2$$

f is L**-smooth**

$\iff$ ∂f L-**Lipschitz** and $\partial f = \nabla f$: $\|\nabla f(x) - \nabla f(y)\| \leq L\|x-y\|$

$\iff$ ∂f $(1/L)$-**cocoercive**: $(\nabla f(x) - \nabla f(y))^T(x-y) \geq (1/L)\|\nabla f(x) - \nabla f(y)\|^2$

Strongly monotone and cocoercive subdifferential

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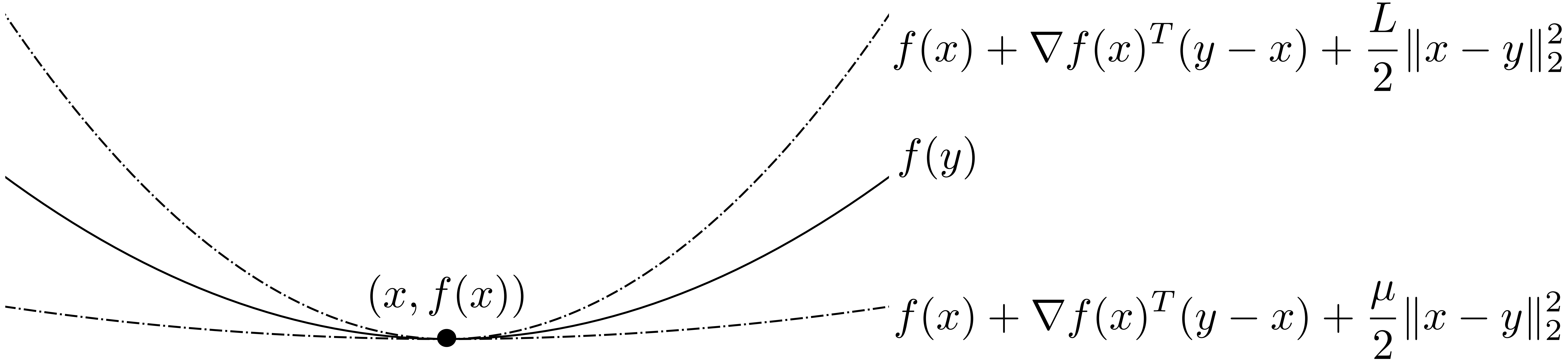

Inverse of subdifferential

If f is CCP, then, $\qquad (\partial f)^{-1} = \partial f^*$

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Proof

$$
\begin{aligned}
(u, v) \in \mathbf{gph}(\partial f)^{-1} &\iff (v, u) \in \mathbf{gph} \partial f \\
&\iff u \in \partial f(v) \\
&\iff 0 \in \partial f(v) - u \\
&\iff v \in \underset{x}{\operatorname{argmin}} \, f(x) - u^T x \\
&\iff f^*(u) = u^T v - f(v)
\end{aligned}
$$

Inverse of subdifferential

If f is CCP, then, $\quad (\partial f)^{-1} = \partial f^*$

Proof

$$
\begin{aligned}
(u, v) \in \mathbf{gph}(\partial f)^{-1} \iff & (v, u) \in \mathbf{gph}\partial f \\
\iff & u \in \partial f(v) \\
\iff & 0 \in \partial f(v) - u \\
\iff & v \in \operatorname*{argmin}_x f(x) - u^T x \\
\iff & f^*(u) = u^T v - f(v)
\end{aligned}
$$

Therefore, $f(v) + f^*(u) = u^T v$. If f is CCP, then $f^{**} = f$ and we can write

$$
f^{**}(v) + f^*(u) = u^T v \quad \iff \quad (u, v) \in \mathbf{gph}\partial f^* \qquad \blacksquare
$$

Strong convexity is the dual of smoothness

f is μ-strongly convex $\iff$ f^* is $(1/\mu)$-smooth

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f is μ-strongly convex $\quad \iff \quad f^*$ is $(1/\mu)$-(smooth)

