

ORF522 – Linear and Nonlinear Optimization

5. The simplex method

Ed forum

- What is the geometric picture of the standard form? Given a standard form P , can we always convert it back to the version defined by halfspaces? (next slides)
- Extent to which these methods are generalizable to infinite-dimensional restrictions e.g. linear difference equations when t goes to infinity.
- Efficient way to deal with inverses? (next lecture)
- How to pick index entering the basis? (this lecture)
- Adjacent solutions why defined that way? (Same active constraints except 1)
- Feasibility LP condition in no strong arbitrage from Arrow-Debreu theory (That's correct! We will discuss feasibility in duality lectures)

Recap

Standard form polyhedra

Definition

Standard form LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

Assumption

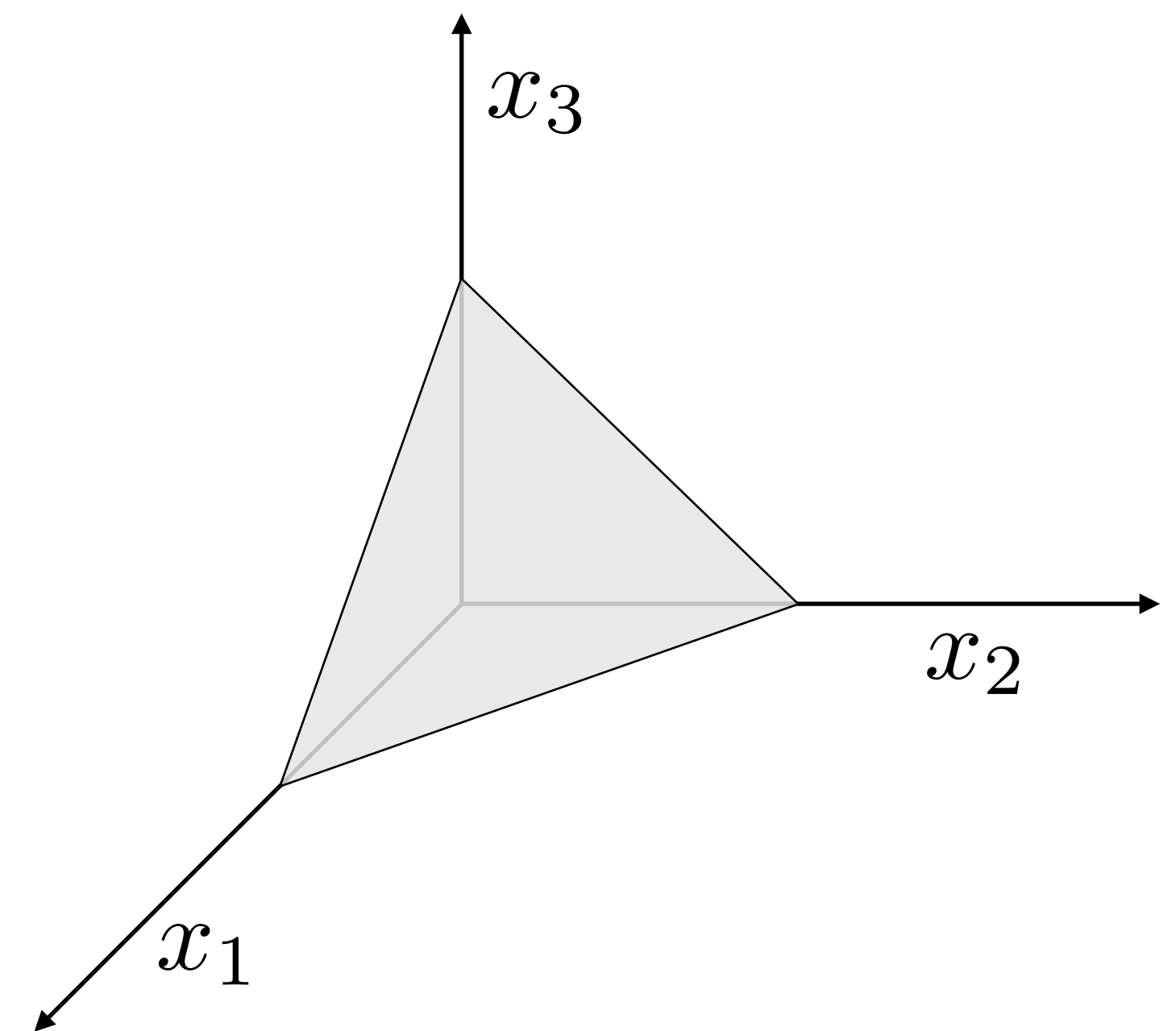
$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P lives in $(n - m)$ -dimensional subspace

