ORF522 – Linear and Nonlinear Optimization

5. The simplex method

Bartolomeo Stellato — Fall 2020

Ed forum

- What is the geometric picture of the standard form? Given a standard form P, can we always convert it back to the version defined by halfspaces? (next slides)
- Extent to which these methods are generalizable to infinite-dimensional restrictions e.g. linear difference equations when t goes to infinity.
- Efficient way to deal with inverses? (next lecture)
- How to pick index entering the basis? (this lecture)
- Adjacent solutions why defined that way? (Same active constraints except 1)
- Feasibility LP condition in no strong arbitrage from Arrow-Debreu theory (That's correct! We will discuss feasibility in duality lectures)

Recap

Standard form polyhedra

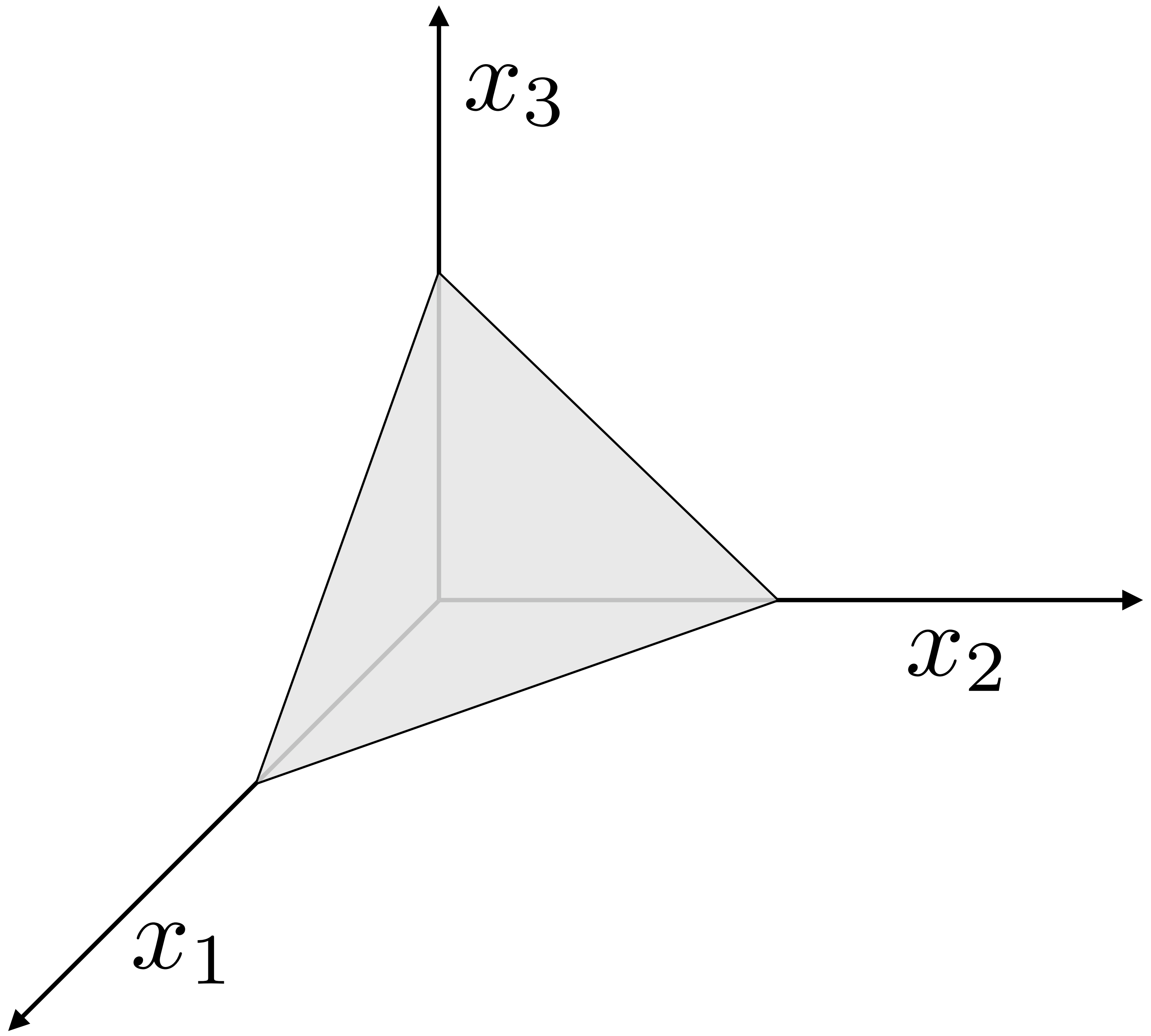
Definition

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Standard form polyhedron

$$P = \{x \mid Ax = b,\ x \geq 0\}$$



Assumption

$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

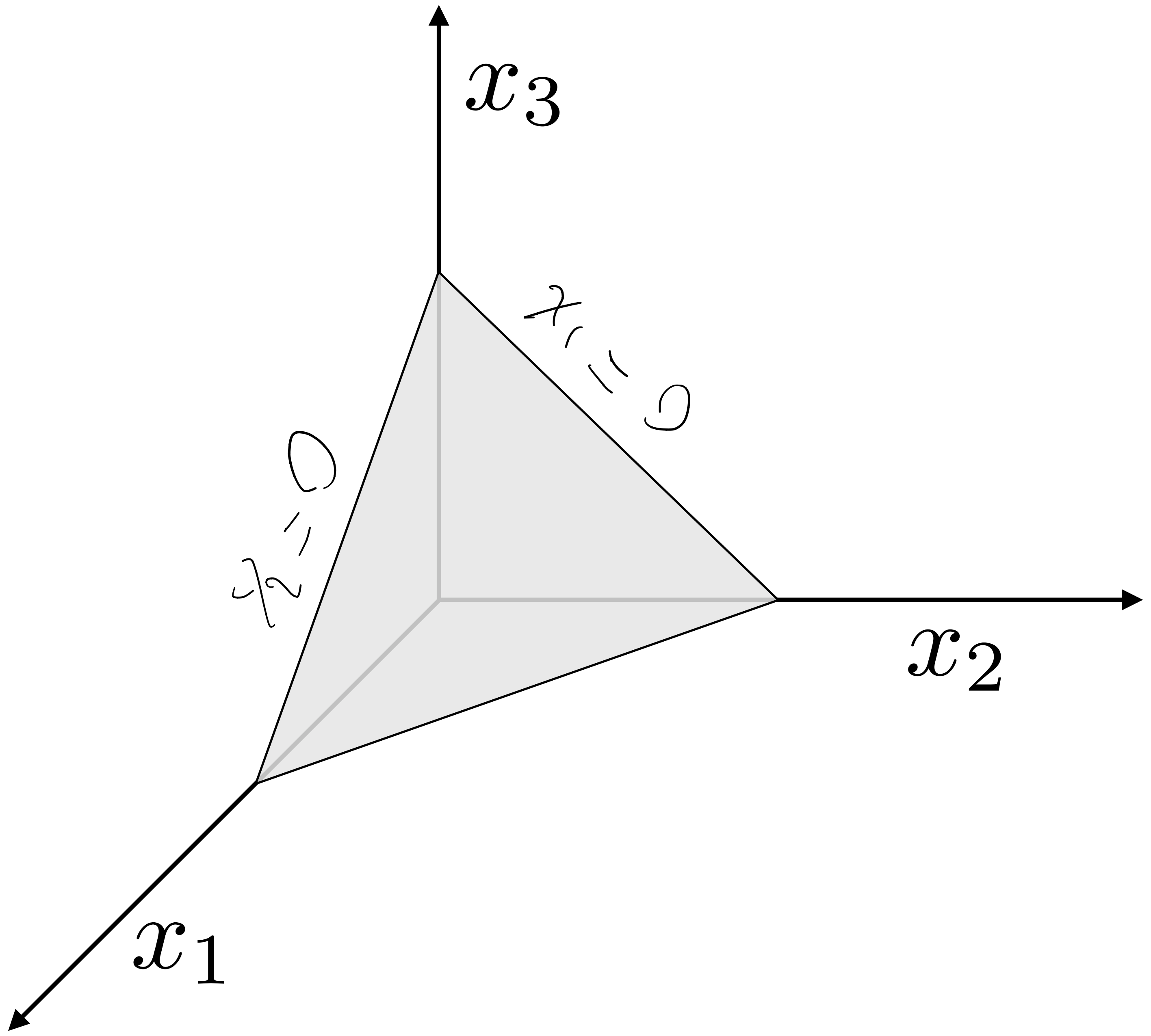
P lives in $(n - m)$-dimensional subspace