Proof

$$
\begin{aligned}
f \quad \mu\text{-strongly convex} \quad &\iff \quad \partial f \quad \mu\text{-strongly monotone} \\
&\iff \quad (\partial f)^{-1} = \partial f^* \quad \mu\text{-cocoercive} \\
&\iff \quad f^* \quad (1/\mu)\text{-smooth} \qquad \blacksquare
\end{aligned}
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$\Longleftrightarrow \quad f^* \quad (1/\mu)$-smooth $\quad \blacksquare$

Remark: strong convexity and (strong) smoothness are **dual**

Operators in optimization problems

KKT operator

minimize $\quad f(x)$

subject to $\quad Ax = b$

→

Lagrangian

$$L(x, y) = f(x) + y^T(Ax - b)$$

KKT operator

minimize $f(x)$

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Lagrangian

$$L(x, y) = f(x) + y^T(Ax - b)$$

KKT operator

STATIONARITY

$$T(x,y) = \begin{bmatrix} \partial_x L(x,y) \\ -\partial_y L(x,y) \end{bmatrix} = \begin{bmatrix} \partial f(x) + A^T y \\ b - Ax \end{bmatrix} = \begin{bmatrix} r^{\text{dual}} \\ -r^{\text{prim}} \end{bmatrix}$$

zero set $\{(x,y) \mid 0 \in T(x,y)\}$ is the set of **primal-dual optimal points**

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Monotonicity

$$T(x,y) = \begin{bmatrix} \partial f(x) \\ b \end{bmatrix} + \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Monotonicity

$$T(x, y) = \begin{bmatrix} \partial f(x) \\ b \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

skew-symmetric

sum of monotone operators

$M + M^T = 0 \succeq 0$

“multiplier to residual” mapping

minimize $f(x)$
subject to $Ax = b$

$\longrightarrow$

Dual problem

maximize $g(y) = -(f^*(-A^T y) - y^T b)$

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Operator

$T(y) = b - Ax$, where $x = \operatorname{argmin}_z L(z, y)$

Monotonicity

If f CCP, then T is monotone

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Operator

$T(y) = b - Ax$, where $x = \operatorname{argmin}_z L(z, y)$

Monotonicity

If f CCP, then T is monotone

Proof

$0 \in \partial f(x) + A^T y \quad \Longleftrightarrow \quad x = (\partial f)^{-1}(-A^T y)$

Therefore, $F(y) = b - A(\partial f)^{-1}(-A^T y) = \partial_y \left(b^T y + f^*(-A^T y)\right) = \partial(-g)$ ■

Operators in algorithms

Forward step operator

The **forward step operator** of T is defined as

$$I - \gamma T$$

In general **monotonicity of** T is not enough for convergence

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$$I - \gamma T$$

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Example

minimize $\quad x$

subject to $\quad x = 0$

KKT operator

$$T(x, y) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Monotone (skew-symmetric)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A + A^T = 0 \succeq 0$$

Forward step operator

The **forward step operator** of T is defined as

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Forward step

$$(I - \gamma T) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -\gamma \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow$$

Expansive

$$\left\| \begin{bmatrix} 1 & -\gamma \\ \gamma & 1 \end{bmatrix} \right\|_2 > 1, \quad \forall \gamma$$