Standard form polyhedron

$$P = \{x \mid Ax = b, x \geq 0\}$$

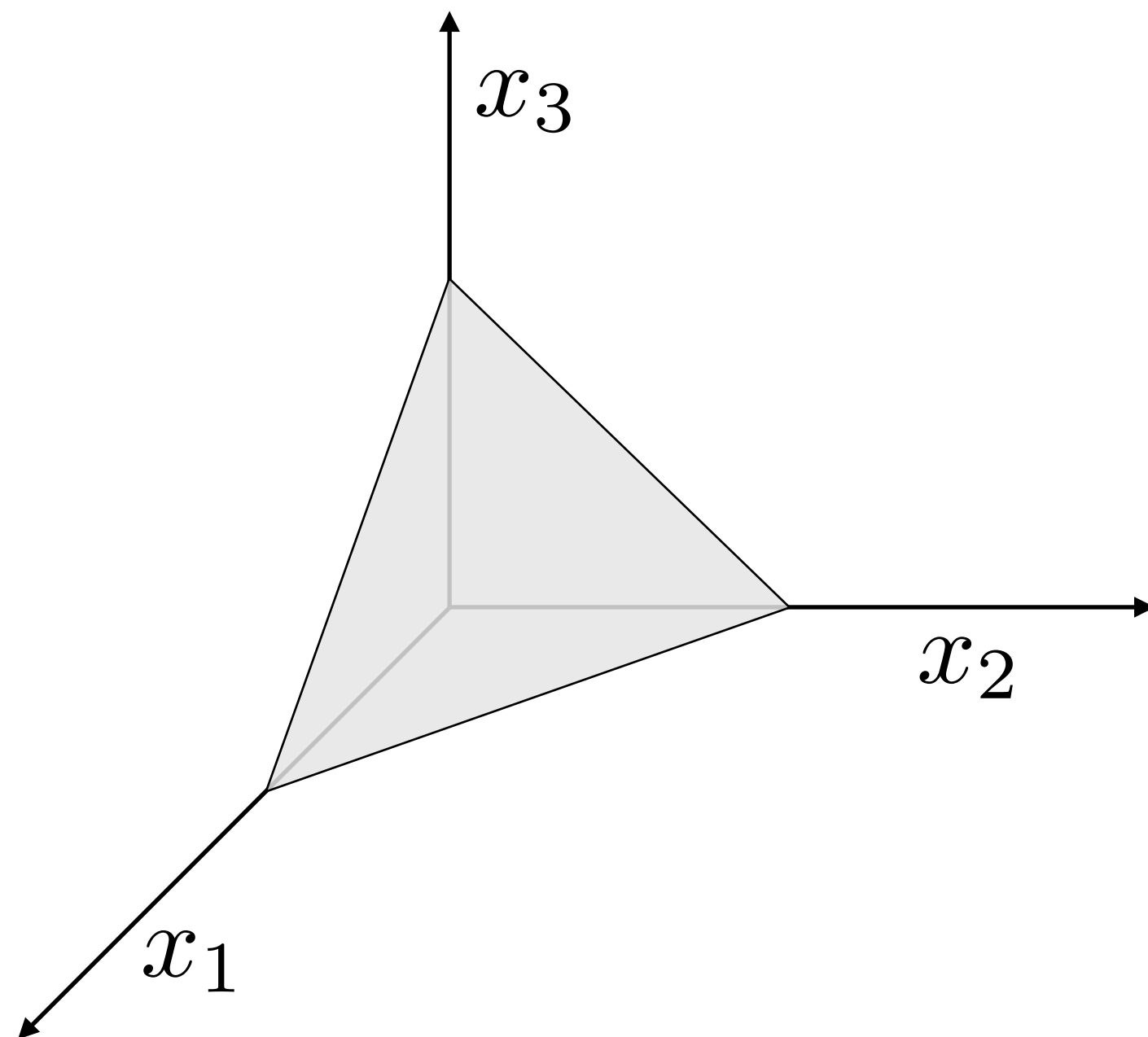


Standard form polyhedra

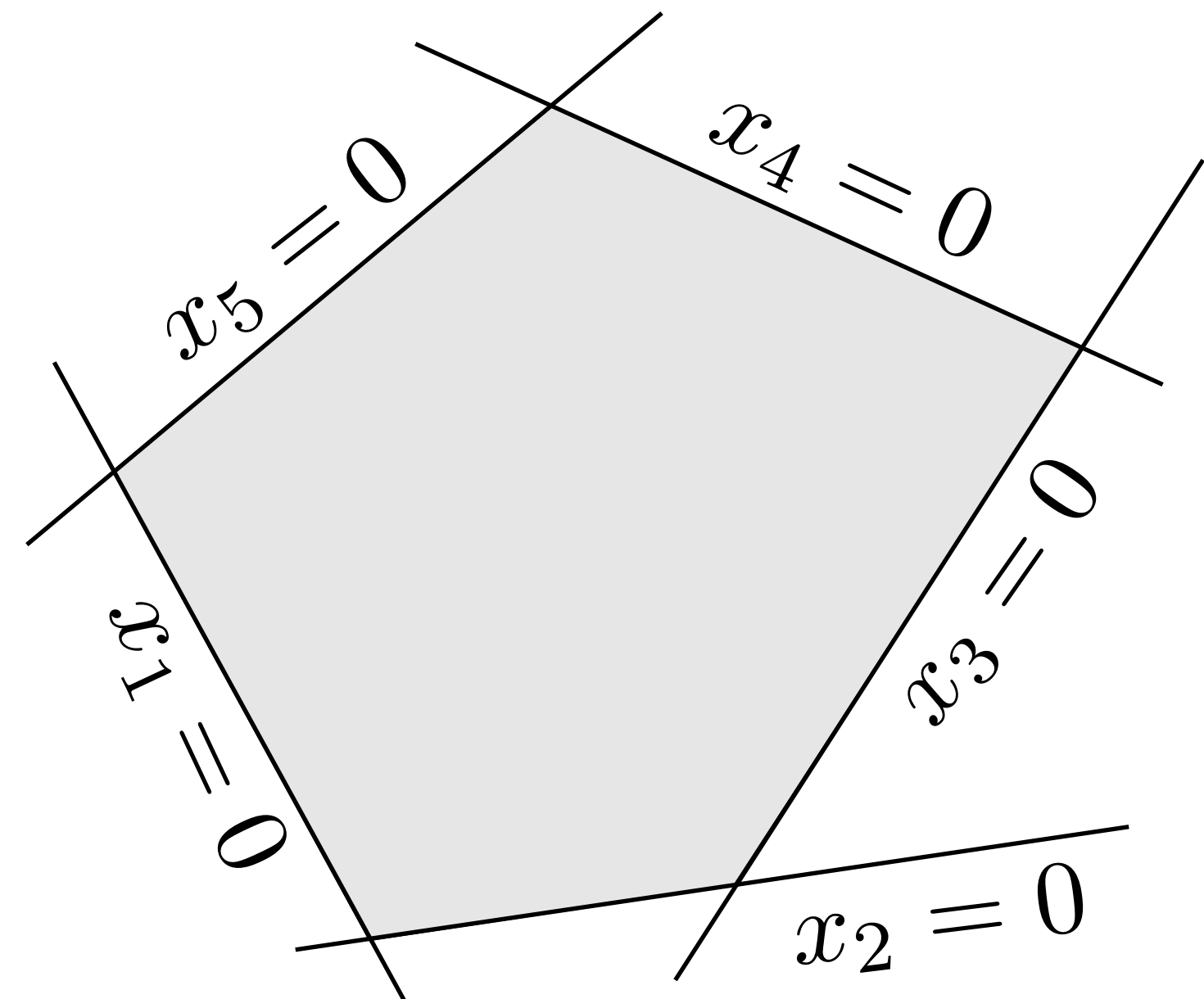
Visualization

$$P = \{x \mid Ax = b, x \geq 0\}, \quad n - m = 2$$

Three dimensions



Higher dimensions



Constructing basic solution

1. Choose any m independent columns of A : $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

$$\begin{array}{c} \text{Basis} \\ \text{matrix} \end{array} \quad \begin{array}{c} \text{Basis columns} \end{array} \quad \begin{array}{c} \text{Basic variables} \end{array}$$

$$B = \left[\begin{array}{c|c|c|c} | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{array} \right], \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow x_B = B^{-1}b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

Feasible directions

Conditions

$$P = \{x \mid Ax = b, x \geq 0\}$$

Given a basis matrix $B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$
we have basic feasible solution x :

- $x_B = B^{-1}b$
- $x_i = 0, \forall i \neq B(1), \dots, B(m)$

Feasible direction d

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

Feasible directions

Computation

Nonbasic indices

- $d_j = 1 \longrightarrow$ **Basic direction**
- $d_k = 0, \forall k \notin \{j, B(1), \dots, B(m)\}$

Basic indices

$$Ad = 0 = \sum_{i=1}^n A_i d_i = Bd_B + A_j = 0 \implies d_B = -B^{-1}A_j$$

Non-negativity (non-degenerate assumption)

- Non-basic variables: $x_i = 0$. Nonnegative direction $d_i \geq 0$
- Basic variables: $x_B > 0$. Therefore $\exists \theta > 0$ such that $x_B + \theta d_B \geq 0$

Stepsize

What happens if some $\bar{c}_j < 0$?

We can decrease the cost by bringing x_j into the basis