Standard form polyhedra

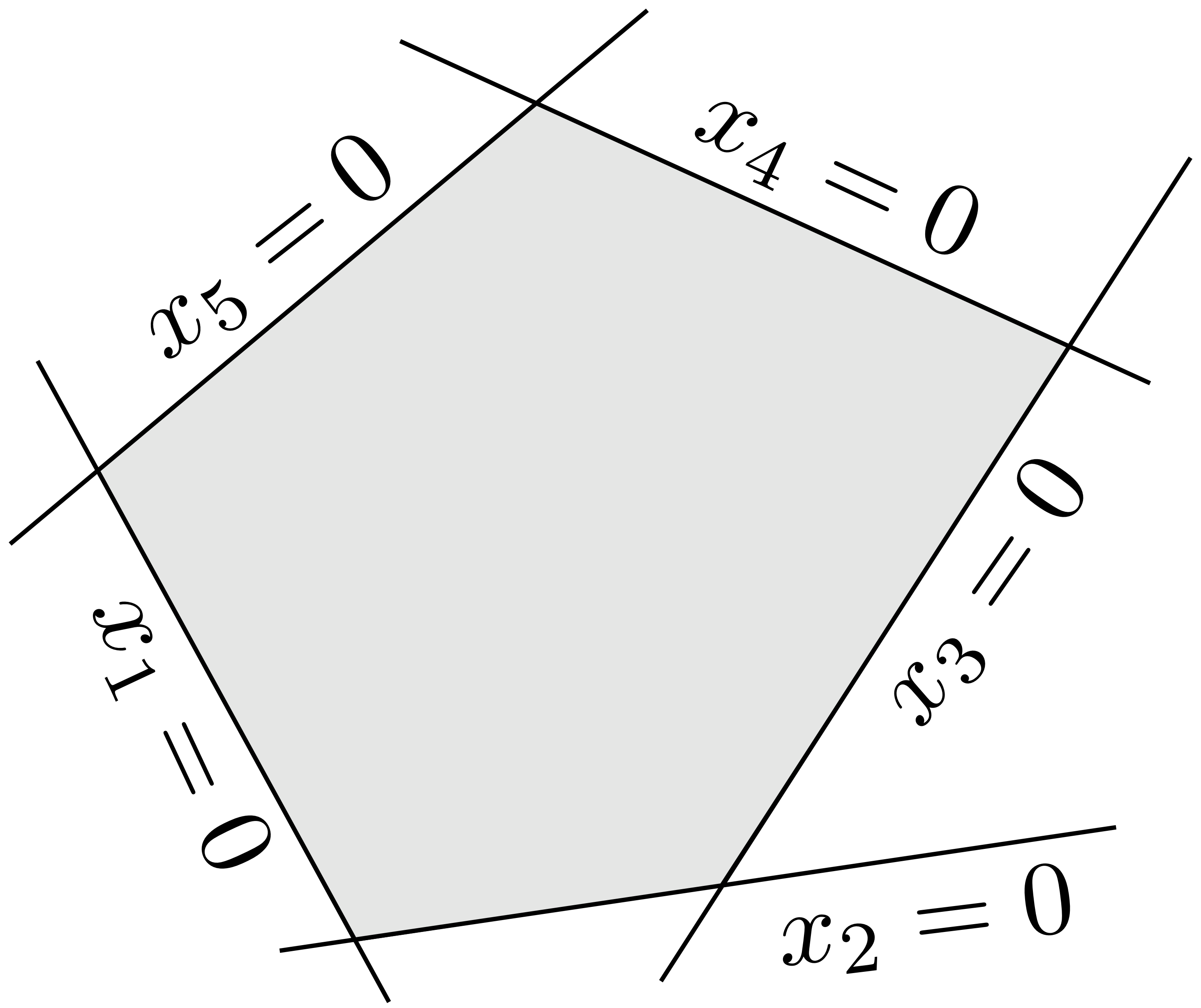
Visualization

$$P = \{x \mid Ax = b,\ x \geq 0\}, \quad n - m = 2$$

Three dimensions



Higher dimensions



Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

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Basis matrix

Basis columns

Basic variables

$$B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \ldots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \quad x_B = B^{-1}b$$

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If $x_B \geq 0$, then x is a **basic feasible solution**

Feasible directions

Conditions

$$P = \{x \mid Ax = b,\ x \geq 0\}$$

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we have basic feasible solution x:

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Feasible direction d

- $A(x + \theta d) = b \Longrightarrow Ad = 0$
- $x + \theta d \geq 0$

Feasible directions

Computation

Nonbasic indices

- $d_j = 1 \xrightarrow{\qquad\qquad}$ **Basic direction**
- $d_k = 0, \ \forall k \notin \{j, B(1), \ldots, B(m)\}$

Feasible directions

Computation

Nonbasic indices

- $d_j = 1 \xrightarrow{\hspace{2cm}}$ **Basic direction**
- $d_k = 0,\ \forall k \notin \{j, B(1), \dots, B(m)\}$

Basic indices

$$Ad = 0 = \sum_{i=1}^{n} A_i d_i = B d_B + A_j = 0 \Longrightarrow d_B = -B^{-1} A_j$$

Feasible directions

Computation

Nonbasic indices

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Non-negativity (non-degenerate assumption)

- Non-basic variables: $x_i = 0$. Nonnegative direction $d_i \geq 0$
- Basic variables: $x_B > 0$. Therefore $\exists \theta > 0$ such that $x_B + \theta d_B \geq 0$

Stepsize

What happens if some $\bar{c}_j$ < 0?

We can decrease the cost by bringing x_j into the basis

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How far can we go?

$\theta^\star = \max\{\theta \mid \theta \geq 0$ and $x + \theta d \geq 0\}$

d is the j-th basic direction

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Unbounded

If $d \geq 0$, then $\theta^\star = \infty$. The LP is unbounded.

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d is the j-th basic direction

Unbounded

If $d \geq 0$, then $\theta^\star = \infty$. The LP is unbounded.

