ORF522 – Linear and Nonlinear Optimization

16. Proximal methods and introduction to operator theory

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Ed forum

- If the **local minima is not unique**, in that there's a set of x* with multiple values that yield the same objective value, would the subgradient method still work? (One example might be LP when there are infinitely many solutions)
- It is mentioned that in subgradient methods, algorithms just find one subgradient g and proceeds with the iteration. But if it has already hit the minimum, **0 \in \partial f**. Can the algorithm **just pick out this 0** and terminate itself?
- In practice, are we ever able to **derive, or approximate, the Lipschitz constant?** Or, is it even necessary to do so? How well do these methods translate over, if they do at all, to non Lipschitz functions? Or do we just make this assumption in order to make the analysis cleaner?
  [Efficient and Accurate Estimation of Lipschitz Constants for Deep Neural Networks, https://arxiv.org/pdf/1906.04893.pdf]
- If f^star is known, we can apply the **Polyak step size** to find the optimal solution. Does it work if I **only have an estimation** of the optimal value? Can it still lead to the right convergence point?
- At each iteration of the subgradient method, we don't need the whole subdifferential. Instead we only need one subgradient. This is kind of reminiscent to picking the entering index for Simplex method. Picking diminishing step size is like avoiding cycling for Simplex. Is there an **analogous rule for picking which subgradient to use**, if I have access to more than one?

Recap

Subgradient method

Convex optimization problem

minimize $f(x)$ (optimal cost $f^\star$)

Iterations

$$x^{k+1} = x^k - t_k g^k, \qquad g^k \in \partial f(x^k)$$

g^k is **any subgradient** of f at x^k

Not a descent method, keep track of the best point

$$f^k_{\text{best}} = \min_{i=1,\ldots,k} f(x^i)$$

Step sizes

Line search can lead to **suboptimal points**

Step sizes ***pre-specified***, not adaptively computed
(different than gradient descent)

Fixed: $t_k = t$ for $k = 0, \ldots$

Diminishing: $\sum_{k=0}^{\infty} t_k^2 < \infty, \quad \sum_{k=0}^{\infty} t_k = \infty$

Square summable but not summable
(goes to 0 but not too fast)
e.g., $t_k = O(1/k)$

Implications for step size rules

$$f_{\text{best}}^k - f^\star \leq \frac{R^2 + G^2 \sum_{i=0}^k t_i^2}{2 \sum_{i=0}^k t_i}$$

Fixed: $t_k = t$ for $k = 0, \ldots$

$$f_{\text{best}}^k - f^\star \leq \frac{R^2 + G^2 (k+1) t^2}{2(k+1)t}$$

May be suboptimal

$$\lim_{k \to \infty} f_{\text{best}}^k \leq f^\star + \frac{G^2 t}{2}$$

Diminishing: $\sum_{k=0}^{\infty} t_k^2 < \infty, \quad \sum_{k=0}^{\infty} t_k = \infty$

e.g., $t_k = \tau/(k+1)$ or $t_k = \tau/\sqrt{k+1}$

Optimal

$$\lim_{k \to \infty} f_{\text{best}}^k = f^\star$$

Summary subgradient method

- Simple
- Handles general nondifferentiable convex functions
- Very slow convergence $O(1/\epsilon^2)$
- No good stopping criterion

Can we do better?

Can we incorporate constraints?

Today's lecture

[Chapter 3 and 6, First-order methods in optimization, Beck]
[Proximal Algorithms, Parikh and Boyd]
[A premier on monotone operator methods, Parikh and Boyd]

Proximal methods and introduction to operators

- Optimality conditions with subdifferentials
- Proximal operators
- Proximal gradient method
- Operator theory
- Fixed point iterations