How far can we go?

$$\theta^* = \max\{\theta \mid \theta \geq 0 \text{ and } x + \theta d \geq 0\} \quad d \text{ is the } j\text{-th basic direction}$$

Unbounded

If $d \geq 0$, then $\theta^* = \infty$. The LP is unbounded.

Bounded

If $d_i < 0$ for some i , then

$$\theta^* = \min_{\{i \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right) = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

(Since $d_i \geq 0$, $i \in N$)

Moving to a new basis

Next feasible solution

$$x + \theta^* d$$

Let $B(\ell) \in \{B(1), \dots, B(m)\}$ be the index such that $\theta^* = -\frac{x_{B(\ell)}}{d_{B(\ell)}}$. Then,

$$x_{B(\ell)} + \theta^* d_{B(\ell)} = 0$$

New solution

- $x_{B(\ell)}$ becomes 0 (exits)
- x_j becomes θ^* (enters)

New basis

$$\bar{B} = \begin{bmatrix} A_{B(1)} & \dots & A_{B(\ell-1)} & A_j & A_{B(\ell+1)} & \dots & A_{B(m)} \end{bmatrix}$$

An iteration of the simplex method

First part

We start with a basic feasible solution x and a basis matrix $B = \begin{bmatrix} A_{B(1)} & \dots, A_{B(m)} \end{bmatrix}$

1. Compute the reduced costs $\bar{c}_j = c_j - c_B^T B^{-1} A_j$ for $j \in N$
2. If $\bar{c}_j \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$

An iteration of the simplex method

Second part

4. Compute search direction components $d_B = -B^{-1}A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^* d$

Today's agenda

- Find initial feasible solution
- Degeneracy
- Complexity

**Find an initial point in simplex
method**

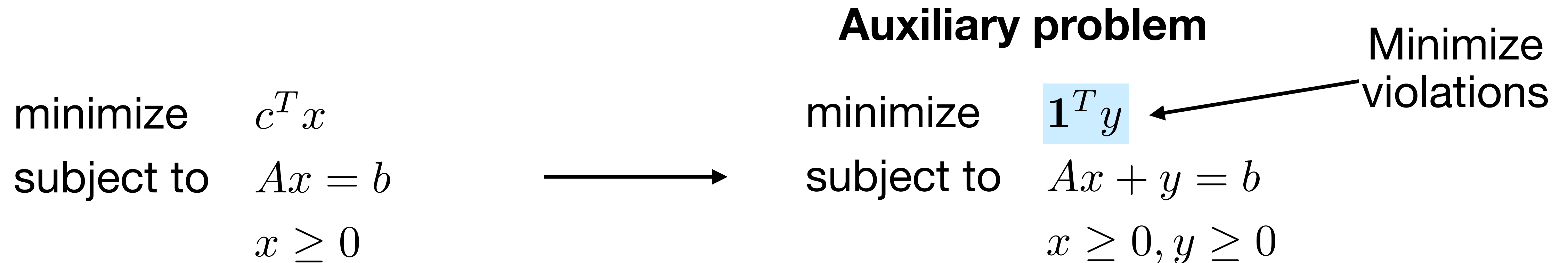
Initial basic feasible solution

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

How do we get an initial **basic feasible solution** x and a **basis** B ?

