ORF522 – Linear and Nonlinear Optimization

7. Linear optimization duality

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Ed forum

- How computationally expensive is it for computers to "detect" special patterns such as sparsity or orthogonality? Or do people input these features and the algorithm just assumes it?
- Do solvers usually check to see what B is before deciding which method to use, or are they usually coded to use some default method? Does one need to create custom code when when the structure is known and the solver can be sped up a bit for some particular problems?
- I recall the final solution having 4 nonzero x values, while the basis consists of 3 elements - is this still a basic feasible solution, since there is a nonbasic variable that has a nonzero value? (It was a typo!)
- If we want to factor A into A1 A2 ... Ak, the matrices it can factor into, and the order they appear, is probably not unique. How does the computer typically do it? Perhaps, a related question is, how does the computer take inverses of large matrices?
- Are there situations, such as in physics, electrical engineering, robotics, etc where one knows for such that B will be tridiagonal and/or positive definite so that one uses a custom simplex method solver to speed up the process by not doing, say, and LU factorization but some faster method that applies to the given situation?

Recap

Linear optimization formulations

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Today's agenda

Readings: [Chapter 4, Bertsimas, Tsitsiklis][Chapter 5, Vanderbei]

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

Obtaining lower bounds

Obtaining lower bounds

A simple example

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + 3x_2 \geq 2 \end{array}$$

What is a **lower bound** on the optimal cost?

A lower bound is 2 because $x_1 + 3x_2 \geq 2$

Obtaining lower bounds

Another example

minimize $x_1 + 3x_2$

subject to $x_1 + x_2 \geq 2$

$x_2 \geq 1$

What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$1 \cdot (x_1 + x_2 \geq 2)$$
$$+ 2 \cdot (x_2 \geq 1)$$
$$= x_1 + 3x_2 \geq 4$$

A lower bound is 4

Obtaining lower bounds

A more interesting example

minimize $x_1 + 3x_2$

subject to $x_1 + x_2 \geq 2$

$x_2 \geq 1$

$x_1 - x_2 \geq 3$

How can we obtain a lower bound?

Add constraints

$$y_1 \cdot (x_1 + x_2 \geq 2)$$
$$+ y_2 \cdot (x_2 \geq 1)$$
$$+ y_3 \cdot (x_1 - x_2 \geq 3)$$
$$= x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3$$

Bound

Match the cost

$$y_1 + y_3 = 1$$
$$y_1 + y_2 - y_3 = 3$$
$$y_1, y_2, y_3 \geq 0$$

Many options

$y = (1, 2, 0) \Rightarrow$ Bound 4
$y = (0, 4, 1) \Rightarrow$ Bound 7

How can we get the **best one**?

Obtaining lower bounds

A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$
\begin{array}{ll}
\text{maximize} & 2y_1 + y_2 + 3y_3 \\
\text{subject to} & y_1 + y_3 = 1 \\
& y_1 + y_2 - y_3 = 3 \\
& y_1, y_2, y_3 \geq 0
\end{array}
$$

This linear optimization problem is called the **dual problem**

The dual problem

Lagrange multipliers

Consider the LP in standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Relax the constraint

$$g(y) = \begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + y^T (Ax - b) \\ \text{subject to} & x \geq 0 \end{array}$$

Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) = c^T x^\star$$

Best lower bound

$$\underset{y}{\text{maximize}} \quad g(y)$$

The dual

Dual function

$$g(y) = \underset{x \geq 0}{\text{minimize}} \left(c^T x + y^T (Ax - b) \right)$$

$$- b^T y + \underset{x \geq 0}{\text{minimize}} \left(c + A^T y \right)^T x$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem (find the best bound)

$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Primal and dual problems

Primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Primal variable $x \in \mathbf{R}^n$

Dual problem

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Dual variable $y \in \mathbf{R}^m$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

Dual of inequality form LP

What if you find an LP with inequalities?

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T (Ax - b)$$

Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) \leq c^T x^\star$$

we must have $y \geq 0$

Dual of LP with inequalities

Derivation

Dual function

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$- b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y = 0 \quad \text{(and } y \geq 0\text{)} \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem (find the best bound)

$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

General forms

Standard form LP

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Inequality form LP

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

LP with inequalities and equalities

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y - d^T z \\ \text{subject to} & A^T y + C^T z + c = 0 \\ & y \geq 0 \end{array}$$

Example from before

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3 \end{aligned}$$

$\longrightarrow$

Inequality form LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$b = (-2, -1, -3)$$

Dual

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c = 0 \\ & y \geq 0 \end{aligned}$$

$\longrightarrow$

$$\begin{aligned} \text{maximize} \quad & 2y_1 + y_2 + 3y_3 \\ \text{subject to} \quad & -y_1 - y_3 = -1 \\ & -y_1 - y_2 + y_3 = -3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

To memorize

Ways to get the dual

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention

- Consider constraints as $g(x) \leq 0$ or $g(x) = 0$
- Each dual variable is associated to a primal constraint
- y free for primal equalities and $y \geq 0$ for primal inequalities

Dual of the dual

Theorem

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

Exercise

Derive dual and dualize again

Primal		**Dual**	
minimize	$c^T x$	maximize	$-b^T y - d^T z$
subject to	$Ax \leq b$	subject to	$A^T y + C^T z + c = 0$
	$Cx = d$		$y \geq 0$

Theorem

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

Weak and strong duality

Optimal objective values

Primal

minimize $c^T x$
subject to $Ax \leq b$

Dual

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y \qquad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Strong duality

Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow$$

- optimal basis B
- optimal solution $x^\star$ with $Bx_B^\star = b$
- reduced costs $\bar{c} = c - A^T B^{-T} c_B \geq 0$

Define $y^\star$ such that $y^\star = -B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

$$-b^T y^\star = -b^T(-B^{-T} c_B) = c_B^T(B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

By weak duality theorem corollary, $y^\star$ is an optimal solution of the dual.
Therefore, $d^\star = p^\star$. $\blacksquare$

Exception to strong duality

Primal

minimize x

subject to $0 \cdot x \leq -1$

Dual

maximize y

subject to $0 \cdot y + 1 = 0$

$y \geq 0$

Optimal value is $p^\star = +\infty$

Optimal value is $d^\star = -\infty$

Both **primal** and **dual infeasible**

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**

Example

Production problem

maximize $x_1 + 2x_2$ ← Profits

subject to
$x_1 \leq 100$
$2x_2 \leq 200$ ← Resources
$x_1 + x_2 \leq 150$
$x_1, x_2 \geq 0$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Dualize

1. Transform in inequality form

minimize $c^T x$
subject to $Ax \leq b$

2. Derive dual

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

Production problem

The dual

$$\begin{aligned} \text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Interpretation

- **Sell your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs $1\times$ resource 1 and 2): $y_1 + y_3 \geq 1$
  - Product 2 (price 2, needs $2\times$ resource 2 and $1\times$ resource 3): $2y_2 + y_3 \geq 2$

Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Interpret** simple dual optimization problems

Next lecture

More on duality:

- Game theoretic interpretation
- Complementary slackness
- Alternative systems