Gradient step: special case of forward step

f L-smooth $\iff$ ∇f $(1/L)$-cocoercive $\iff$ $I - (2/L)\nabla f$ nonexpansive

Gradient step: special case of forward step

f L-smooth $\iff$ ∇f $(1/L)$-cocoercive $\iff$ $I - (2/L)\nabla f$ nonexpansive

Construct averaged iterations

FORWARD STEP

$$I - \gamma \nabla f = (1-\alpha)I + \alpha(I - (2/L)\nabla f)$$

where $\alpha = \gamma L/2 \in (0,1)$ $\iff$ $\gamma \in (0, L/2)$

Gradient step: special case of forward step

f L-smooth $\iff$ ∇f $(1/L)$-cocoercive $\iff$ $I - (2/L)\nabla f$ nonexpansive

Construct averaged iterations

$$I - \gamma \nabla f = (1-\alpha)I + \alpha(I - (2/L)\nabla f)$$

where $\alpha = \gamma L/2 \in (0,1) \iff \gamma \in (0, L/2)$

Remark

- Only smoothness assumption gives **sublinear convergence**
- Similar result we obtained in gradient descent lecture

Resolvent and Cayley operators

The **resolvent** of operator A is defined as

$$R_A = (I + A)^{-1}$$

The **Cayley (reflection) operator** of A is defined as

$$C_A = 2R_A - I = 2(I + A)^{-1} - I$$

Properties

- If A is maximal monotone, $\mathbf{dom}\, R_A = \mathbf{dom}\, C_A = \mathbf{R}^n$ (Minty's theorem)
- If A is **monotone**, R_A and C_A are **nonexpansive** (thus functions)
- **Zeros** of A are **fixed points** of R_A and C_A

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- **Zeros** of A are **fixed points** of R_A and C_A

Key result we can solve $0 \in A(x)$ by finding fixed points of C_A or R_A

Fixed points of R_A and C_A are zeros of A

Proof

$$R_A = (I + A)^{-1}$$

$x \in \mathbf{fix}\, R_A$

$$
\begin{aligned}
0 \in A(x) &\iff x \in (I + A)(x) \\
&\iff (I + A)^{-1}(x) = x \\
&\iff x = R_A(x) \qquad (R_A \text{ is a function})
\end{aligned}
$$

Fixed points of R_A and C_A are zeros of A

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$x \in \mathbf{fix}\, C_A$

$$C_A(x) = 2R_A(x) - I(x) = 2x - x = x \qquad \blacksquare$$

If A is monotone, then R_A is nonexpansive

Proof

If $(x, u) \in R_A$ and $(y, v) \in R_A$, then

$$u + A(u) \ni x, \qquad v + A(v) \ni y$$

If A is monotone, then R_A is nonexpansive

Proof

If $(x, u) \in \text{gph}\, R_A$ and $(y, v) \in \text{gph}\, R_A$, then

$$u + A(u) \ni x, \qquad v + A(v) \ni y$$

Subtract to get $u - v + (A(u) - A(v)) \ni x - y$

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Subtract to get $u - v + (A(u) - A(v)) \ni x - y$

Multiply by $(u - v)^T$ and use monotonicity of A (being also a function) to get

$$\|u - v\|^2 \leq (x - y)^T (u - v)$$

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If $(x, u) \in R_A$ and $(y, v) \in R_A$, then

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Subtract to get $u - v + (A(u) - A(v)) \ni x - y$

Multiply by $(u - v)^T$ and use monotonicity of A (being also a function) to get

$$\|u - v\|^2 \leq (x - y)^T (u - v)$$

Apply Cauchy-Schwarz and divide by $\|u - v\|$ to get

$$\|u - v\| \leq \|x - y\| \qquad \blacksquare$$

If A is monotone, then C_A is nonexpansive

Proof

Given $u = R_A(x)$ and $v = R_A(y)$ (R_A is a function)

$$\begin{aligned}
\|C(x) - C(y)\|^2 &= \|(2u - x) - (2v - y)\|^2 \\
&= \|2(u - v) - (x - y)\|^2 \\
&= 4\|u - v\|^2 - 4(u - v)^T(x - y) + \|x - y\|^2 \\
&\leq \|x - y\|^2
\end{aligned}$$

Note from previous slide: $\|u - v\|^2 \leq (u - v)^T(x - y)$ ■

If A is monotone, then C_A is nonexpansive

Proof

Given $u = R_A(x)$ and $v = R_A(y)$ (R_A is a function)

