ORF522 – Linear and Nonlinear Optimization

8. Linear optimization duality

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Ed forum

- I am wondering is there a general intepretation of the practical meaning of the Dual Problem to its Primal ?
- Is it always possible to find an interpretation of the dual problem?
- Also the dual of dual is primal, they should be mathematically related. Can you please explain this if you remember?
- Can the strong duality be deduced at some point in the primal simplex algorithm or do we need both primal and dual simplex to run to check the strong duality?
- I'm still not to sure why dual is important to LP problems compared to convex problems
- If primal optimal solution x is unique, would we also have a unique optimal solution y in the dual?
- When do we decide whether we want to solve the primal or the dual?

Recap

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**

Today's agenda

Readings: [Chapter 4, Bertsimas, Tsitsiklis][Chapter 11, Vanderbei]

- Two-person zero-sum games
- Farkas lemma
- Complementary slackness
- Dual simplex method

Two-person zero-sum games

Rock paper scissors

Rules

At count to three declare one of: Rock, Paper, or Scissors

Winners

Identical selection is a draw, otherwise:

- Rock beats (“dulls”) scissors
- Scissors beats (“cuts”) paper
- Paper beats (“covers”) rock

Extremely popular: world RPS society, USA RPS league, etc.

Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of m actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of n actions)

Two players make their choice independently

Rule

Player 1 pays A_{ij} to player 2

$A \in \mathbf{R}^{m \times n}$ is the **payoff matrix**

Rock, Paper, Scissors

$$A = \begin{array}{c} \\ \mathrm{R} \\ \mathrm{P} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{P} & \mathrm{S} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

Mixed (randomized) strategies

Deterministic strategies can be **systematically defeated**

Randomized strategies

- P1 chooses randomly according to distribution x:

  $x_i =$ probability that P1 selects action i

- P2 chooses randomly according to distribution y:

  $y_i =$ probability that P2 selects action j

Expected payoff (from P1 P2), if they use mixed-strategies x and y,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

Mixed strategies and probability simplex

Probability simplex in $\mathbf{R}^k$
$P_k = \{p \in \mathbf{R}^k \mid p \geq 0, \quad \mathbf{1}^T p = 1\}$

Mixed strategy

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $\longrightarrow$ $x \in P_m, \quad y \in P_n$

Optimal mixed strategies

P1: optimal strategy $x^\star$ is the solution of

minimize	$\max_{y \in P_n} x^T A y$	$\longrightarrow$	minimize	$\max_{j=1,\dots,n} (A^T x)_j$
subject to	$x \in P_m$		subject to	$x \in P_m$

Inner problem over deterministic strategies (**vertices**)

P2: optimal strategy $y^\star$ is the solution of

maximize	$\min_{x \in P_m} x^T A y$	$\longrightarrow$	maximize	$\min_{i=1,\dots,m} (Ay)_i$
subject to	$y \in P_n$		subject to	$y \in P_n$

Optimal strategies $x^\star$ and $y^\star$ can be computed using **linear optimization**

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

The optimal $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \geq w\mathbf{1} \\ & \mathbf{1}^T y = 1 \\ & y \geq 0 \end{array}$$

The two LPs are **duals** and by **strong duality** the equality follows. ■

Nash equilibrium

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Consequence

The pair of mixed strategies $(x^\star, y^\star)$ attains the **Nash equilibrium** of the two-person matrix game, i.e.,

$$x^T A y^\star \geq x^{\star T} A y^\star \geq x^{\star T} A y, \quad \forall x \in P_m, \; \forall y \in P_n$$

Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

Optimal mixed strategies

$$x^\star = (0.37, 0.33, 0.3), \qquad y^\star = (0.4, 0, 0.13, 0.47)$$

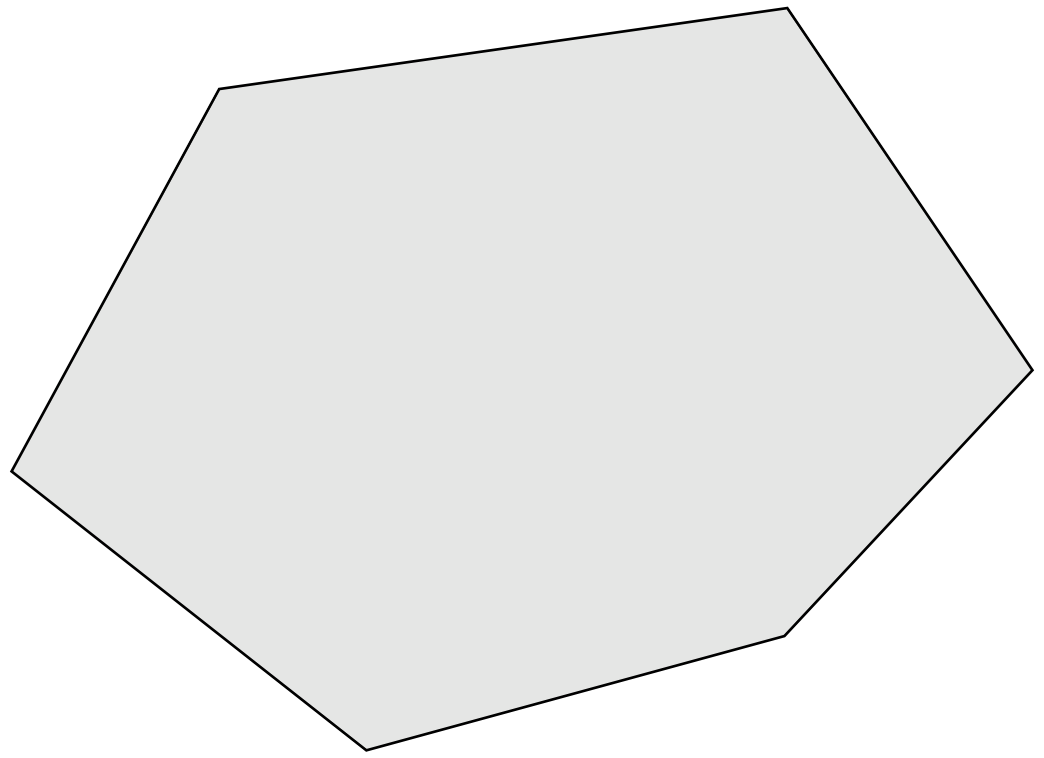
Expected payoff

$$x^{\star T} A y^\star = 0.2$$

Farkas lemma

Feasibility of polyhedra

$$P = \{x \mid Ax = b, \quad x \geq 0\}$$



How to show that P is **feasible**?

Easy: we just need to provide an $x \in P$, i.e., a **certificate**

How to show that P is **infeasible**?

Farkas lemma

Theorem

Given A and b, exactly one of the following statements is true:

1. There exists an x with $Ax = b$, $x \geq 0$
2. There exists a y with $A^T y \geq 0$, $b^T y < 0$

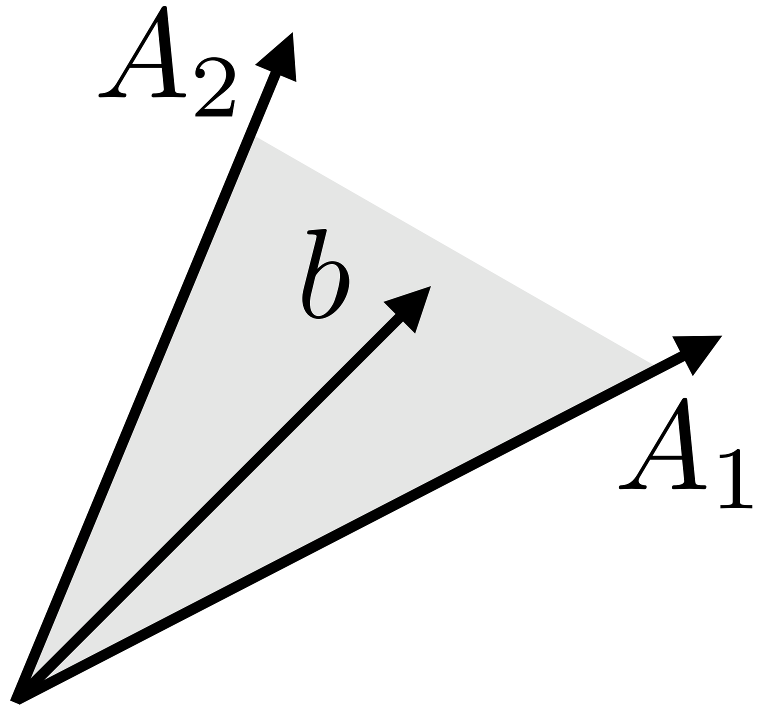
Farkas lemma

Geometric interpretation

1. First alternative

$$b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0,\ i = 1, \ldots, n$$

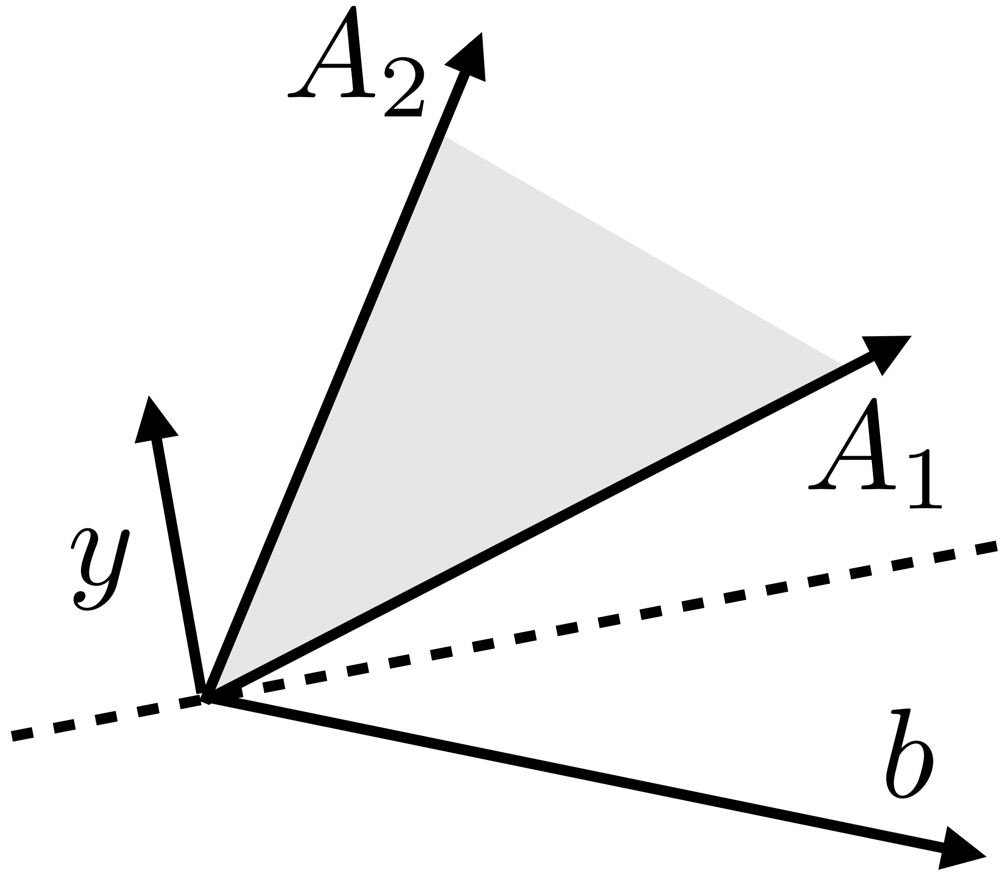
b is in the cone generated by the columns of A



2. Second alternative

$$y^T A_i \geq 0, \quad i = 1, \ldots, m, \qquad y^T b < 0$$

The hyperplane $y^T z = 0$ separates b from $A_1, \ldots, A_n$



Farkas lemma

Proof

1 and 2 cannot be both true (easy)

$x \geq 0$, $Ax = b$ and $y^T A \geq 0$ $\longrightarrow$ $y^T b = y^T A x \geq 0$

1 and 2 cannot be both false (duality)

Primal		**Dual**	
minimize	0	maximize	$-b^T y$
subject to	$Ax = b$	subject to	$A^T y \geq 0$
	$x \geq 0$		

$\longleftarrow$ $y = 0$ always feasible

$d^\star \neq -\infty, \quad p^\star = d^\star$

Alternative 1: primal feasible $p^\star = d^\star = 0$

$b^T y \geq 0$ for all y such that $A^T y \geq 0$

Alternative 2: primal infeasible $p^\star = d^\star = +\infty$

There exists y such that $A^T y \geq 0$ and $b^T y < 0$

y is an **infeasibility certificate**

Complementary slackness

Optimality conditions

Primal

minimize $\quad c^T x$

subject to $\quad Ax \leq b$

Dual

maximize $\quad -b^T y$

subject to $\quad A^T y + c = 0$

$\quad y \geq 0$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax \leq b$
- y is **dual feasible:** $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Can we **relate** x and y (not only the objective)?

Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Theorem

Primal,dual feasible x, y are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and y have a **complementary sparsity** pattern:

$$y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$$
$$a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$$

Complementary slackness

Primal

minimize $c^T x$
subject to $Ax \leq b$

Dual

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

Proof

The duality gap at primal feasible x and dual feasible y can be written as

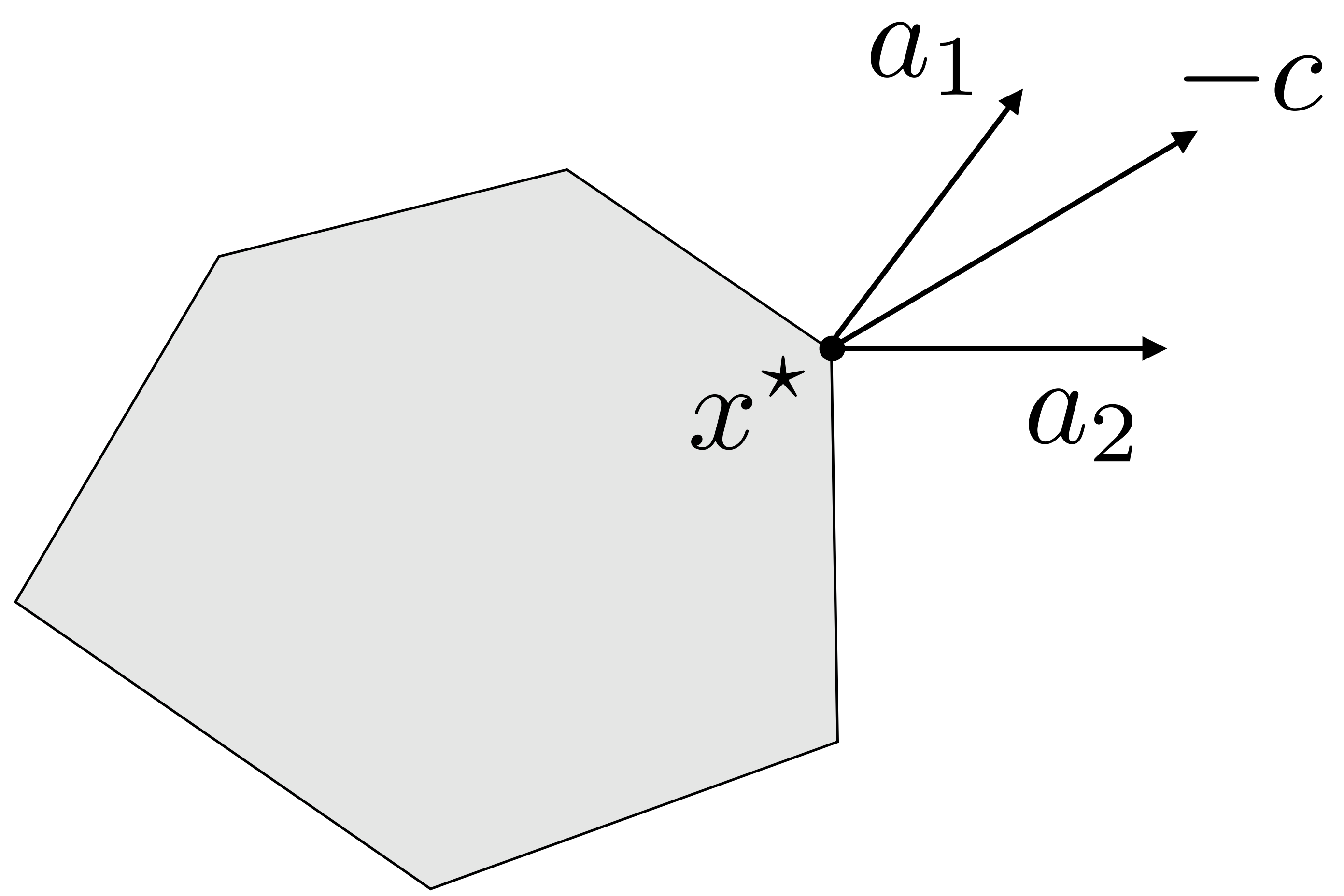
$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0 ■

For **feasible** x and y **complementary slackness** = **zero duality gap**

Geometric interpretation

Example in $\mathbf{R}^2$



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Optimal dual solution y satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$