Optimality conditions with subdifferentials

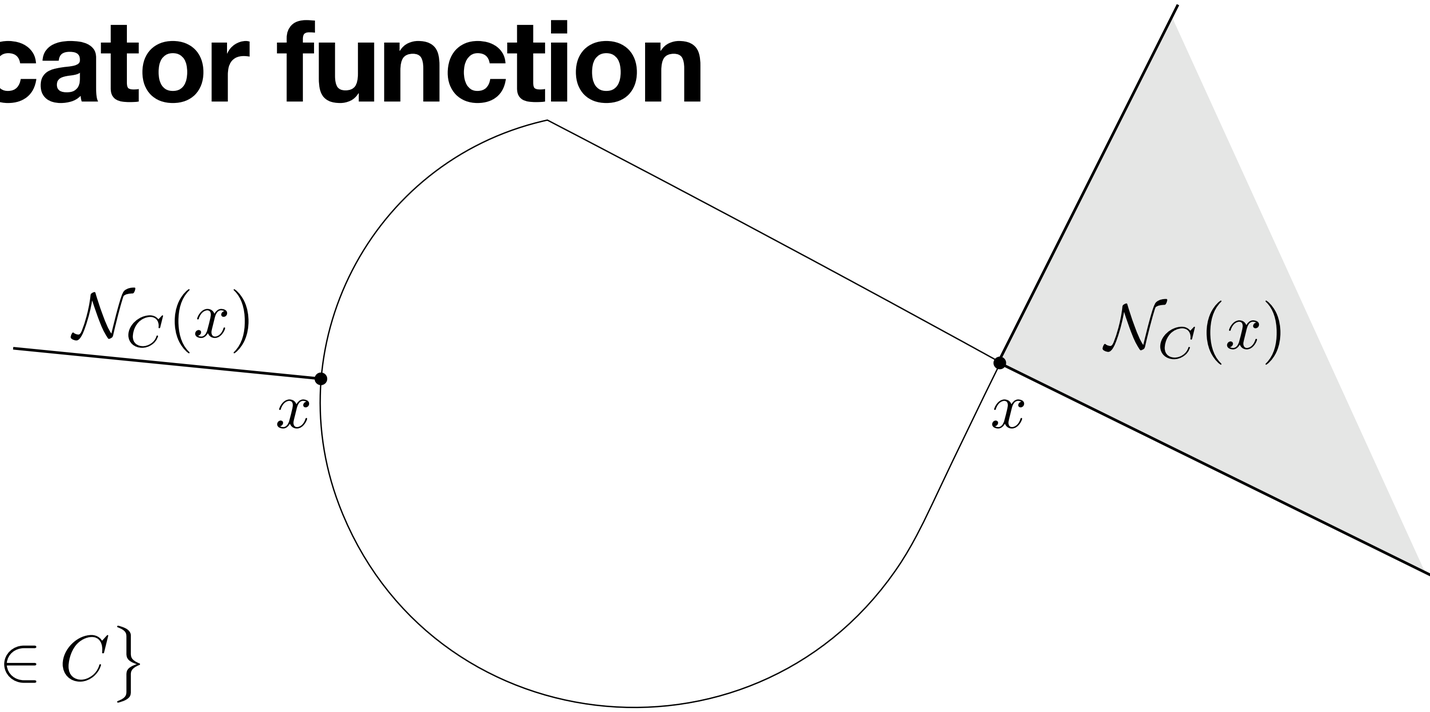
Subgradient of indicator function

The subdifferential of the **indicator functon** is the **normal cone**

$$\partial \mathcal{I}_C(x) = \mathcal{N}_C(x)$$

where,

$$\mathcal{N}_C(x) = \{g \mid g^T(y-x) \leq 0, \quad \text{for all } y \in C\}$$



Proof

By definition of subgradient g, $\mathcal{I}_C(y) \geq \mathcal{I}_C(x) + g^T(y-x), \quad \forall y$

$$y \notin C \quad \Longrightarrow \quad \mathcal{I}_C(y) = \infty$$

$$y \in C \quad \Longrightarrow \quad 0 \leq g^T(y-x)$$

■

Constrained optimization

Indicator function
of a convex set

$$\mathcal{I}_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

Constrained form

minimize $f(x)$

subject to $x \in C$

$\longleftrightarrow$

Unconstrained form

minimize $f(x) + \mathcal{I}_C(x)$

First-order optimality conditions from subdifferentials

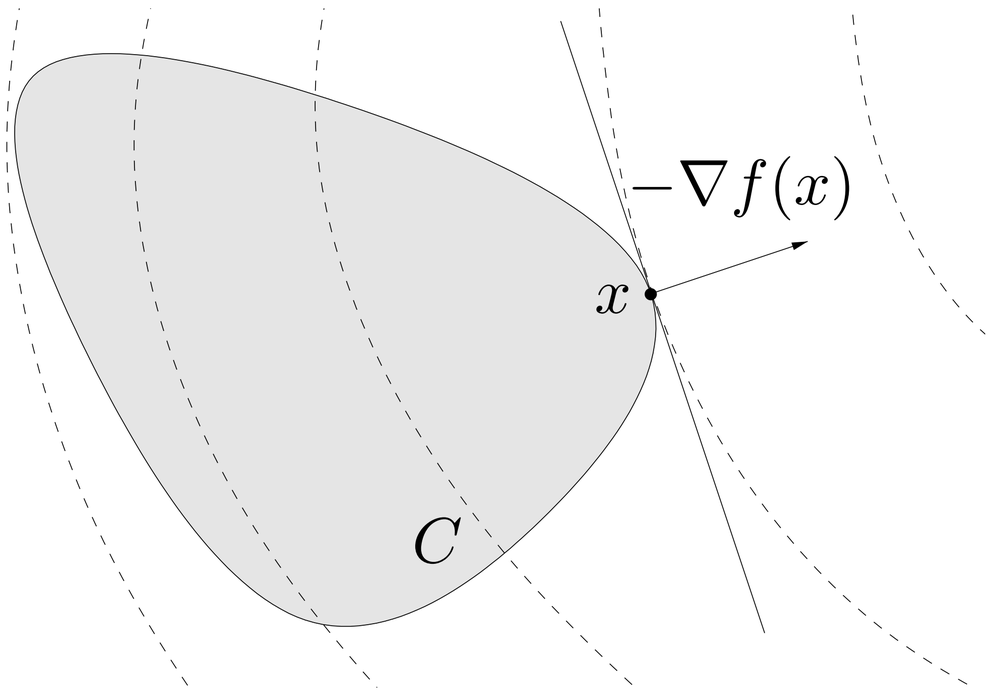
minimize $\quad f(x) + \mathcal{I}_C(x)$ $\qquad$ (f smooth, C convex)

Fermat's optimality condition

$$0 \in \partial(f(x) + \mathcal{I}_C(x))$$

$$\iff \; 0 \in \{\nabla f(x)\} + \mathcal{N}_C(x)$$

$$\iff \; -\nabla f(x) \in \mathcal{N}_C(x)$$



Equivalent to

$$\nabla f(x)^T (y - x) \geq 0, \quad \forall y \in C$$

Example: KKT of a quadratic program

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x \\ \text{subject to} & Ax \leq b \end{array} \longrightarrow \text{minimize} \quad (1/2)x^T P x + q^T x + \mathcal{I}_{\{Ax \leq b\}}(x)$$

Gradient

$$\nabla f(x) = Px + q$$

Normal cone to polyhedron

$$\mathcal{N}_{\{Ax \leq b\}}(x) = \{A^T y \mid y \geq 0 \quad \text{and} \quad y_i(a_i^T x - b_i) = 0\}$$

First-order optimality condition

$$-\nabla f(x) \in \partial \mathcal{I}_{\{Ax \leq b\}}(x) = \mathcal{N}_{\{Ax \leq b\}}(x)$$

$$\longleftrightarrow$$

KKT Optimality conditions

$$\begin{aligned} Px + q + A^T y &= 0 \\ y &\geq 0 \\ Ax - b &\leq 0 \\ y_i(a_i^T x - b_i) &= 0, \quad i = 1, \ldots, m \end{aligned}$$

Proximal operators

Composite models

$$\text{minimize} \quad f(x) + g(x)$$

$f(x)$ convex and smooth
$g(x)$ convex (may be not differentiable)

Examples

- Regularized regression: $g(x) = \|x\|_1$
- Constrained optimization: $g(x) = \mathcal{I}_C(x)$

Proximal operator

Definition

The **proximal operator** of the function $g : \mathbf{R}^n \to \mathbf{R}$ is

$$\mathbf{prox}_g(x) = \underset{z}{\operatorname{argmin}} \left(g(z) + \frac{1}{2}\|z - x\|_2^2 \right)$$

Optimality conditions of prox

$$0 \in \partial g(z) + z - x \quad \Longrightarrow \quad x - z \in \partial g(z)$$

Properties

- It involves solving an optimization problem (not always easy!)
- Easy to evaluate for many standard functions, i.e. **proxable functions**
- Generalizes many well-known algorithms