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Note from previous slide: $\|u - v\|^2 \leq (u - v)^T(x - y)$ ■

Remark

R_A is nonexpansive since it is the average of I and C_A:

$$R_A = (1/2)I + (1/2)C_A = (1/2)I + (1/2)(2R_A - 1)$$

Role of maximality

We mostly consider **maximal** operators A because of

Theory: A, R_A and C_A do not bring iterates outside their domains

Practice: hard to compute R_A and C_A for non-maximal monotone operators, e.g., when $A = \partial f(x)$ where f nonconvex.

Resolvent of subdifferential: proximal operator

$$\mathbf{prox}_f = R_{\partial f} = (I + \partial f)^{-1}$$

Proof

Let $z = \mathbf{prox}_f(x)$, then

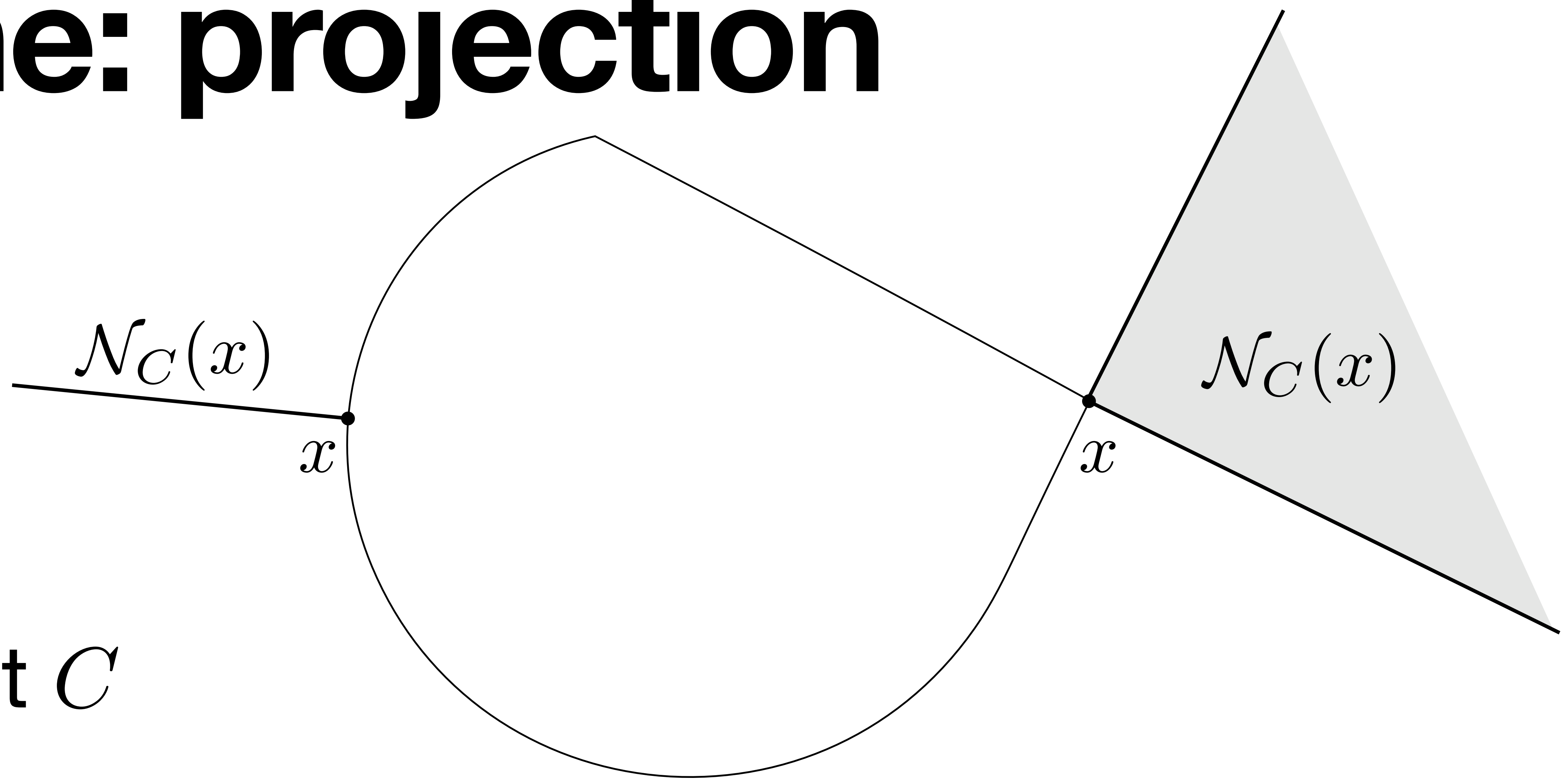
$$z = \underset{u}{\operatorname{argmin}}\, f(u) + \frac{1}{2}\|u - x\|^2$$