Bounded

If $d_i < 0$ for some i, then

$$\theta^\star = \min_{\{i \mid d_i < 0\}} \left(-\frac{x_i}{d_i}\right) = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i}\right)$$

(Since $d_i \geq 0,\ i \in N$)

Moving to a new basis

Next feasible solution

$$x + \theta^\star d$$

Moving to a new basis

Next feasible solution

$$x + \theta^\star d$$

Let $B(\ell) \in \{B(1), \ldots, B(m)\}$ be the index such that $\theta^\star = -\dfrac{x_{B(\ell)}}{d_{B(\ell)}}$. Then,

$$x_{B(\ell)} + \theta^\star d_{B(\ell)} = 0$$

Moving to a new basis

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New solution

- $x_{B(\ell)}$ becomes 0 (exits)
- x_j becomes $\theta^\star$ (enters)

Moving to a new basis

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New solution

- $x_{B(\ell)}$ becomes 0 (exits)
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New basis

$$\bar{B} = \begin{bmatrix} A_{B(1)} & \dots & A_{B(\ell-1)} & A_j & A_{B(\ell+1)} & \dots & A_{B(m)} \end{bmatrix}$$

An iteration of the simplex method

First part

We start with a basic feasible solution x and a basis matrix $B = \begin{bmatrix} A_{B(1)} & \ldots, A_{B(m)} \end{bmatrix}$

An iteration of the simplex method

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We start with a basic feasible solution x and a basis matrix $B = \begin{bmatrix} A_{B(1)} & \ldots, A_{B(m)} \end{bmatrix}$

1. Compute the reduced costs $\bar{c}_j = c_j - c_B^T B^{-1} A_j$ for $j \in N$
2. If $\bar{c_j} \geq 0$, x **optimal**. **break**
3. Choose j such that $\bar{c}_j < 0$

An iteration of the simplex method

Second part

4. Compute search direction components $d_B = -B^{-1}A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min\limits_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^\star d$

Today’s agenda

- Find initial feasible solution
- Degeneracy
- Complexity

Find an initial point in simplex method

Initial basic feasible solution

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

How do we get an initial **basic feasible solution** x and a **basis** B ?

Does it **exist**?

Finding an initial basic feasible solution

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
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Finding an initial basic feasible solution

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \longrightarrow \quad \begin{array}{ll} \multicolumn{2}{c}{\textbf{Auxiliary problem}} \\ \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & Ax + y = b \\ & x \geq 0, y \geq 0 \end{array}$$

Finding an initial basic feasible solution

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Auxiliary problem

Minimize violations

Finding an initial basic feasible solution

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

→

Auxiliary problem

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Minimize violations

Assumption $b \geq 0$ w.l.o.g. (if not multiply constraint by -1)

Trivial basic feasible solution: $x = 0, y = b$

Finding an initial basic feasible solution

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Auxiliary problem

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Possible outcomes

- **Feasible problem** (cost = 0): $y^\star = 0$ and $x^\star$ is a basic feasible solution
- **Infeasible problem** (cost > 0): $y^\star > 0$ are the violations

Two-phase simplex method

Phase I

1. Construct **auxiliary problem** such that $b \geq 0$
2. Solve auxiliary problem using simplex method starting from $(x, y) = (0, b)$
3. If the optimal value is greater than 0, **problem infeasible**. **break**.

Phase II

1. Recover original problem (drop variables y and restore original cost)
2. Solve original problem starting from the solution x ~~and~~ of the auxiliary problem and its basis B.

Big-M method

$$\begin{array}{ll} \text{minimize} & c^T x + M\mathbf{1}^T y \\ \text{subject to} & Ax + y = b \\ & x \geq 0, y \geq 0 \end{array}$$

Big-M method

Very large constant

$$\begin{array}{ll} \text{minimize} & c^T x + M \mathbf{1}^T y \\ \text{subject to} & Ax + y = b \\ & x \geq 0, y \geq 0 \end{array}$$

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Incorporate penalty in the cost

- We can still use $y = b \geq 0$ as initial basic feasible solution
- If the problem is **feasible**, y will not be in the basis.

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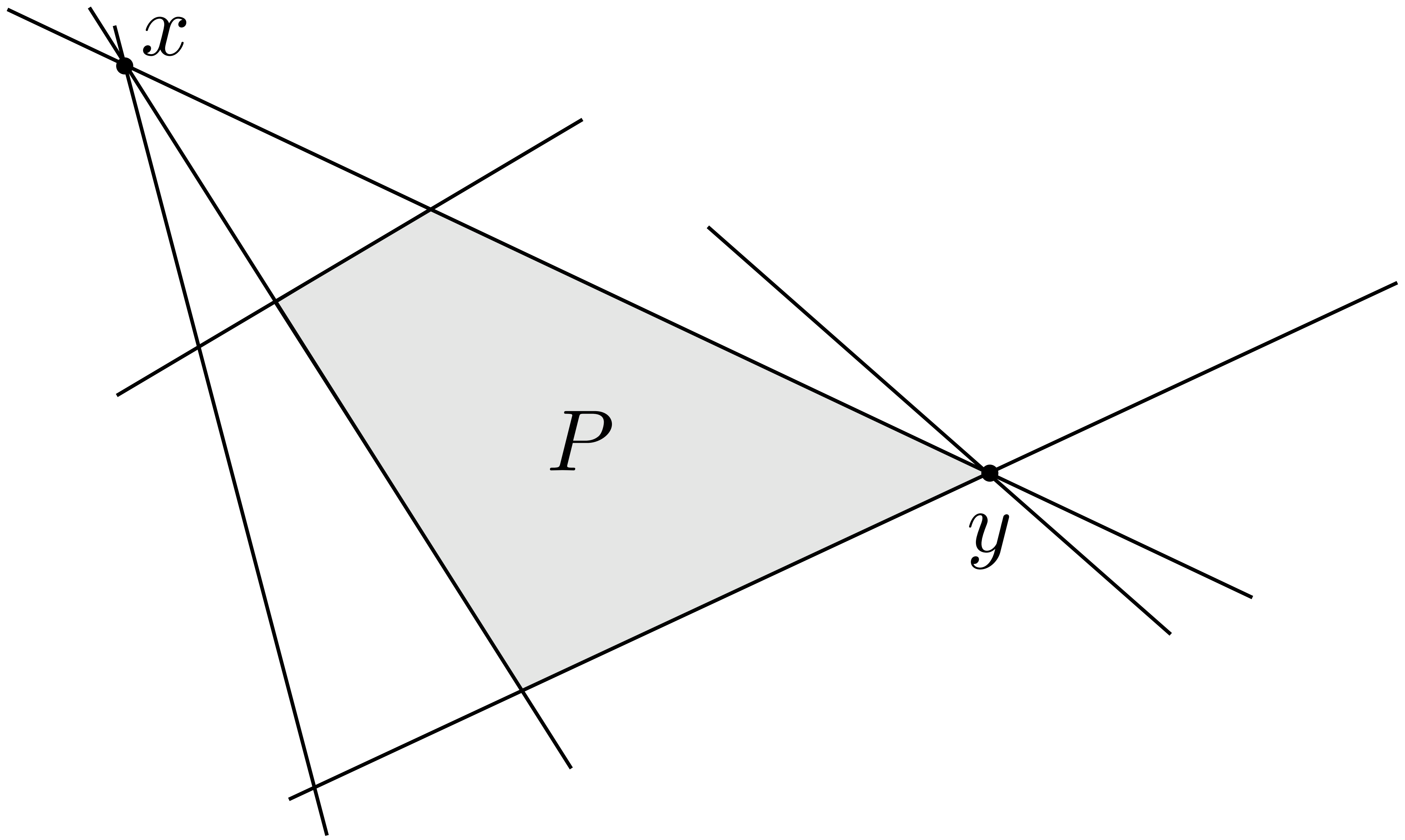
Remarks

- **Pro:** need to solve only one LP
- **Con:** it is not easy to pick M and it makes the problem badly scaled

Degeneracy

Degenerate basic feasible solutions

A solution $\bar{x}$ is degenerate if $|\mathcal{I}(\bar{x})| > n$



Degenerate basic feasible solutions

Definition

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we have basic feasible solution x:

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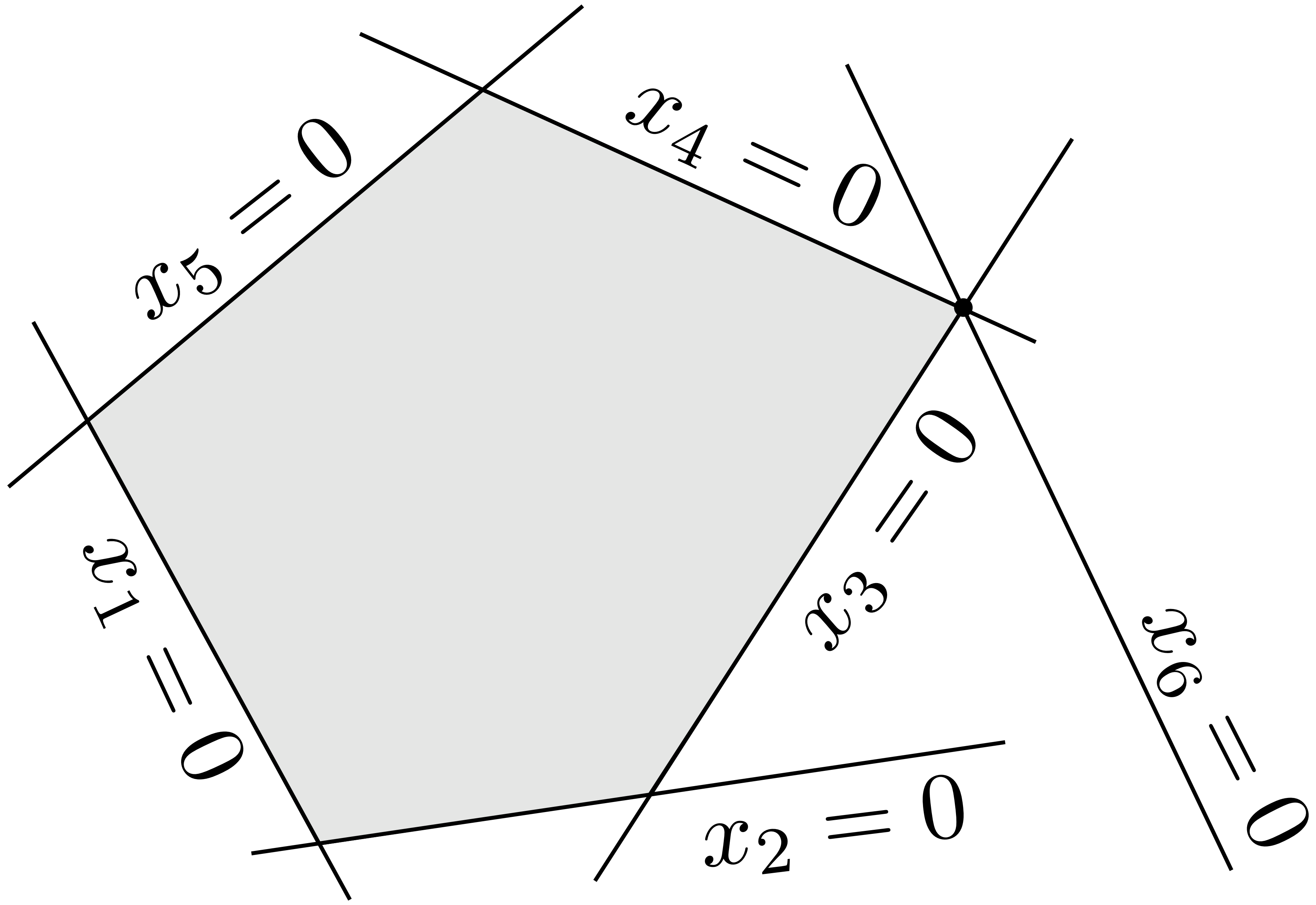

Degenerate basic feasible solutions

Example

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -x_1 + x_2 - x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

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Degenerate solutions

Basis $B = \{1, 2\}$ $\longrightarrow$ $x = (0, 1, 0)$

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Degenerate solutions

Basis $B = \{1, 2\}$ $\longrightarrow$ $x = (0, 1, 0)$

Basis $B = \{2, 3\}$ $\longrightarrow$ $y = (0, 1, 0)$

Cycling

Stepsize

$$\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

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$$\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right) \longrightarrow$$

If $i \in B$, $d_i < 0$ and $x_i = 0$ (degenerate)

$$\theta^\star = 0$$

Cycling

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Therefore $y = x + \theta^\star x = x$ and $\boxed{B} = \boxed{\bar{B}}$

Same solution and cost
Different basis

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Different basis

Finite termination **no longer guaranteed**!

How can we fix it?

Cycling

Stepsize

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Pivoting rules

Pivoting rules

Choose the index entering the basis

Simplex iterations

3. Choose j such that $\bar{c}_j < 0$

Pivoting rules

Choose the index entering the basis

Simplex iterations

3. Choose j such that $\bar{c}_j < 0$ ⟶ **Which j?**

Pivoting rules

Choose the index entering the basis

Simplex iterations

3. Choose j such that $\bar{c}_j < 0$ $\longrightarrow$ **Which j?**

Possible rules

- **Smallest subscript:** smallest j such that $\bar{c}_j < 0$
- **Most negative:** choose j with the most negative $\bar{c}_j$
- **Largest cost decrement:** choose j with the largest $\theta^\star|\bar{c}_j|$

Pivoting rules

Choose index exiting the basis

Simplex iterations

6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$

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We can have more than one i for which $x_i = 0$ (**next** solution is **degenerate**)

Which i?

Pivoting rules

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Simplex iterations

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We can have more than one i for which $x_i = 0$ (**next** solution is **degenerate**)

Which i?

Smallest index rule

Smallest i such that $\theta^\star = -\frac{x_i}{d_i}$

Bland’s rule to avoid cycles

Theorem

If we use the **smallest index rule** for choosing both the j entering the basis and the i leaving the basis, then **no cycling will occur**.