Geometric interpretation: $-c$ lies in the cone generated by a_1 and a_2

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at x $\longrightarrow$ $y = (0, y_2, 0, y_4)$

$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \text{and} \qquad y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Complementary slackness is useful to recover $y^\star$ from $x^\star$

The dual simplex

Optimality conditions

Primal problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax = b$ and $x \geq 0$
- y is **dual feasible**: $A^T y + c \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Primal and dual basic feasible solutions

Primal problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

Given a **basis** matrix B

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = B^{-1} b \geq 0$

Reduced costs

Dual feasible: $A^T y + c \geq 0.$ If $y = -B^{-T} c_B \quad \Rightarrow \quad c - A^T B^{-T} c_B \geq 0$

Zero duality gap: $c^T x + b^T y = c_B x_B - b^T B^{-T} c_B = c_B x_B - c_B^T B^{-1} b = 0$

(by construction)

The primal (dual) simplex method

Primal problem

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

Dual problem

maximize $-b^T y$

subject to $A^T y + c \geq 0$

Primal simplex

- Primal feasibility
- Zero duality gap

↓

Dual feasibility

Dual simplex

- Dual feasibility
- Zero duality gap

↓

Primal feasibility

Feasible dual directions

Conditions

$$P = \{y \mid A^T y + c \geq 0\}$$

Given a basis matrix $B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

we have dual feasible solution y:

$$\bar{c} = A^T y + c \geq 0$$

Feasible direction d

$$y + \theta d$$

Reduced cost change

$$c + A^T(y + \theta d) \geq 0 \quad \Rightarrow \quad \bar{c} + \theta z \geq 0$$

$A^T d = z$ (subspace restriction)

Feasible directions

Computation

Subspace restriction

$$A^T d = z \longrightarrow \begin{matrix} B^T d = z_B \\ N^T d = z_N \end{matrix}$$

Basic indices

$z_B = e_j \longrightarrow B(\ell) = j$ **exits the basis**

Get d by solving $B^T d = e_j$

Nonbasic indices

$z_N = N^T d = N^T B^{-T} e_j$

Non-negativity of reduced costs (non-degenerate assumption)

- Basic variables: $\bar{c}_B = 0$. Nonnegative direction $z_B \geq 0$.
- Nonbasic variables: $\bar{c}_N > 0$. Therefore $\exists \theta > 0$ such that $\bar{c}_N + \theta z_N \geq 0$

Stepsize

How far can we go?

$$\theta^\star = \max\{\theta \mid \theta \geq 0 \text{ and } \bar{c} + \theta z \geq 0\}$$

Unbounded

If $z \geq 0$, then $\theta^\star = \infty$. The dual problem is unbounded (primal infeasible).

Bounded

If $z_i < 0$ for some i, then $$\theta^\star = \min_{\{i \mid z_i < 0\}} \left(-\frac{\bar{c}_i}{z_i}\right) = \min_{\{i \in N \mid z_i < 0\}} \left(-\frac{\bar{c}_i}{z_i}\right)$$

(Since $z_i \geq 0,\ i \in B$)

Moving to a new basis

Next reduced cost

$$\bar{c} + \theta^\star z$$

Let $i \notin \{B(1), \ldots, B(m)\}$ be the index such that $\theta^\star = -\dfrac{\bar{c}_i}{z_i}$. Then,

$$\bar{c}_i + \theta^\star z_i = 0$$

New basis

$$\bar{B} = \begin{bmatrix} A_{B(1)} & \ldots & A_{B(\ell-1)} & A_i & A_{B(\ell+1)} & \ldots & A_{B(m)} \end{bmatrix}$$

New solution

$$x = \bar{B}^{-1} b$$

Dual simplex method

Initialization

1. Given basic dual feasible solution y, i.e., $A^T y + c \geq 0$
2. Factor basis matrix $B = \begin{bmatrix} A_{B(1)} & \dots, A_{B(m)} \end{bmatrix}$

Iterations

1. Solve $Bx_B = b$, $\quad$ ($O(m^2)$)
2. Get x from x_B ($x_i = 0,\ i \notin$ basis)
3. If $x \geq 0$, x **feasible. break**
4. Choose j such that $x_j < 0$

Dual simplex method

Iterations (continued)