$$\iff \; 0 \in \partial f(z) + z - x \qquad \text{(optimality conditions)}$$

$$\iff \; x \in (I + \partial f)(z)$$

$$\iff \; z = (I + \partial f)^{-1}(x) \qquad \blacksquare$$

Resolvent of normal cone: projection



$$R_{\partial \mathcal{I}_C} = \Pi_C(x)$$

Proof

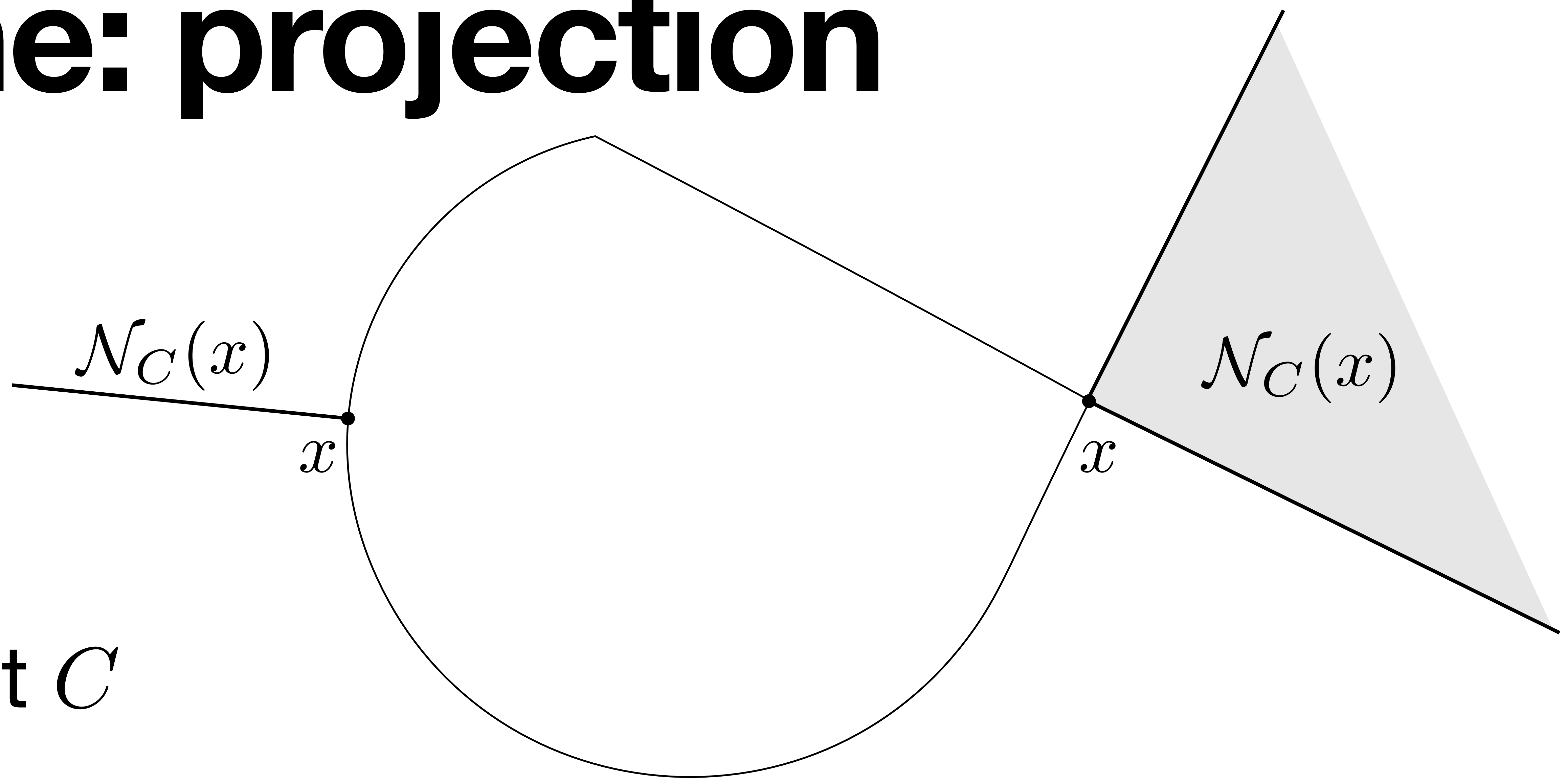
Let $f = \mathcal{I}_C$, the indicator function of a convex set C