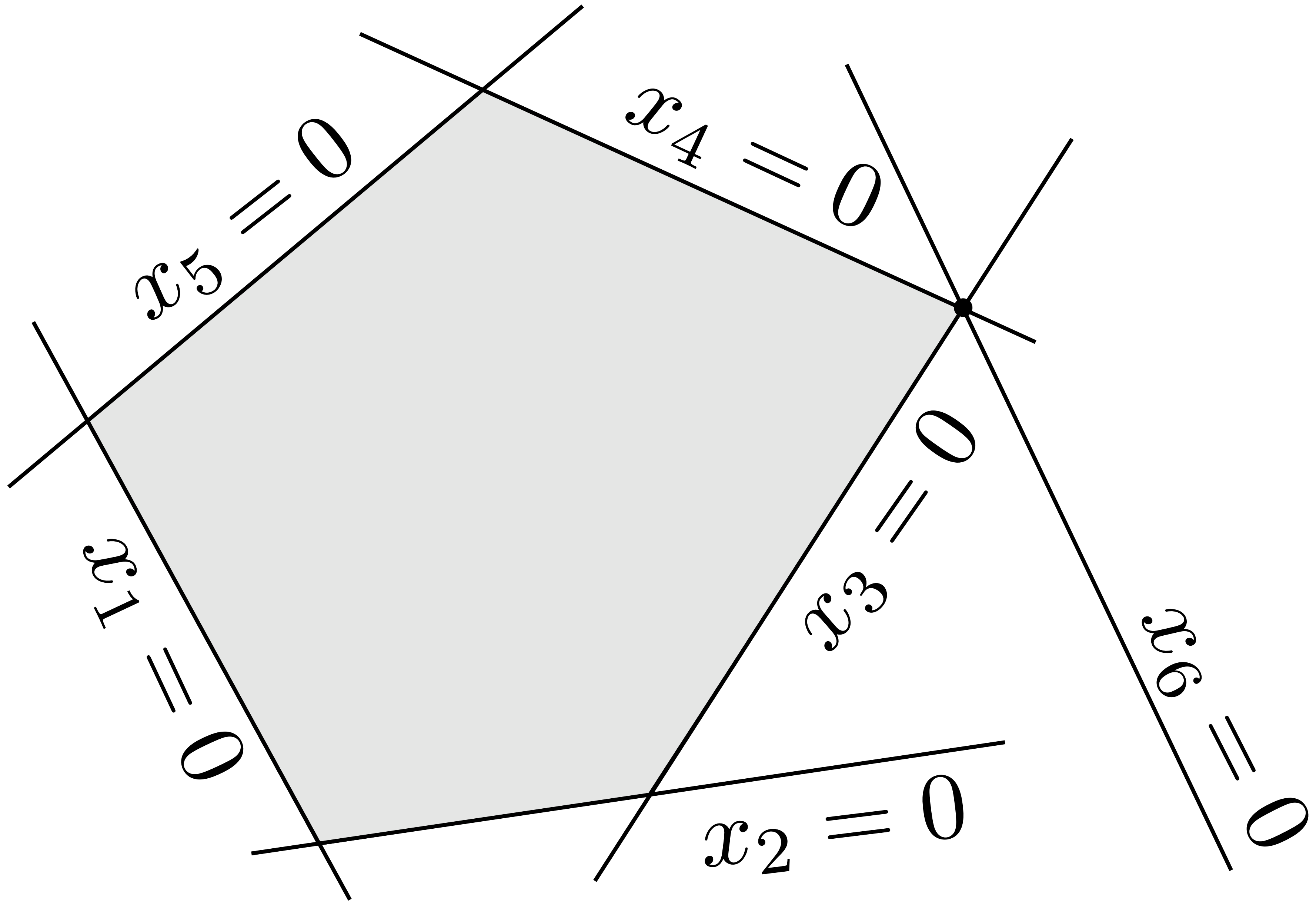
Bland’s rule to avoid cycles

Theorem
If we use the **smallest index rule** for choosing both the j entering the basis and the i leaving the basis, then **no cycling will occur**.

Proof idea (left as exercise)

- Assume that Bland’s rule is applied and there exists a cycle with different bases.
- Obtain same basis.

Perturbation approach to avoid cycles



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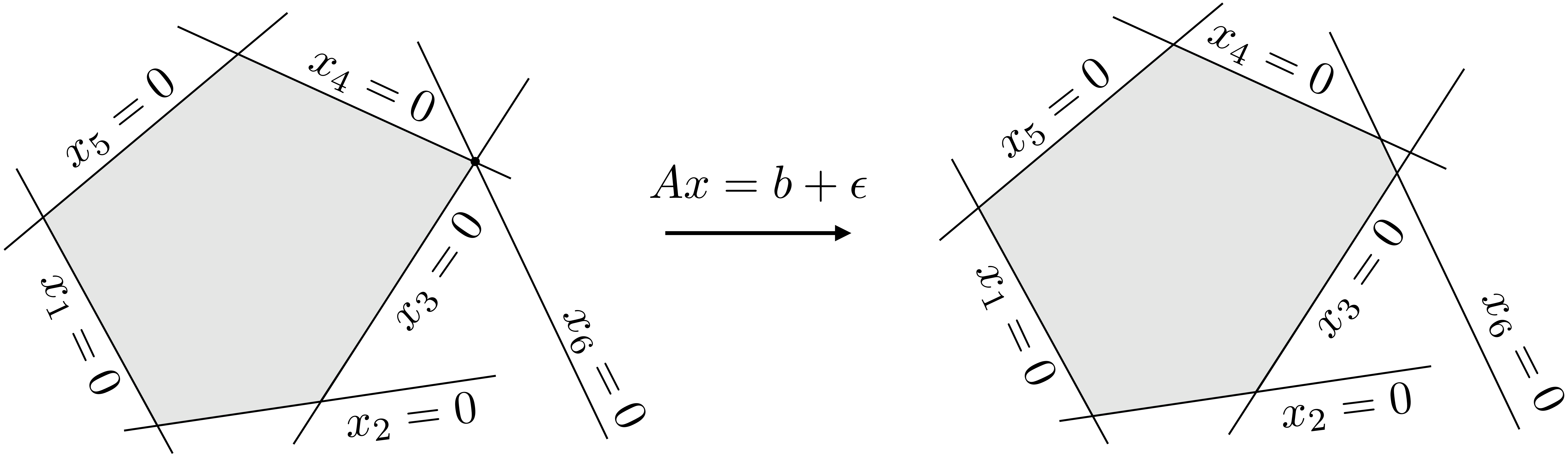

Complexity

Complexity

Basic operation: one simplex iteration

Estimate complexity of an algorithm

- Write number of basic operations as a **function of problem dimensions**
- Simplify and keep only leading terms

Complexity

Notation

We write $g(x) \sim O(f(x))$ if and only if there exist $c > 0$ and an x_0 such that

$$|g(x)| \leq cf(x), \quad \forall x \geq x_0$$

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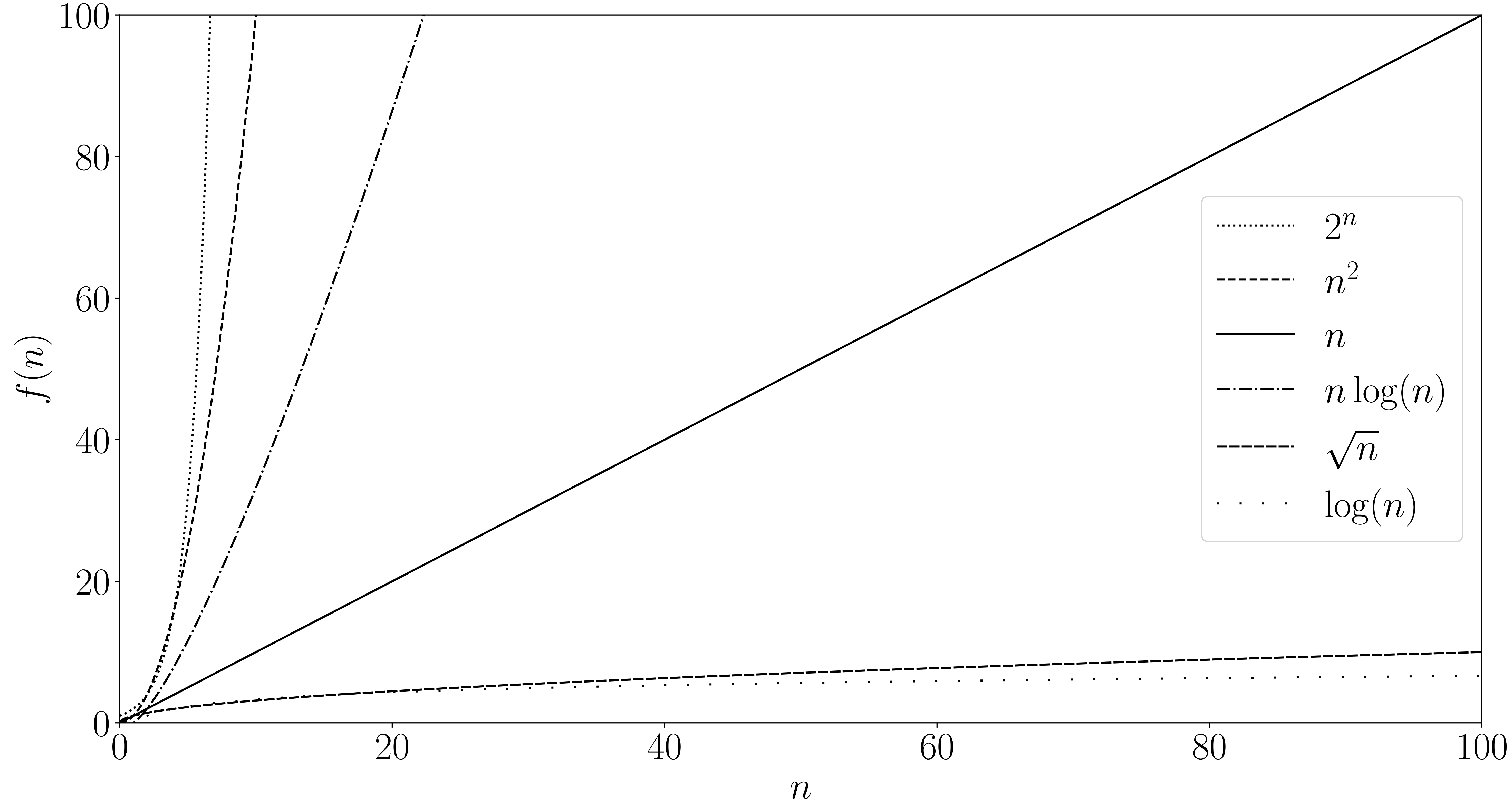