Does it **exist**?

Finding an initial basic feasible solution



Assumption $b \geq 0$ w.l.o.g. (if not multiply constraint by -1)

Trivial basic feasible solution: $x = 0, y = b$

Possible outcomes

- **Feasible problem** (cost = 0): $y^* = 0$ and x^* is a basic feasible solution
- **Infeasible problem** (cost > 0): $y^* > 0$ are the violations

Two-phase simplex method

Phase I

1. Construct **auxiliary problem** such that $b \geq 0$
2. Solve auxiliary problem using simplex method starting from $(x, y) = (0, b)$
3. If the optimal value is greater than 0, **problem infeasible. break.**

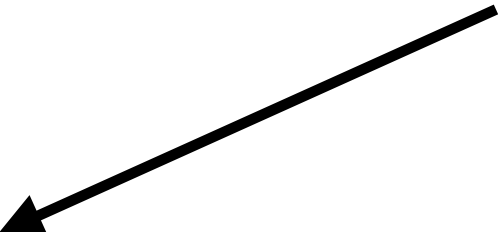
Phase II

1. Recover original problem (drop variables y and restore original cost)
2. Solve original problem starting from the solution x and its basis B .

Big-M method

$$\begin{array}{ll}\text{minimize} & c^T x + M \mathbf{1}^T y \\ \text{subject to} & Ax + y = b \\ & x \geq 0, y \geq 0\end{array}$$

Very large constant



Incorporate penalty in the cost

- We can still use $y = b \geq 0$ as initial basic feasible solution
- If the problem is **feasible**, y will not be in the basis.

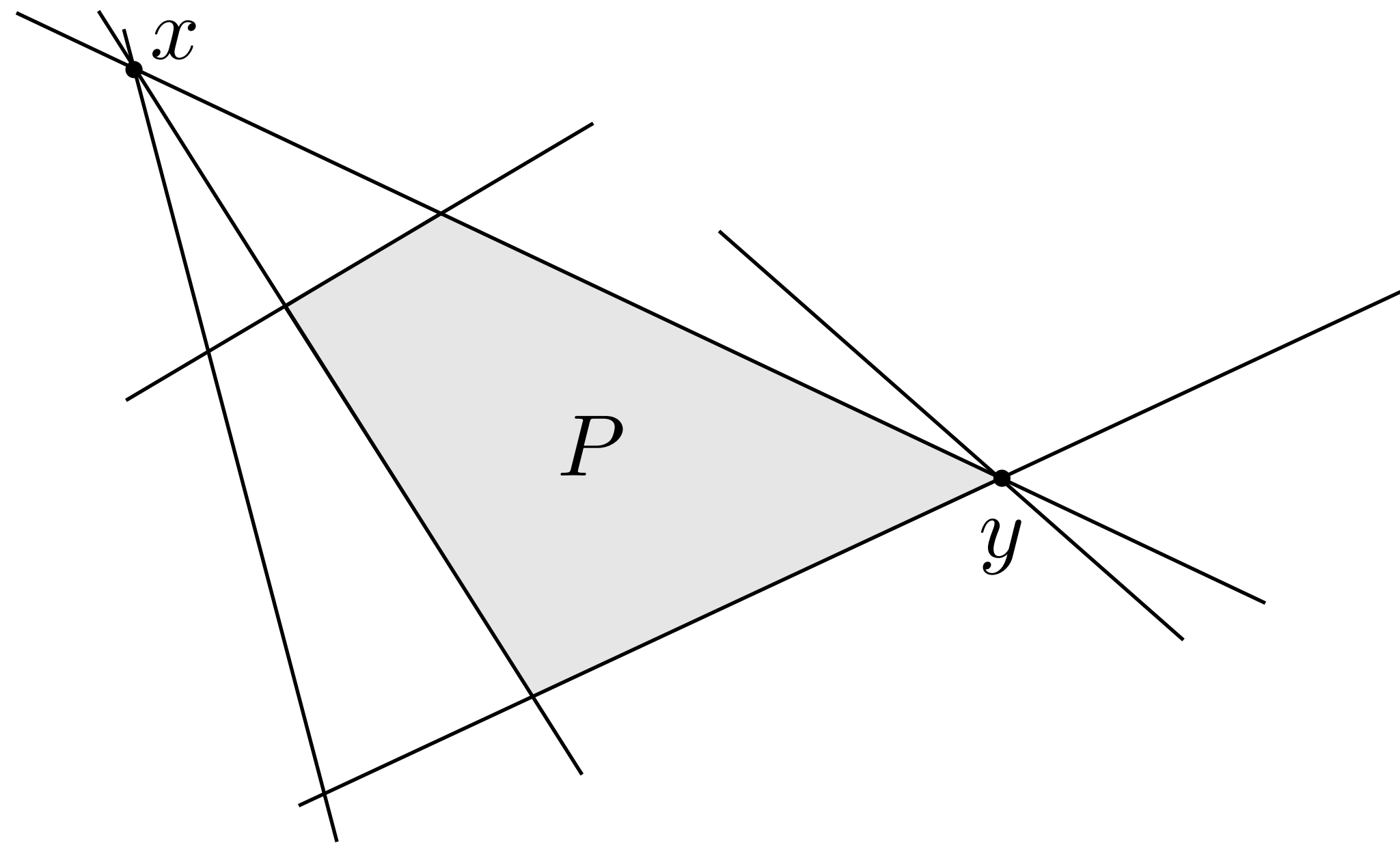
Remarks

- **Pro:** need to solve only one LP
- **Con:** it is not easy to pick M and it makes the problem badly scaled