Recall: $\partial \mathcal{I}_C(x) = \mathcal{N}_C(x)$ **normal cone operator**

$$u = (I + \partial \mathcal{I}_C)^{-1}(x) \quad \Longleftrightarrow \quad u = \underset{z}{\operatorname{argmin}} \ \mathcal{I}_C(u) + (1/2)\|z - x\|^2 = \Pi_C(x) \quad \blacksquare$$

Resolvent of normal cone: projection

$R_{\partial I_C} = \Pi_C(x)$



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Recall: $\partial \mathcal{I}_C(x) = \mathcal{N}_C(x)$ **normal cone operator**

$$u = (I + \partial \mathcal{I}_C)^{-1}(x) \quad \Longleftrightarrow \quad u = \underset{z}{\operatorname{argmin}} \ \mathcal{I}_C(u) + (1/2)\|z - x\|^2 = \Pi_C(x) \qquad \blacksquare$$

$\mathcal{N}_C$ monotone $\implies$ Π_C nonexpansive

Proof

$$u \in \mathcal{N}_C(x) \ \Rightarrow \ u^T(z - x) \le 0, \ \forall z \in C \ \Rightarrow \ u^T(y - x) \le 0$$

$$v \in \mathcal{N}_C(y) \ \Rightarrow \ v^T(z - y) \le 0, \ \forall z \in C \ \Rightarrow \ v^T(x - y) \le 0$$

⟶ add to obtain monotonicity $\blacksquare$

Building contractions

Forward step contractions

Given T L-Lipschitz and μ-strongly monotone, then $I - \gamma T$ converges linearly at rate $\sqrt{1 - 2\gamma\mu + \gamma^2 L^2}$, with optimal step $\gamma = \mu/L^2$.

Forward step contractions

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converges linearly at rate $\sqrt{1 - 2\gamma\mu + \gamma^2 L^2}$, with optimal step $\gamma = \mu / L^2$.