Polynomial
Practical

Exponential
Impractical!

$\mathcal{P}$ and $\mathcal{NP}$

Complexity class $\mathcal{P}$
There exists a polynomial time algorithm to solve it.

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Given a candidate solution, there exists a polynomial time algorithm to verify it.

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Complexity class $\mathcal{NP}$-hard
The problem is at least as hard as the hardest problem in $\mathcal{NP}$.

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↓

We don’t know any **polynomial time algorithm**

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↓

We don't know any **polynomial time algorithm**

Million dollar problem: $\mathcal{P} = \mathcal{NP}$?

- We know that $\mathcal{P} \subset \mathcal{NP}$
- Does it exist a polynomial time algorithm for $\mathcal{NP}$-hard problems?

Complexity of the simplex method

Example of worst-case behavior

Innocent-looking problem

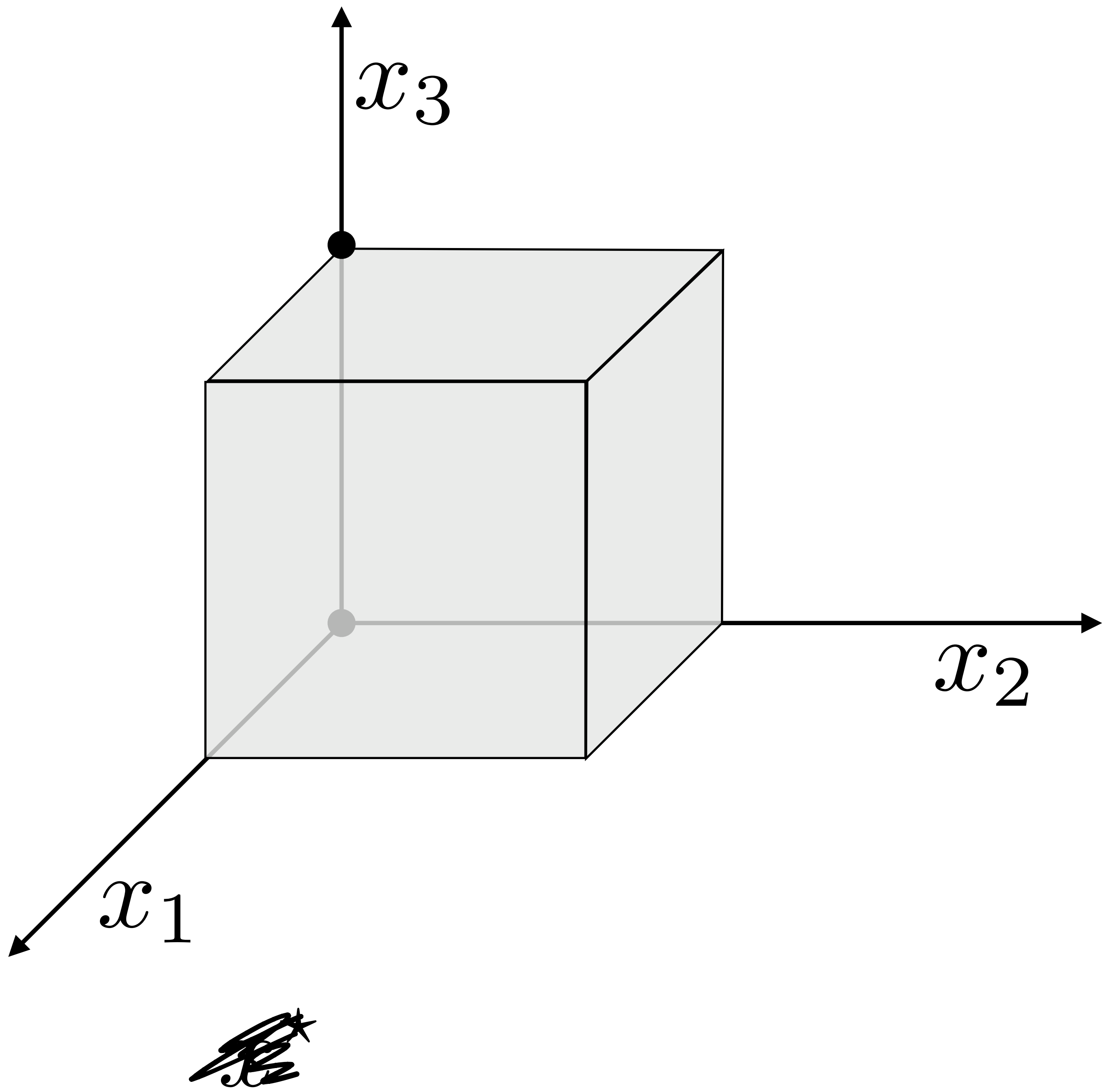
minimize $\quad -x_n$

subject to $\quad 0 \leq x \leq 1$

2^n **vertices**

$2^n/2$ vertices: cost = 1

$2^n/2$ vertices: cost = 0



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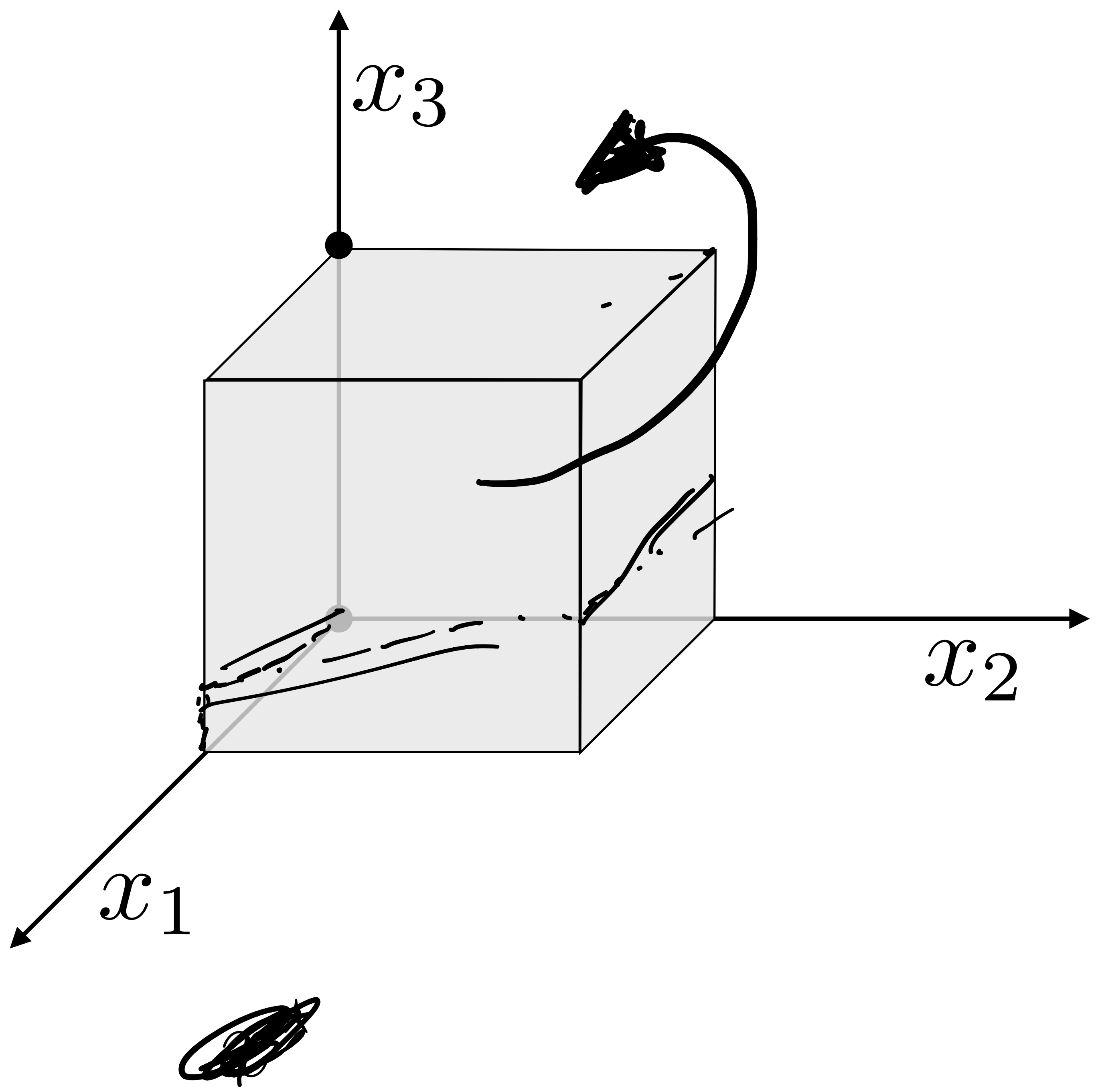

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minimize $\quad -x_n$

subject to $\quad 0 \leq x \leq 1$

2^n **vertices**

$2^n/2$ vertices: cost = 1

$2^n/2$ vertices: cost = 0

Perturb unit cube

minimize $\quad -x_n$

subject to $\quad \epsilon \leq x_1 \leq 1$

$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n$$

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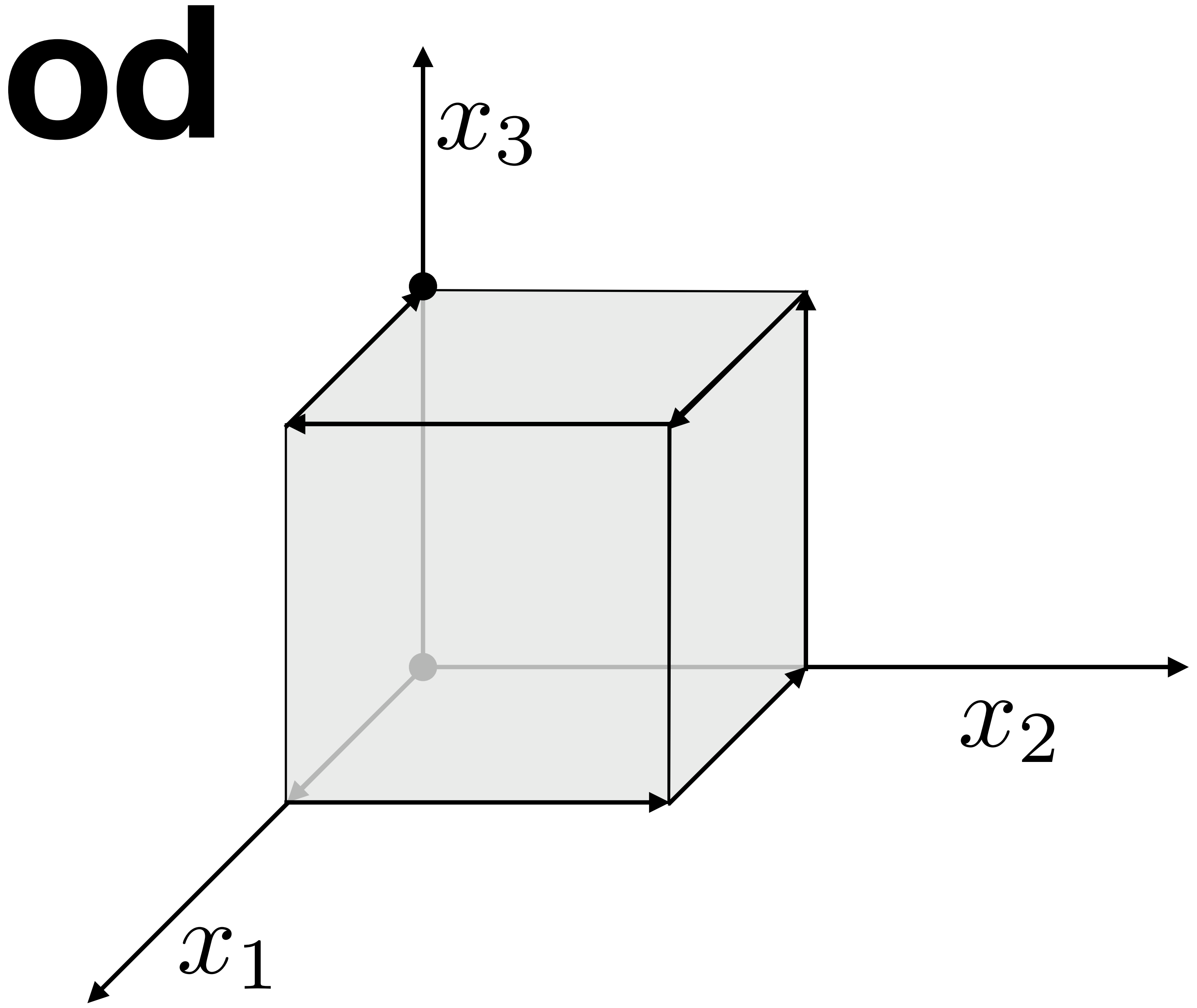
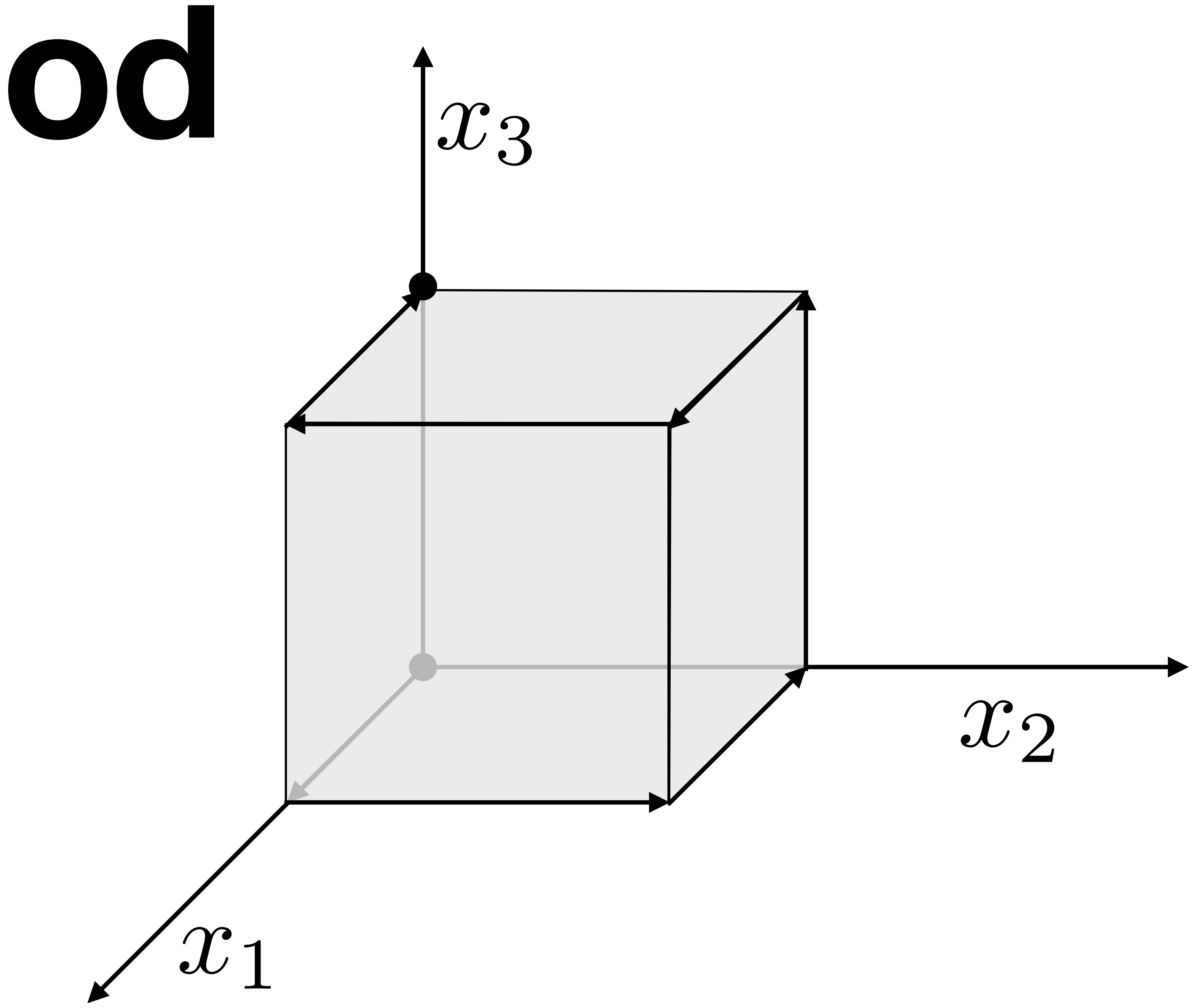