Degeneracy

Degenerate basic feasible solutions

A solution \bar{x} is degenerate if $|\mathcal{I}(\bar{x})| > n$



Degenerate basic feasible solutions

Definition

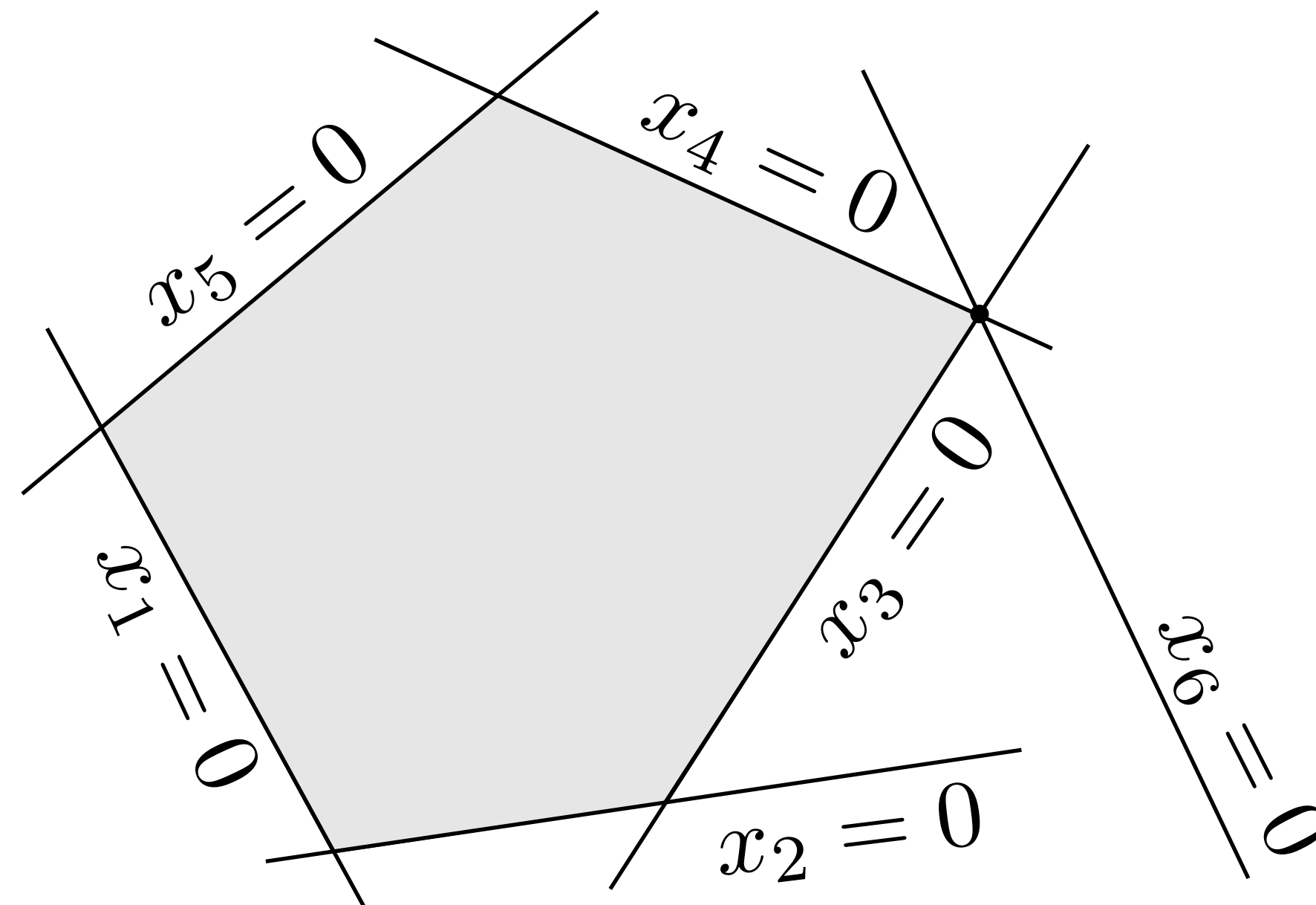
Given a basis matrix $B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

we have basic feasible solution x :

- $x_B = B^{-1}b$
- $x_i = 0, \forall i \neq B(1), \dots, B(m)$



If some of the $x_B = 0$, then it is a **degenerate solution**



Degenerate basic feasible solutions

Example

$$x_1 + x_2 + x_3 = 1$$

$$-x_1 + x_2 - x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Degenerate solutions

$$\text{Basis } B = \{1, 2\} \longrightarrow x = (0, 1, 0)$$

$$\text{Basis } B = \{2, 3\} \longrightarrow y = (0, 1, 0)$$

Cycling

Stepsize

$$\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right) \longrightarrow \begin{array}{l} \text{If } i \in B, d_i < 0 \text{ and } x_i = 0 \text{ (degenerate)} \\ \theta^* = 0 \end{array}$$

Therefore $y = x + \theta^* x = x$ and $B = \bar{B}$ **Same solution and cost**
Different basis

Finite termination **no longer guaranteed!**

How can we fix it?