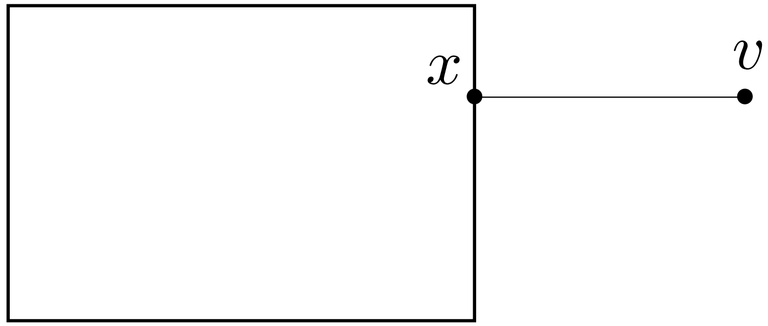
Generalized projection

The $\mathbf{prox}$ operator of the indicator function $\mathcal{I}_C$ is the projection onto C

$$\mathbf{prox}_{\mathcal{I}_C}(v) = \underset{x \in C}{\operatorname{argmin}} \|x - v\|_2 = \Pi_C(v)$$

Example projection onto a box $C = \{x \mid l \leq x \leq u\}$

$$\Pi_C(v)_i = \begin{cases} l_i & v_i \leq l_i \\ v_i & l_i \leq v_i \leq u_i \\ u_i & v_i \geq u_i \end{cases}$$



Remarks

- Easy for many common sets (e.g., closed form)
- Can be hard for surprisingly simple lets, e.g., $C = \{Ax \leq b\}$

Projections at [p. 156, First order methods in optimization, A Beck]

Quadratic functions

If $g(x) = (1/2)x^T P x + q^T x + r$ with $P \succeq 0$, then

$$\mathbf{prox}_g(v) = (I + P)^{-1}(v - q)$$

Remarks

- Closed-form always solvable (even with P not full rank)
- Symmetric, positive definite and usually sparse linear system
- Can prefactor $I + P$ and solve for different v

Separable sum

If $g(x)$ is block separable, i.e., $$g(x) = \sum_{i=1}^{N} g_i(x_i)$$

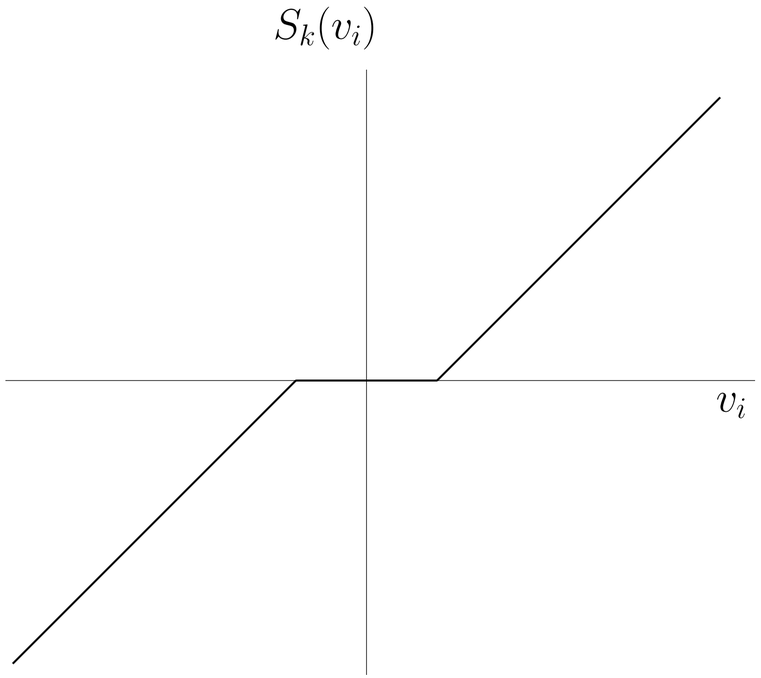
then, $$(\mathbf{prox}_g(v))_i = \mathbf{prox}_{g_i}(v_i), \quad i = 1, \ldots, N$$

(key to parallel/distributed proximal algorithms)

Example: $g(x) = \lambda \|x\|_1 = \sum_{i=1}^{n} \lambda |x_i|$

soft-thresholding

$$(\mathbf{prox}_g(v))_i = \mathbf{prox}_{\lambda|\cdot|}(v_i) = S_\lambda(v_i) = \begin{cases} v_i - \lambda & v_i > \lambda \\ 0 & |v_i| \leq \lambda \\ v_i + \lambda & v_i < -\lambda \end{cases}$$



Basic rules

Examples

- **Scaling and translation:** $g(x) = ah(x) + b$ with $a > 0$, then

$$\mathbf{prox}_g(x) = \mathbf{prox}_{ah}(x)$$

- **Affine addition:** $g(x) = h(x) + a^T x + b$, then

$$\mathbf{prox}_g(x) = \mathbf{prox}_h(x - a)$$

- **Affine transformation:** $g(x) = h(ax + b)$, with $a \neq 0, a \in \mathbf{R}$,

$$\mathbf{prox}_g(x) = \frac{1}{a}\left(\mathbf{prox}_{a^2 h}(ax + b) - b\right)$$

Proofs (exercise):

- Rearrange proximal term: $(1/2)\|z - x\|_2^2$
- Apply $\mathbf{prox}$ optimality conditions

Many more examples at [p. 156, First order methods in optimization, A Beck]

Proximal gradient method

Gradient descent interpretation

Problem

$$\text{minimize} \quad f(x)$$

Iterations

$$x^{k+1} = x^k - t\nabla f(x^k)$$

Quadratic approximation, replacing Hessian $\nabla^2 f(x^k)$ with $\frac{1}{t}I$

$$x^{k+1} = \underset{z}{\text{argmin}}\ f(x^k) + \nabla f(x^k)^T(z - x^k) + \frac{1}{2t}\|z - x^k\|_2^2$$

Let's exploit the smooth part

minimize $\quad f(x) + g(x)$

$f(x)$ convex and smooth
$g(x)$ convex (may be not differentiable)

Quadratic approximation of f while keeping g

$$x^{k+1} = \underset{z}{\text{argmin}}\ g(z) + f(x^k) + \nabla f(x^k)^T (z - x^k) + \frac{1}{2t}\|z - x^k\|_2^2 \longleftarrow$$

same as gradient descent

Equivalent to

Proximal operator

$$x^{k+1} = \underset{z}{\text{argmin}}\ tg(z) + \frac{1}{2}\left\|z - (x^k - t\nabla f(x^k))\right\|_2^2 = \mathbf{prox}_{tg}\left(x^k - t\nabla f(x^k)\right)$$