Proof

$$\|(I - \gamma T)(x) - (I - \gamma T)(y)\|^2 = \|x - y + \gamma T(x) - \gamma T(y)\|^2$$

$$= \|x - y\|^2 - 2\gamma (T(x) - T(y))^T (x - y) + \gamma^2 \|T(x) - T(y)\|^2$$

$$\leq (1 - 2\gamma\mu + \gamma^2 L^2) \|x - y\|^2$$

■

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\begin{aligned}
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&\leq (1 - 2\gamma\mu + \gamma^2 L^2)\|x - y\|^2
\end{aligned}
$$

strongly monotone

Lipschitz

■

Forward step contractions

Given T L-Lipschitz and μ-strongly monotone, then $\quad I - \gamma T$
converges linearly at rate $\sqrt{1 - 2\gamma\mu + \gamma^2 L^2}$, with optimal step $\gamma = \mu/L^2$.

Proof

$$\|(I-\gamma T)(x) - (I-\gamma T)(y)\|^2 = \|x - y + \gamma T(x) - \gamma T(y)\|^2$$

$$= \|x-y\|^2 - 2\gamma (T(x) - T(y))^T (x-y) + \gamma^2 \|T(x) - T(y)\|^2$$

$$\le (1 - 2\gamma\mu + \gamma^2 L^2)\|x-y\|^2$$

strongly monotone

Lipschitz

■

Remarks

- It applies to **gradient descent** with L-smooth and μ-strongly convex f
- Better rate in gradient descent lecture.
  Bound derivative: $\|D(I - \gamma \nabla^2 f(x))\|_2 \le \max\{|1-\gamma L|, |1-\gamma\mu|\}$.
  Optimal step $\gamma = 2/(\mu + L)$ and factor $(\mu/L - 1)(\mu/L + 1)$.

Resolvent contractions

If A is μ-strongly monotone, then

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is a contraction with Lipschitz parameter $1/(1+\mu)$

Resolvent contractions

If A is μ-strongly monotone, then

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Proof

$$\begin{aligned}
A\ \mu\text{-strongly monotone} \quad &\implies \quad (I + A) \quad (1+\mu)\text{-strongly monotone} \\
&\implies \quad R_A = (I + A)^{-1} \quad (1+\mu)\text{-cocoercive} \\
&\implies \quad R_A \quad (1/(1+\mu))\text{-Lipschitz} \qquad \blacksquare
\end{aligned}$$

Cayley contractions

If A is μ-strongly monotone and L-Lipschitz, then

$$C_{\gamma A} = 2R_{\gamma A} - I = 2(I + \gamma A)^{-1} - I$$

is a contraction with factor $\sqrt{1 - 4\gamma\mu/(1+\gamma L)^2}$

Proof
[Page 20, A premier on monotone operator methods, Parikh and Boyd]

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REMARK: NEED ALSO LIPSCHITZ CONDITION

Proof

[Page 20, A premier on monotone operator methods, Parikh and Boyd]

If, in addition, $A = \partial f$ where f is CCP, then $C_{\gamma A}$ converges with factor $(\sqrt{\mu/L} - 1)/(\sqrt{\mu/L} + 1)$ and optimal step $\gamma = 1/\sqrt{\mu L}$

Proof

[Linear Convergence and Metric Selection for Douglas-Rachford Splitting and ADMM, Giselsson and Boyd]

Requirements for contractions

	Operator A	**Function** f ($A = \partial f$)
Forward step $I - \gamma A$	μ-strongly monotone	μ-strongly convex L-smooth
Resolvent $R_A = (I + A)^{-1}$	μ-strongly monotone	μ-strongly convex L-smooth
Cayley $C_A = 2(I + A)^{-1} - I$	μ-strongly monotone L-Lipschitz	μ-strongly convex L-smooth

faster convergence

Key to contractions: strong monotonicity/convexity

Operator theory

Today, we learned to:

- **Use** conjugate functions to define duality
- **Define** monotone and cocoercive operators and their relations
- **Relate** subdifferential operator and monotonicity
- **Recognize** monotone operators in optimization problems
- **Apply** operators in algorithms: forward step, resolvent, Cayley
- **Understand requirements** for building contractions

Next lecture

- Operator splitting algorithms