Pivoting rules

Pivoting rules

Choose the index entering the basis

Simplex iterations

3. Choose j such that $\bar{c}_j < 0$ \longrightarrow Which j ?

Possible rules

- **Smallest subscript:** smallest j such that $\bar{c}_j < 0$
- **Most negative:** choose j with the most negative \bar{c}_j
- **Largest cost decrement:** choose j with the largest $\theta^* |\bar{c}_j|$

Pivoting rules

Choose index exiting the basis

Simplex iterations

6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right) \longrightarrow$

We can have more than one i for which $x_i = 0$
(**next** solution is **degenerate**)

Which i ?

Smallest index rule

Smallest i such that $\theta^* = -\frac{x_i}{d_i}$

Bland's rule to avoid cycles

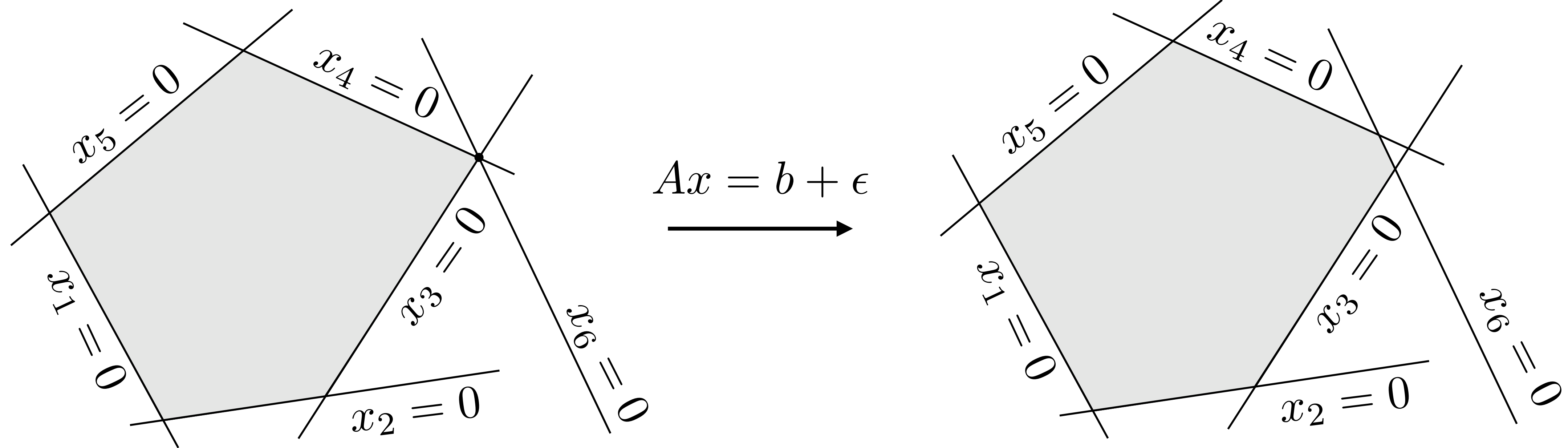
Theorem

If we use the **smallest index rule** for choosing both the j entering the basis and the i leaving the basis, then **no cycling will occur**.

Proof idea (left as exercise)

- Assume that Bland's rule is applied and there exists a cycle with different bases.
- Obtain same basis.

Perturbation approach to avoid cycles



Complexity

Complexity

Basic operation: one simplex iteration

Estimate complexity of an algorithm

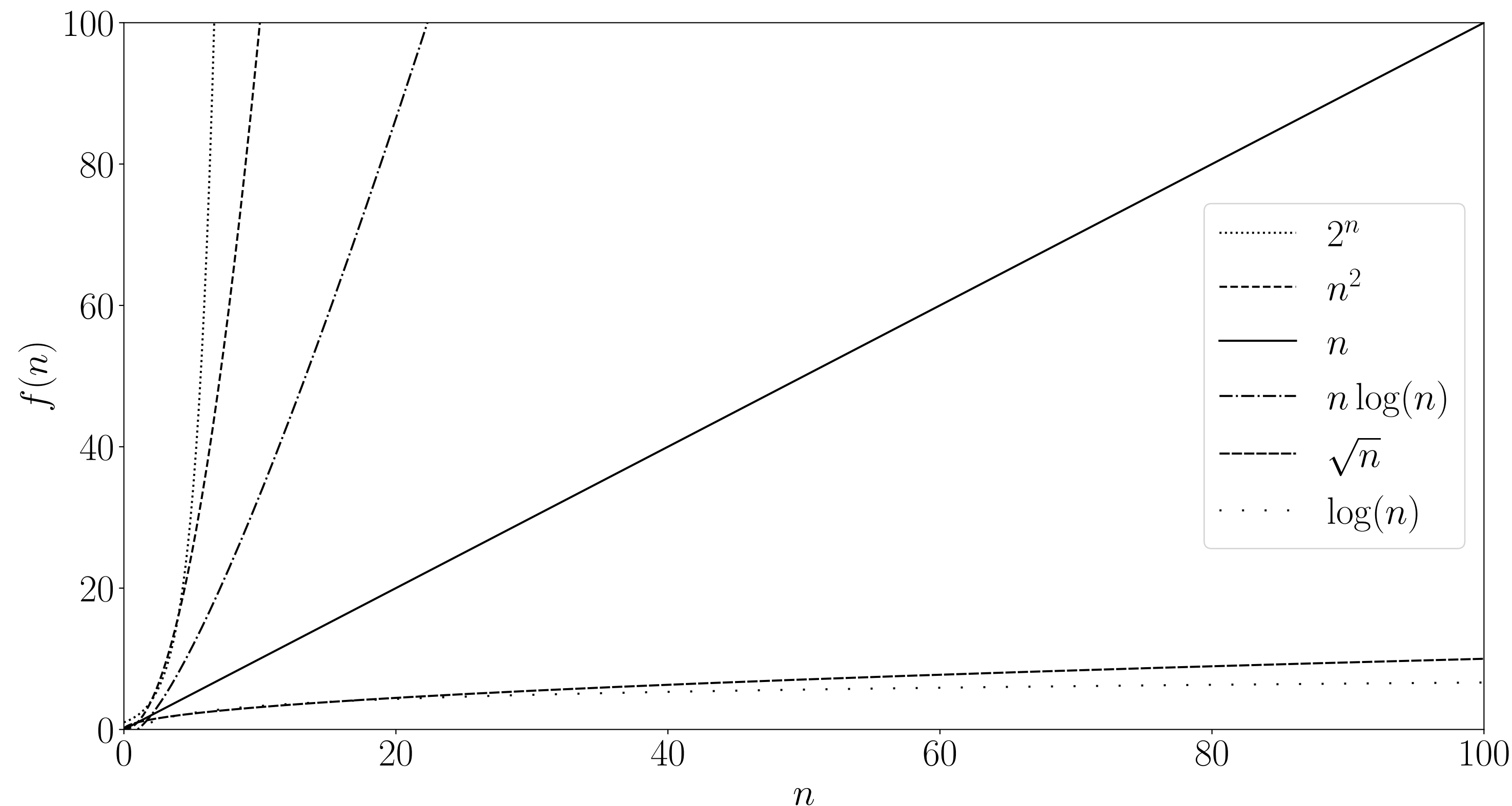
- Write number of basic operations as a **function of problem dimensions**
- Simplify and keep only leading terms