$\uparrow$ make g small

$\uparrow$ stay close to gradient update

Proximal gradient method

minimize $\quad f(x) + g(x)$

$f(x)$ convex and smooth
$g(x)$ convex (may be not differentiable)

Iterations

$$x^{k+1} = \mathbf{prox}_{tg}\left(x^k - t\nabla f(x^k)\right)$$

Properties

- Alternates between gradient updates of f and proximal updates on g
- Useful if $\mathbf{prox}_{tg}$ is inespensive
- Can handle nonsmooth and constrained problems

Special cases

Generalized gradient descent

Problem

minimize $\quad f(x) + g(x)$

Iterations

$$x^{k+1} = \mathbf{prox}_{tg}\left(x^k - t\nabla f(x^k)\right)$$

Smooth

$$g(x) = 0 \quad \Longrightarrow \quad \mathbf{prox}_{tg}(x) = x$$

Gradient descent

$$\Longrightarrow \quad x^{k+1} = x^k - t\nabla f(x^k)$$

Constraints

$$g(x) = \mathcal{I}_C(x) \quad \Longrightarrow \quad \mathbf{prox}_{tg}(x) = \Pi_C(x)$$

Projected gradient descent

$$\Longrightarrow \quad x^{k+1} = \Pi_C(x^k - t\nabla f(x^k))$$

Non smooth

$$f(x) = 0$$

Proximal minimization

$$\Longrightarrow \quad x^{k+1} = \mathbf{prox}_{tg}(x^k)$$

Note: useful if $\mathbf{prox}_{tg}$ is cheap

What happens if we cannot evaluate the prox?

At every iteration, it can be very expensive to evaluate

$$\mathbf{prox}_g(x) = \underset{z}{\operatorname{argmin}} \left(g(z) + \frac{1}{2}\|z - x\|_2^2 \right)$$

Idea: solve it approximately!

If you precisely control the $\mathbf{prox}_g(x)$ evaluation errors
you can obtain the same convergence guarantees (and rates)
as the exact evaluations.

[Schmidt et al. (2011), “Convergence rates of inexact proximal-gradient methods for convex optimization”]

Example: Lasso

Iterative Soft Thresholding Algorithm (ISTA)

$$\text{minimize} \quad \underbrace{(1/2)\|Ax-b\|_2^2}_{f(x)} + \underbrace{\lambda\|x\|_1}_{g(x)}$$

Proximal gradient descent

$$x^{k+1} = \mathbf{prox}_{tg}\left(x^k - t\nabla f(x^k)\right)$$

$$\nabla f(x) = A^T(Ax-b)$$

$$\mathbf{prox}_{tg}(x) = S_{\lambda t}(x)$$

(component wise soft-thresholding)

Closed-form iterations

$$x^{k+1} = S_{\lambda t}\left(x^k - tA^T(Ax^k - b)\right)$$

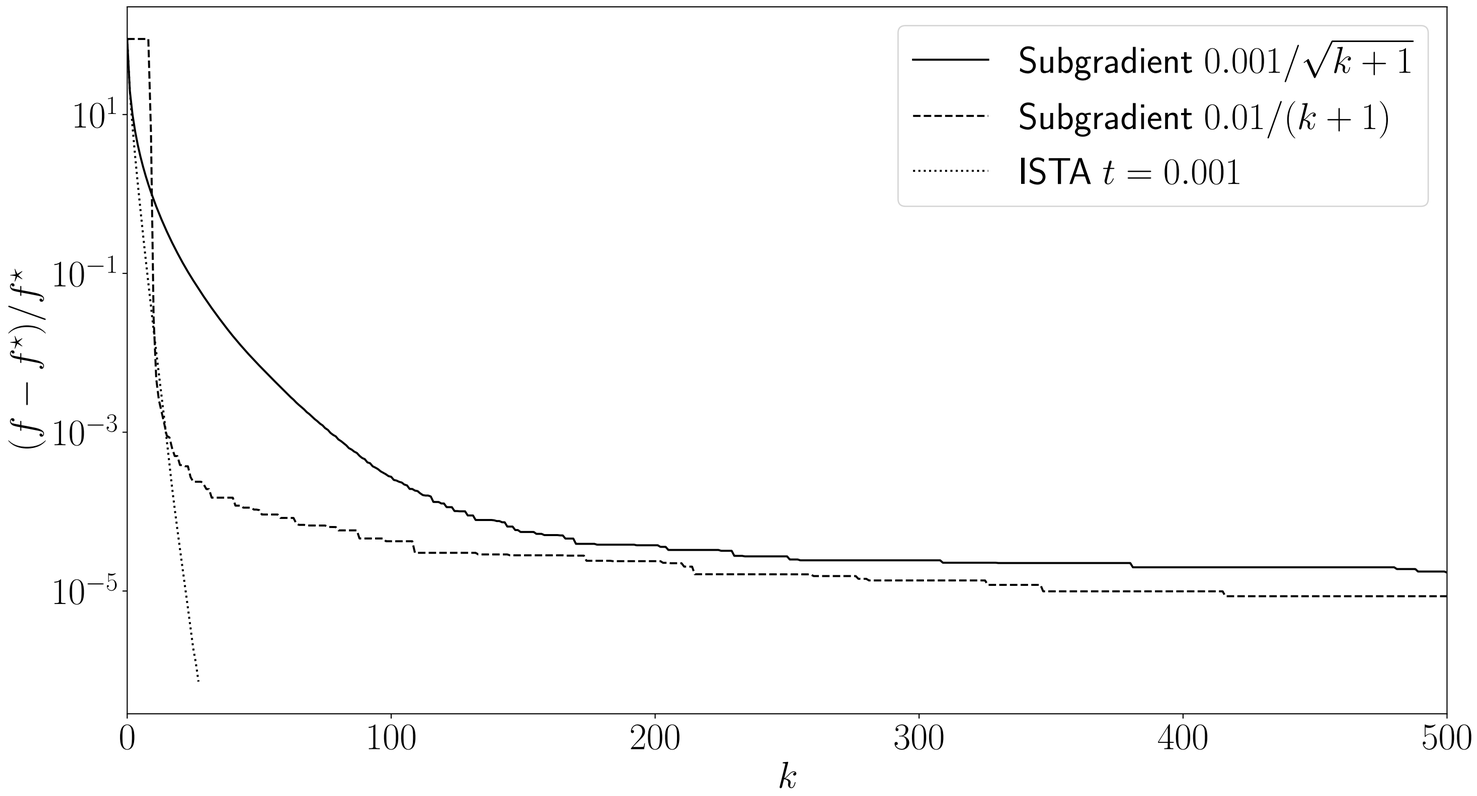
Example: Lasso

Iterative Soft Thresholding Algorithm (ISTA)

minimize $\quad (1/2)\|Ax - b\|_2^2 + \lambda\|x\|_1$

Closed-form iterations

$$x^{k+1} = S_{\lambda t}\left(x^k - tA^T(Ax^k - b)\right)$$



Better convergence

Can we prove convergence generally?