$$\begin{array}{ll} \text{minimize} & -x_n \\ \text{subject to} & \epsilon \le x_1 \le 1 \\ & \epsilon x_{i-1} \le x_i \le 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n \end{array}$$

Complexity of the simplex method

Example of worst-case behavior



$$\begin{array}{ll} \text{minimize} & -x_n \\ \text{subject to} & \epsilon \leq x_1 \leq 1 \\ & \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n \end{array}$$

Theorem

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations** $O(2^n)$

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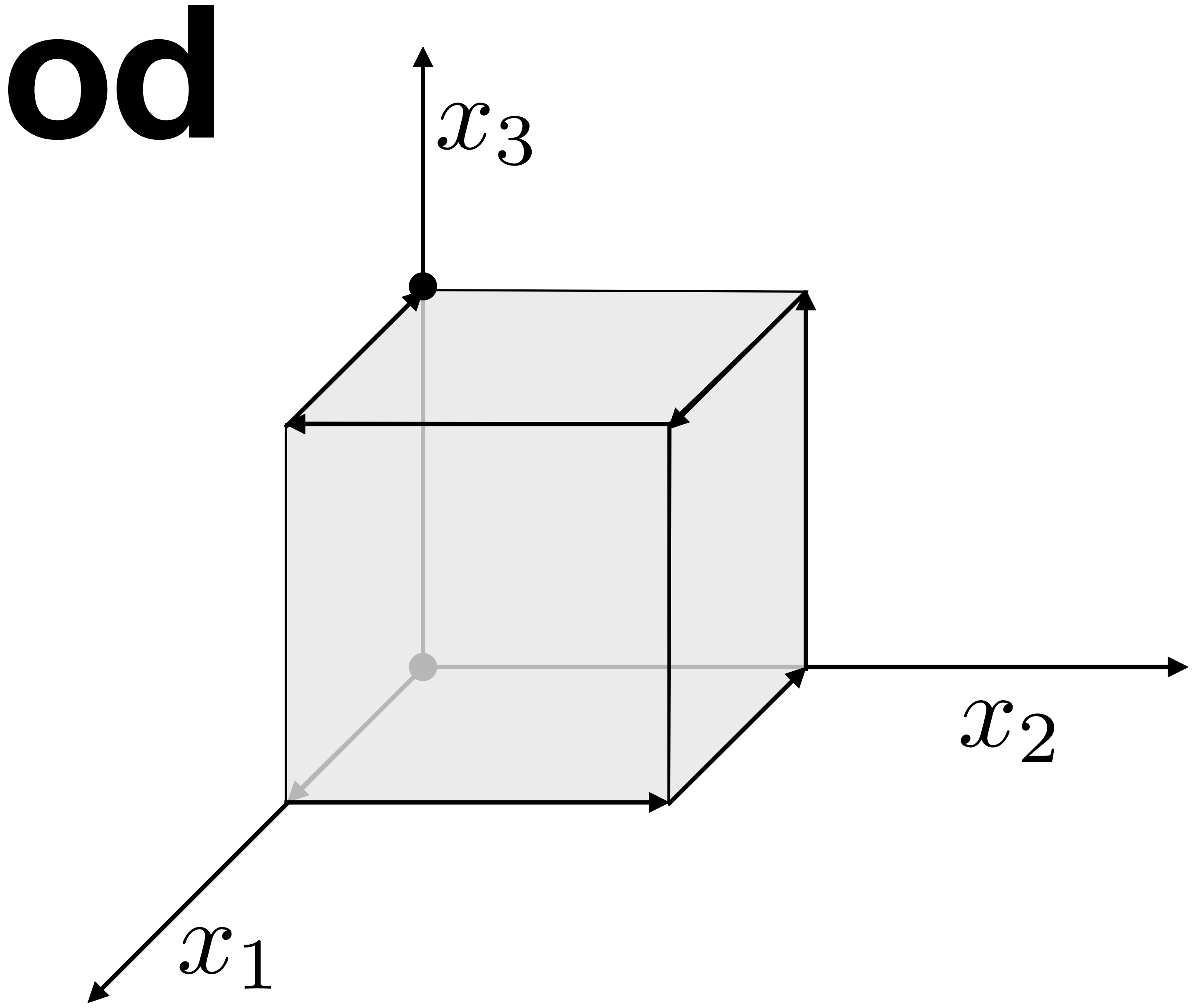

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Remark

- A **different pivot rule** would have converged in one iteration.
- We have a bad example for every pivot rule.

Complexity of the simplex method

We do **not know any polynomial version of the simplex method**, no matter which pivoting rule we pick.

→

Still **open research question!**

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Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions n and m: $O(2^n)$

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Good news: average-case

Practical performance is very good. On average, it stops in $O(n)$ iterations.

The simplex method

Today, we learned to:

- **Formulate** auxiliary problem to find starting simplex solutions
- **Apply** pivoting rules to avoid cycling in degenerate linear programs
- **Analyze** complexity of the simplex method

Next lecture

- Numerical linear algebra
- “Realistic" simplex implementation
- Examples