Complexity

Notation

We write $g(x) \sim O(f(x))$ if and only if there exist $c > 0$ and an x_0 such that

$$|g(x)| \leq cf(x), \quad \forall x \geq x_0$$



Polynomial
Practical

Exponential
Impractical!

\mathcal{P} and \mathcal{NP}

Complexity class \mathcal{P}

There exists a polynomial time algorithm to solve it.

Complexity class \mathcal{NP}

Given a candidate solution, there exists a polynomial time algorithm to verify it.

Complexity class \mathcal{NP} -hard

The problem is at least as hard as the hardest problem in \mathcal{NP} .



We don't know any **polynomial time algorithm**

Million dollar problem: $\mathcal{P} = \mathcal{NP}$?

- We know that $\mathcal{P} \subset \mathcal{NP}$
- Does it exist a polynomial time algorithm for \mathcal{NP} -hard problems?

Complexity of the simplex method

Example of worst-case behavior

Innocent-looking problem

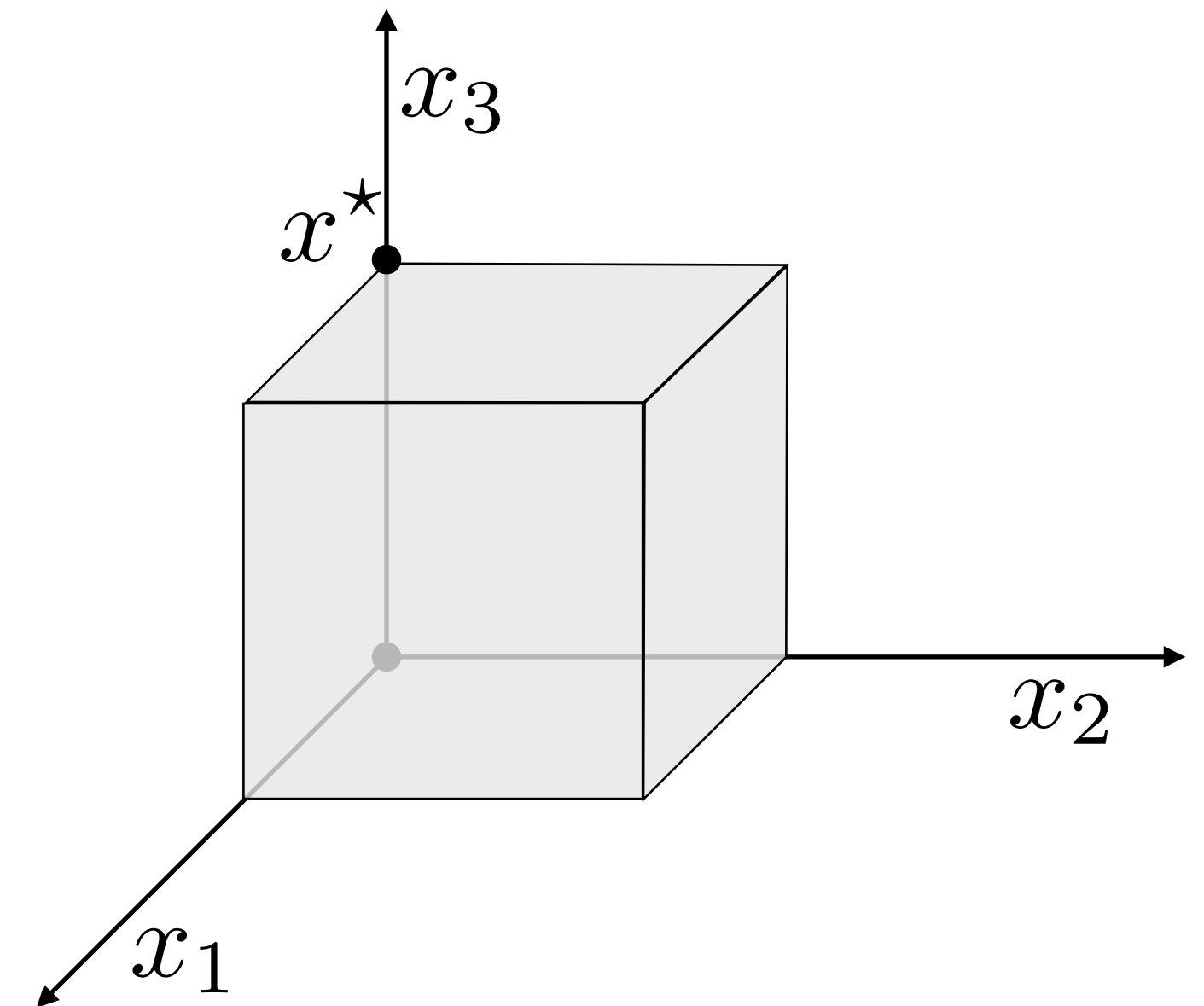
minimize $-x_n$

subject to $0 \leq x \leq 1$

2^n vertices

$2^n/2$ vertices: cost = 1

$2^n/2$ vertices: cost = 0



Perturb unit cube

minimize $-x_n$

subject to $\epsilon \leq x_1 \leq 1$