Can we combine different operators?

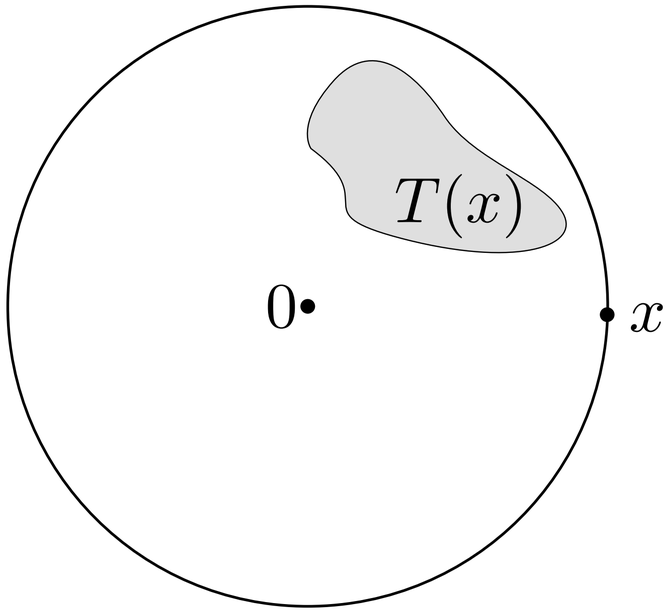
Introduction to operators

Operators

An operator T maps each point in $\mathbf{R}^n$ to a subset of $\mathbf{R}^n$

- **set valued** $T(x)$ returns a set
- **single-valued** $T(x)$ (function) returns a singleton

The **domain** of T is the set $\mathbf{dom}\, T = \{x \mid T(x) \neq \emptyset\}$



Example

- The subdifferential ∂f is a set-valued operator
- The gradient ∇f is a single-valued operator

Graph and inverse operators

Graph

The graph of an operator T is defined as

$$\mathbf{gph}T = \{(x, y) \mid y \in T(x)\}$$

In other words, all the pairs of points (x, y) such that $y \in T(x)$.

Inverse

The graph of the inverse operator T^{-1} is defined as

$$\mathbf{gph}T^{-1} = \{(y, x) \mid (x, y) \in \mathbf{gph}T\}$$

Therefore, $y \in T(x)$ if and only if $x \in T^{-1}(y)$.

Zeros

Zero

x is a **zero** of T if $\quad 0 \in T(x)$

Zero set

The set of all the zeros $\quad T^{-1}(0) = \{x \mid 0 \in T(x)\}$

Example
If $T = \partial f$ and $f : \mathbf{R}^n \to \mathbf{R}$, then
$0 \in T(x)$ means that x minimizes f

Many problems
can be posed as finding zeros
of an operator

Fixed points

$\bar{x}$ is a **fixed-point** of a single-valued operator T if

$$\bar{x} = T(\bar{x})$$

Set of fixed points $\mathbf{fix}\, T = \{x \in \mathbf{dom}\, T \mid x = T(x)\} = (I - T)^{-1}(0)$

Examples

- **Identity** $T(x) = x$. Any point is a fixed point
- **Zero operator** $T(x) = 0$. Only 0 is a fixed point

Lipschitz operators

An operator T is L-Lipschitz if

$$\|T(x) - T(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbf{dom}\, T$$

Fact If T is Lipschitz, then it is single-valued

Proof If $y = T(x), z = T(x)$, then $\|y - z\| \leq L\|x - x\| = 0 \Longrightarrow y = z$ $\blacksquare$

For $L = 1$ we say T is **nonexpansive**

For $L < 1$ we say T is **contractive** (with contraction factor L)

Lipschitz operators examples

Lipschitz affine functions

$T(x) = Ax + b$

$\longleftrightarrow$

maximum singular value

$L = \|A\|_2 = \lambda_{\max}(A^T A)$

Lipschitz differentiable functions

T such that there exists derivative DT

$\longleftrightarrow$

derivative is bounded

$\|DT\|_2 \leq L$

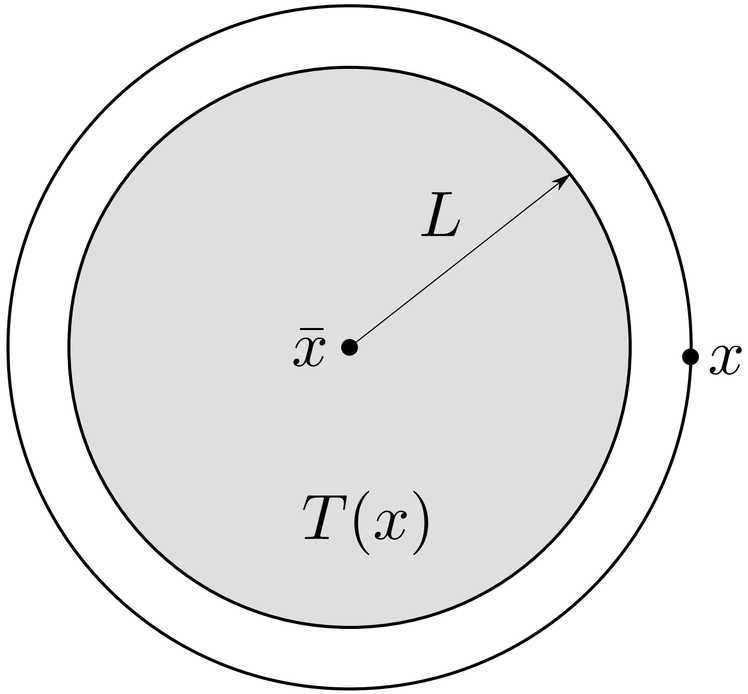
Lipschitz operators and fixed points

Given a L-Lipschitz operator T and a fixed point $\bar{x} = T\bar{x}$,

$$\|Tx - \bar{x}\| = \|Tx - T\bar{x}\| \leq L\|x - \bar{x}\|$$

A contractive operator ($L < 1$) can have at most one fixed point, i.e., $\mathbf{fix}\, T = \{\bar{x}\}$