5. Search direction: $z_j = 1$, solve $B^T d = e_j$ and compute $z_N = N^T d \quad$ ($O(m^2)$)
6. If $z_N \geq 0$, the dual problem is **unbounded** and the optimal value is $+\infty$. **break**
7. Compute step length $\theta^\star = \min\limits_{\{i \in N | z_i < 0\}} \left(-\frac{\bar{c}_i}{z_i} \right)$ and pick i entering the basis
8. Compute new point $y + \theta^\star d$
9. Get new basis $\bar{B} = B + (A_i - A_j)e_j^T$ and perform rank-1 factor update. $\quad$ ($O(m^2)$)

Remark: reduced costs nonnegative ⟶ dual objective non-decreasing

Example

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 2 \\ & x_1 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

$$c = (1, 1, 0, 0)$$

$$A = \begin{bmatrix} -1 & -2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (-2, -1)$$

Initialize

$y = (0, 0)$ Basis $\{3, 4\}$

Example

Iteration 1

$$y = (0, 0)$$
$$-b^T y = 0$$
$$\bar{c} = c + A^T y = (1, 2, 0, 0)$$
Basis $\{3, 4\}$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$c = (1, 1, 0, 0)$$
$$A = \begin{bmatrix} -1 & -2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$
$$b = (-2, -1)$$

Primal solution $x = (0, 0, -2, -1)$

Solve $Bx_B = b \quad \Rightarrow \quad x_B = (-2, -1)$

Direction $z = (-1, -2, 1, 0), \quad j = 3$

Solve $B^T d = e_j \quad \Rightarrow \quad d = (1, 0)$

Get $z_N = N^T d = (-1, -2)$

Step $\theta^\star = 0.5, \quad i = 2$

$$\theta^\star = \min_{\{i | z_i < 0\}} (-\bar{c}_i / z_i) = \{1, 0.5\}$$

New $y \leftarrow y + \theta^\star d = (0.5, 0)$

Example
Iteration 2

$$y = (0.5, 0)$$
$$-b^T y = 1$$
$$\bar{c} = c + A^T y = (0.5, 0, 0.5, 0)$$
Basis $\{2, 4\}$
$$B = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$c = (1, 1, 0, 0)$$
$$A = \begin{bmatrix} -1 & -2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$
$$b = (-2, -1)$$

Primal solution $x = (0, 1, 0, -1)$

Solve $Bx_B = b \quad \Rightarrow \quad x_B = (1, -1)$

Direction $z = (-1, 0, 0, 1), \quad j = 4$

Solve $B^T d = e_j \quad \Rightarrow \quad d = (0, 1)$

Get $z_N = N^T d = (-1, 0)$

Step $\theta^\star = 0.5, \quad i = 1$

$$\theta^\star = \min_{\{i | z_i < 0\}} (-\bar{c}_i / z_i) = \{0.5\}$$

New $y \leftarrow y + \theta^\star d = (0.5, 0.5)$

Example

Iteration 3

$y = (0.5, 0.5)$
$-b^T y = 1.5$
$\bar{c} = c + A^T y = (0, 0, 0.5, 0.5)$
Basis $\{1, 2\}$

$$B = \begin{bmatrix} -1 & -2 \\ -1 & 0 \end{bmatrix}$$

$c = (1, 1, 0, 0)$

$$A = \begin{bmatrix} -1 & -2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$b = (-2, -1)$

Primal solution $x = (1, 1.5, 0, 0)$

Solve $Bx_B = b \quad \Rightarrow \quad x_B = (1, 1.5)$

$x \geq 0$ ⟶ **Optimal solution** $x^\star = (1, 1.5, 0, 0)$

Equivalence and symmetry

The dual simplex is equivalent to the primal simplex applied to the dual problem.

Dual problem

maximize $\quad -b^T y$

subject to $\quad A^T y + c \geq 0$

$\longrightarrow$

Symmetrized dual

minimize $\quad b^T y$

subject to $\quad A^T y + c = w$

$\quad w \geq 0$

$w \geq 0$ are the **reduced costs**

Dual simplex efficiency

Sequence of problems with varying feasible region

previous y still dual feasible $\longrightarrow$ **warm-start**

Often applied in **mixed-integer optimization** to **solve subproblems**
(more later in the course…)

Linear optimization duality

Today, we learned to:

- **Interpret** linear optimization duality using game theory
- **Prove** Farkas lemma using duality
- **Geometrically link** primal and dual solutions with complementary slackness
- **Implement** the dual simplex method

Next lecture

- Sensitivity analysis