$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n$

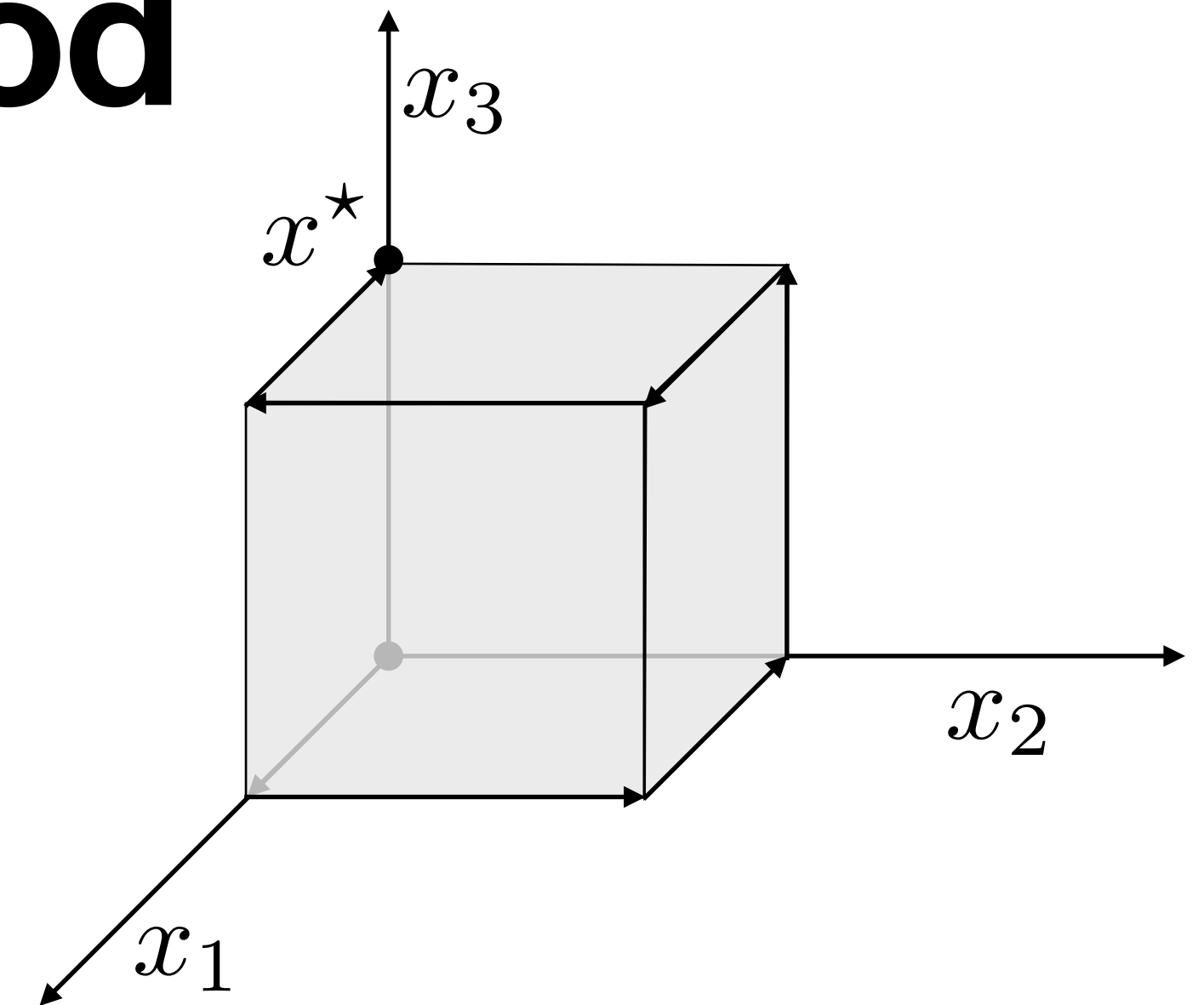
Complexity of the simplex method

Example of worst-case behavior

minimize $-x_n$

subject to $\epsilon \leq x_1 \leq 1$

$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n$$



Theorem

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations**

Remark

- A **different pivot rule** would have converged in one iteration.
- We have a bad example for every pivot rule.

Complexity of the simplex method

We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick.



Still open research question!

Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions n and m : $O(2^n)$

Good news: average-case

Practical performance is very good. On average, it stops in $O(n)$ iterations.

The simplex method

Today, we learned to:

- **Formulate** auxiliary problem to find starting simplex solutions
- **Apply** pivoting rules to avoid cycling in degenerate linear programs
- **Analyze** complexity of the simplex method

Next lecture

- Numerical linear algebra
- “Realistic” simplex implementation
- Examples