Proof
If $\bar{x}, \bar{y} \in \mathbf{fix}\, T$ and $\bar{x} \neq \bar{y}$ then
$\|\bar{x} - \bar{y}\| = \|T(\bar{x}) - T(\bar{y})\| < \|x - y\|$ (contradiction) ■



A nonexpansive operator ($L = 1$) need not have a fixed point

Example $T(x) = x + 2$

Combining Lipschitz operators

T_1 is L_1-Lipschitz and T_2 is L_2-Lipschitz

The **composition** $T_1 T_2$ is $L_1 L_2$-Lipschitz

Proof $\|T_1 T_2 x - T_1 T_2 y\|_2 \leq L_1 \|T_2 x - T_2 y\|_2 \leq L_1 L_2 \|x - y\|_2$ ■

- Composition of *nonexpansive* is nonexpansive
- Composition of *nonexpansive* and *contractive* is contractive

The **weighted average** $\theta T_1 + (1-\theta) T_2,\ \theta \in (0,1)$ is $(\theta L_1 + (1-\theta) L_2)$-Lipschitz

Proof (exercise)

- Weighted average of *nonexpansive* is nonexpansive
- Weighted average of *nonexpansive* and *contractive* is contractive

Fixed point iterations

Fixed point iteration

Apply operator

$$x^{k+1} = T(x^k)$$

until you reach $\bar{x} \in \mathbf{fix}\, T$

Main approach

1. Find a suitable T such that $\bar{x} \in \mathbf{fix}\, T$ solve your problem
2. Show that the fixed point iteration converges

Fixed point residual to terminate

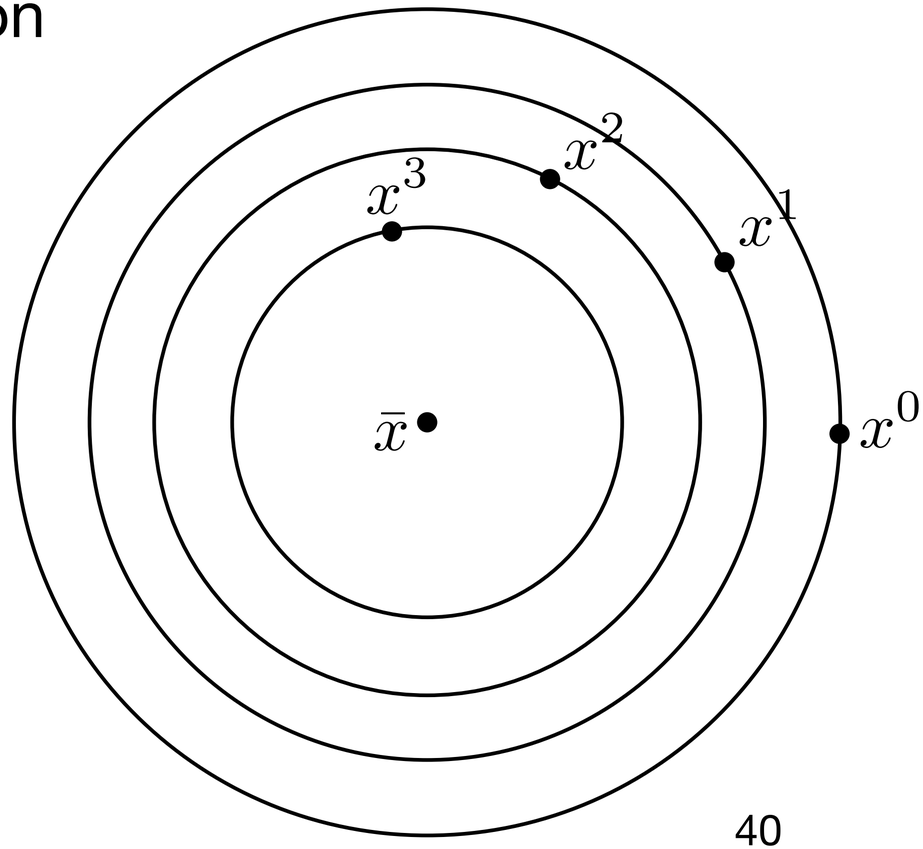
$$r^k = T(x^k) - x^k$$

Contractive fixed point iterations

Contraction mapping theorem
If T is L-Lipschitz with $L < 1$ (contraction), the iteration

$$x^{k+1} = T(x^k)$$

converges to $\bar{x}$, the unique fixed point of T



Properties

- Distance to $\bar{x}$ decreases at each step

$$\|x^{k+1} - \bar{x}\| \leq L\|x^k - \bar{x}\|$$

  (iteration is **Fejer monotone**)

- Linear convergence rate L

Contraction mapping theorem

Proof

The sequence x^k is Cauchy

$$\begin{aligned}
\|x^{k+\ell} - x^k\| &\leq \|x^{k+\ell} - x^{k+\ell-1}\| + \cdots + \|x^{k+1} - x^k\| && \text{(Lipschitz constant)} \\
&\leq (L^{\ell-1} + \cdots + 1)\|x^{k+1} - x^k\| \\
&\leq \frac{1}{1-L}\|x^{k+1} - x^k\| && \text{(geometric series)} \\
&\leq \frac{L^k}{1-L}\|x^1 - x^0\|
\end{aligned}$$

Therefore it converges to a point $\bar{x}$ which must be the (unique) fixed point of T

The convergence is linear (geometric) with rate L

$$\|x^k - \bar{x}\| = \|T(x^{k-1}) - T(\bar{x})\| \leq L\|x^{k-1} - \bar{x}\| \leq L^k\|x^0 - x^\star\| \qquad \blacksquare$$

Nonexpansive fixed point iterations

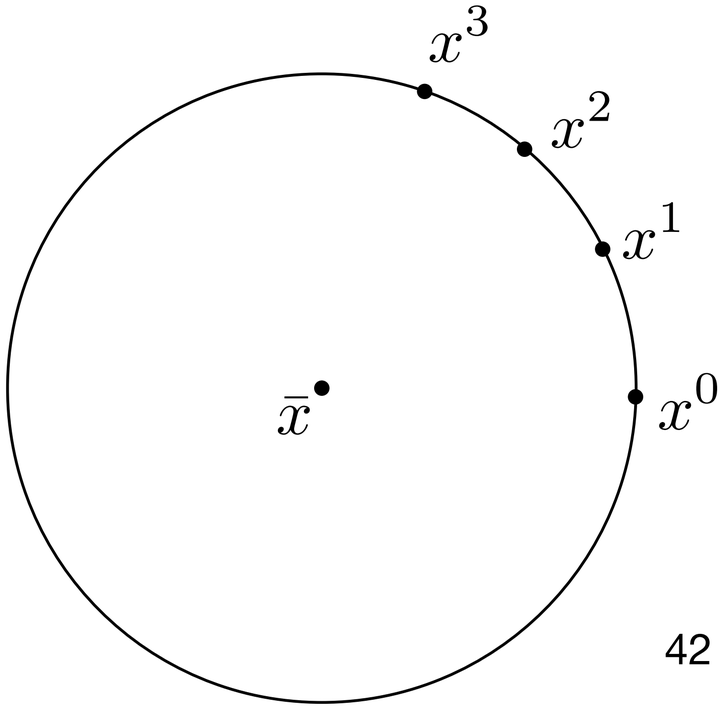
If T is L-Lipschitz with $L = 1$ (nonexpansive), the iteration

$$x^{k+1} = T(x^k)$$

need not converge to a fixed point, even if one exists.

Example

- Let T be a rotation around the origin
- T is nonexpansive and has a fixed point $\bar{x} = 0$
- $\|x^k\|$ never decreases

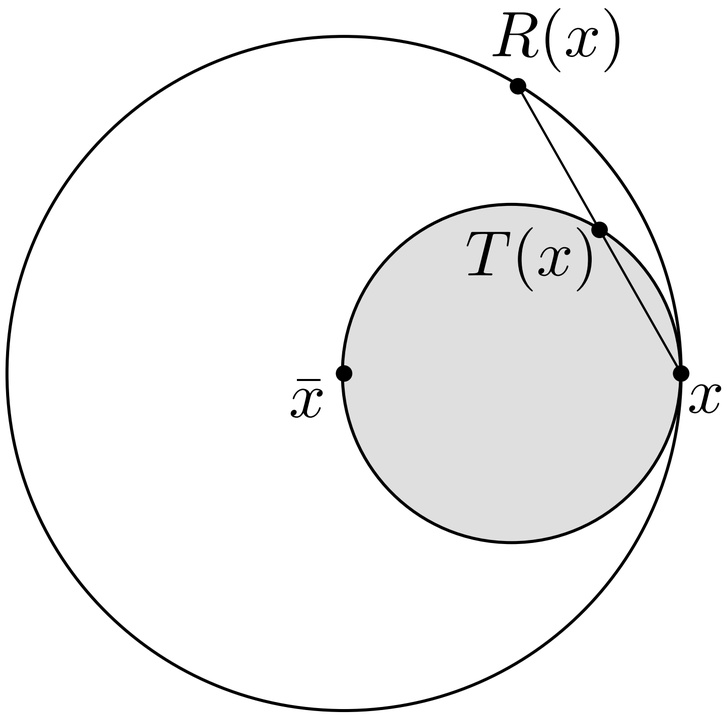


Averaged operators

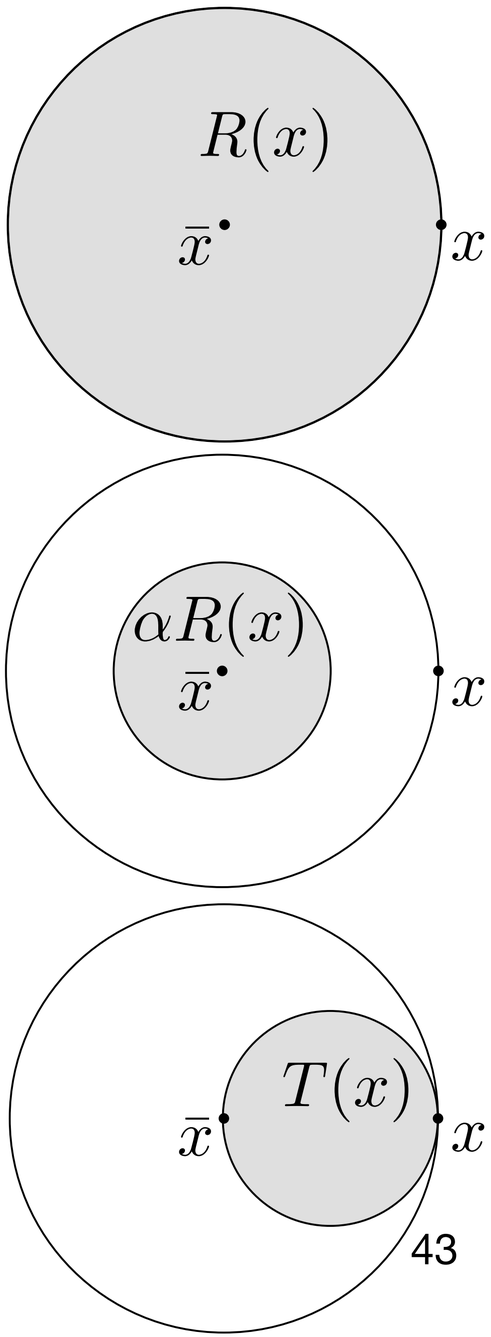
We say that an operator T is $\alpha-$**averaged** with $\alpha \in (0,1)$ if

$$T = (1-\alpha)I + \alpha R$$

and R is nonexpansive.



Example $\alpha = 1/2$



Averaged operators fixed points

We say that an operator T is $\alpha-$**averaged** with $\alpha \in (0,1)$ if

$$T = (1-\alpha)I + \alpha R$$

Fact If T is α-averaged, then $\mathbf{fix}\, T = \mathbf{fix}\, R$

Proof

$$\begin{aligned} \bar{x} = T(\bar{x}) &= (1-\alpha)I(\bar{x}) + \alpha R(\bar{x}) \\ &= (1-\alpha)\bar{x} + \alpha R(\bar{x}) \\ \iff \quad & \alpha\bar{x} = \alpha R(\bar{x}) \\ \iff \quad & \bar{x} = R(\bar{x}) \end{aligned}$$

$\blacksquare$

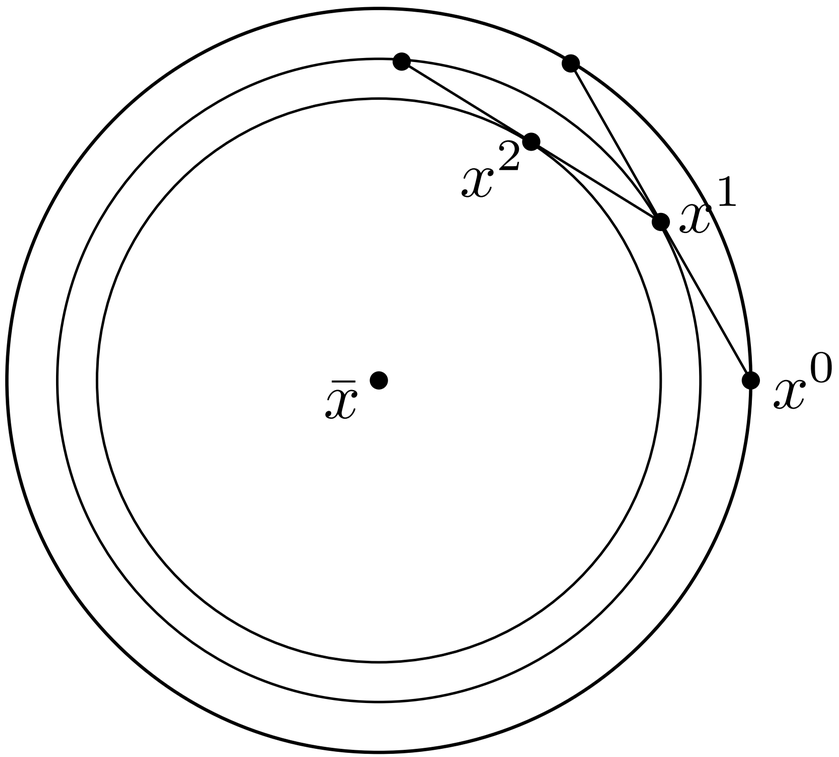
Averaged fixed point iterations

If $T = (1-\alpha)I + \alpha R$ is α-averaged
($\alpha \in (0,1)$ and R nonexpansive), the iteration

$$x^{k+1} = T(x^k)$$

converges to $\bar{x} \in \mathbf{fix}\, T$

(also called damped, averaged
or Mann-Krasnosel'skii iteration)



Properties

- Distance to $\bar{x}$ decreases at each step (**Fejer monotone**)
- Sublinear convergence to fixed-point residual

$$\|R(x^k) - x^k\| \leq \frac{1}{\sqrt{(k+1)\alpha(1-\alpha)}}\|x^0 - \bar{x}\|$$

Averaged fixed point iterations

Proof

Use the identity (proof by expanding)

$$\|(1-\alpha)a + \alpha b\|^2 = (1-\alpha)\|a\|^2 + \alpha\|b\|^2 - \alpha(1-\alpha)\|a-b\|^2$$

and apply it to

$$x^{k+1} - \bar{x} = (1-\alpha)(\underbrace{x^k - \bar{x}}_{a}) + \alpha(\underbrace{R(x^k) - \bar{x}}_{b})$$

obtaining

$$\begin{aligned}
\|x^{k+1} - \bar{x}\|^2 &= (1-\alpha)\|x^k - \bar{x}\|^2 + \alpha\|R(x^k) - \bar{x}\|^2 - \alpha(1-\alpha)\|x^k - R(x^k)\|^2 \\
&\leq (1-\alpha)\|x^k - \bar{x}\|^2 + \alpha\|x^k - \bar{x}\|^2 - \alpha(1-\alpha)\|x^k - R(x^k)\|^2 \quad \text{(nonexpansive)} \\
&= \|x^k - \bar{x}\|^2 \underbrace{- \alpha(1-\alpha)\|x^k - R(x^k)\|^2}_{\leq 0}
\end{aligned}$$

Iterations are Fejer monotone

Averaged fixed point iterations

Proof (continued)

iterate righthand side over k steps

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^0 - \bar{x}\|^2 - \alpha(1-\alpha)\sum_{i=0}^{k} \|x^i - R(x^i)\|^2$$

Since $\|x^{k+1} - \bar{x}\|^2 \geq 0$, we have $$\sum_{i=0}^{k} \|x^i - R(x^i)\|^2 \leq \frac{1}{\alpha(1-\alpha)}\|x^0 - \bar{x}\|^2$$

Using $\sum_{i=0}^{k} \|x^i - R(x^i)\|^2 \geq (k+1) \min_{i=0,\ldots,k} \|x^i - R(x^i)\|^2$, we obtain

$$\min_{i=0,\ldots,k} \|x^i - R(x^i)\|^2 \leq \frac{1}{(k+1)\alpha(1-\alpha)}\|x^0 - \bar{x}\|^2$$

Since R is nonexpansive, $\|x^k - R(x^k)\|^2 \leq \frac{1}{(k+1)\alpha(1-\alpha)}\|x^0 - \bar{x}\|^2$ ■

Average fixed point iteration convergence rates

$$\|R(x^k) - x^k\| \leq \frac{1}{\sqrt{(k+1)\alpha(1-\alpha)}}\|x^0 - \bar{x}\|$$

Righthand side minimized when $\alpha = 1/2$

$$\|R(x^k) - x^k\| \leq \frac{2}{\sqrt{k+1}}\|x^0 - \bar{x}\|$$

Iterations

$$x^{k+1} = (1/2)x^k + (1/2)R(x^k)$$

Remarks

- Sublinear convergence (same as subgrad method), in general not the actual rate
- $\alpha = 1/2$ is very common for averaged operators

How to design an algorithm

Problem

$$\text{minimize} \quad f(x)$$

Algorithm (operator) construction

1. Find a suitable T such that $\bar{x} \in \mathbf{fix}\, T$ solve your problem
2. Show that the fixed point iteration converges

If T is contractive $\implies$ **linear convergence**
If T is averaged $\implies$ **sublinear convergence**

Most first order algorithms can be constructed in this way

Proximal methods and introduction to operators

Today, we learned to:

- **Derive** optimality conditions for constrained optimization problems using subdifferentials
- **Define** and **evaluate** proximal operators for various common functions
- **Apply** proximal operators to generalize gradient descent (vanilla, projected, proximal)
- **Use operator theory** to construct general fixed-point iterations and prove their convergence

Next lecture

- Monotone operators and operator splitting algorithms