

ORF522 – Linear and Nonlinear Optimization

3. Geometry and polyhedra

Bartolomeo Stellato – Fall 2020

Ed forum

Questions and notes

- Is piecewise linear optimization always better than solving a convex problem?
- 1-norm LP ($\text{minimize}_{Ax=b} \|x\|_1$) or LASSO ($\text{minimize } \|Ax - b\|_2 + \|x\|_1$)?
- $\ell_1 - \ell_0$ equivalence? Not always!
- “random matrix” on slide 21 (Sparse Signal Recovery example) means “randomly-generated” matrix.
- Robust curve fitting. What does “robustness” mean?
- Would a convex combination of ℓ_1 and ℓ_0 norm work better than any one of them; and be computationally feasible to work with?

Today's agenda

Readings [Chapter 2, Bertsimas and Tsitsiklis]

- Polyhedra and linear algebra
- Corners: extreme points, vertices, basic feasible solutions
- Constructing basic solutions
- Existence and optimality of extreme points

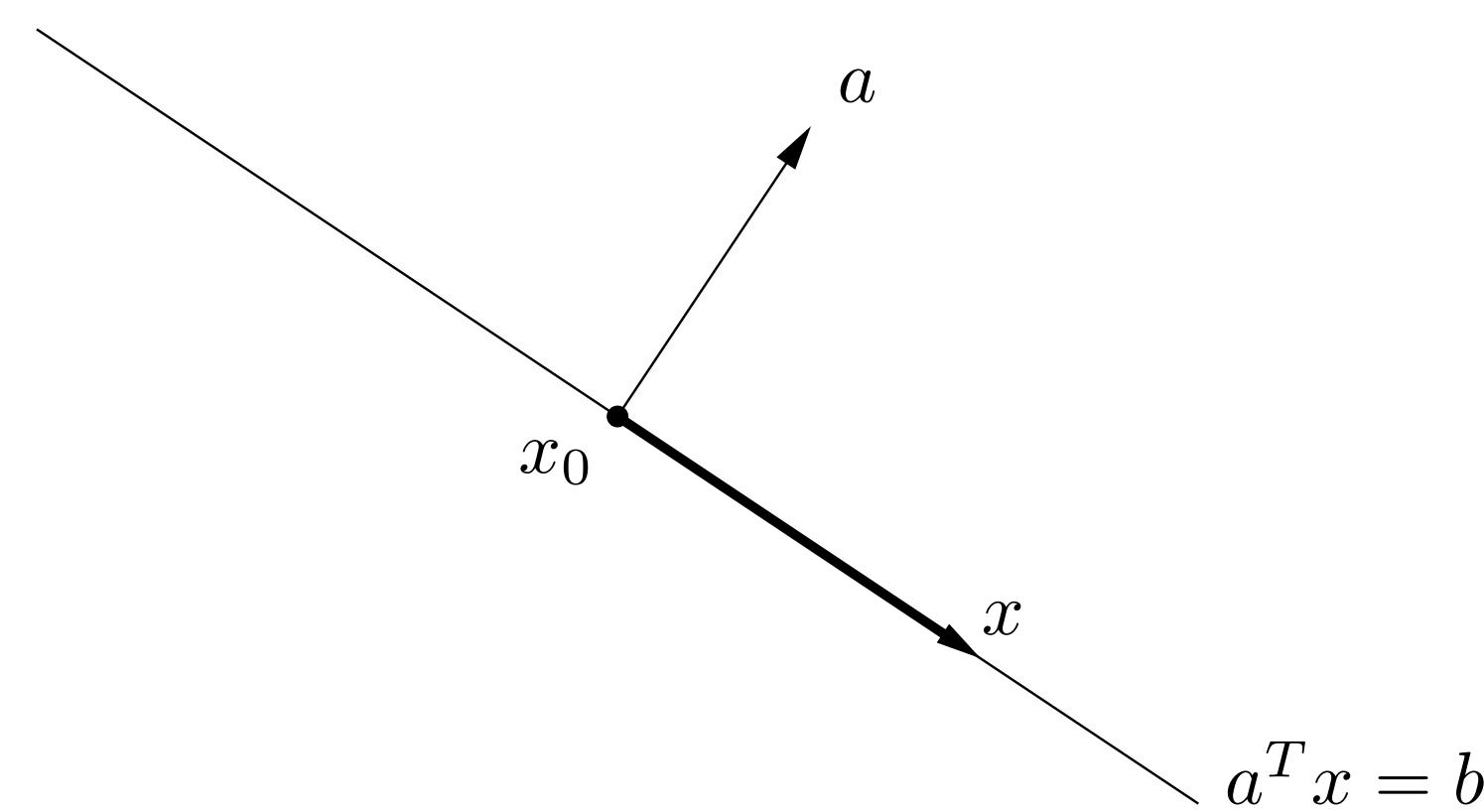
Polyhedra and linear algebra

Hyperplanes and halfspaces

Definitions

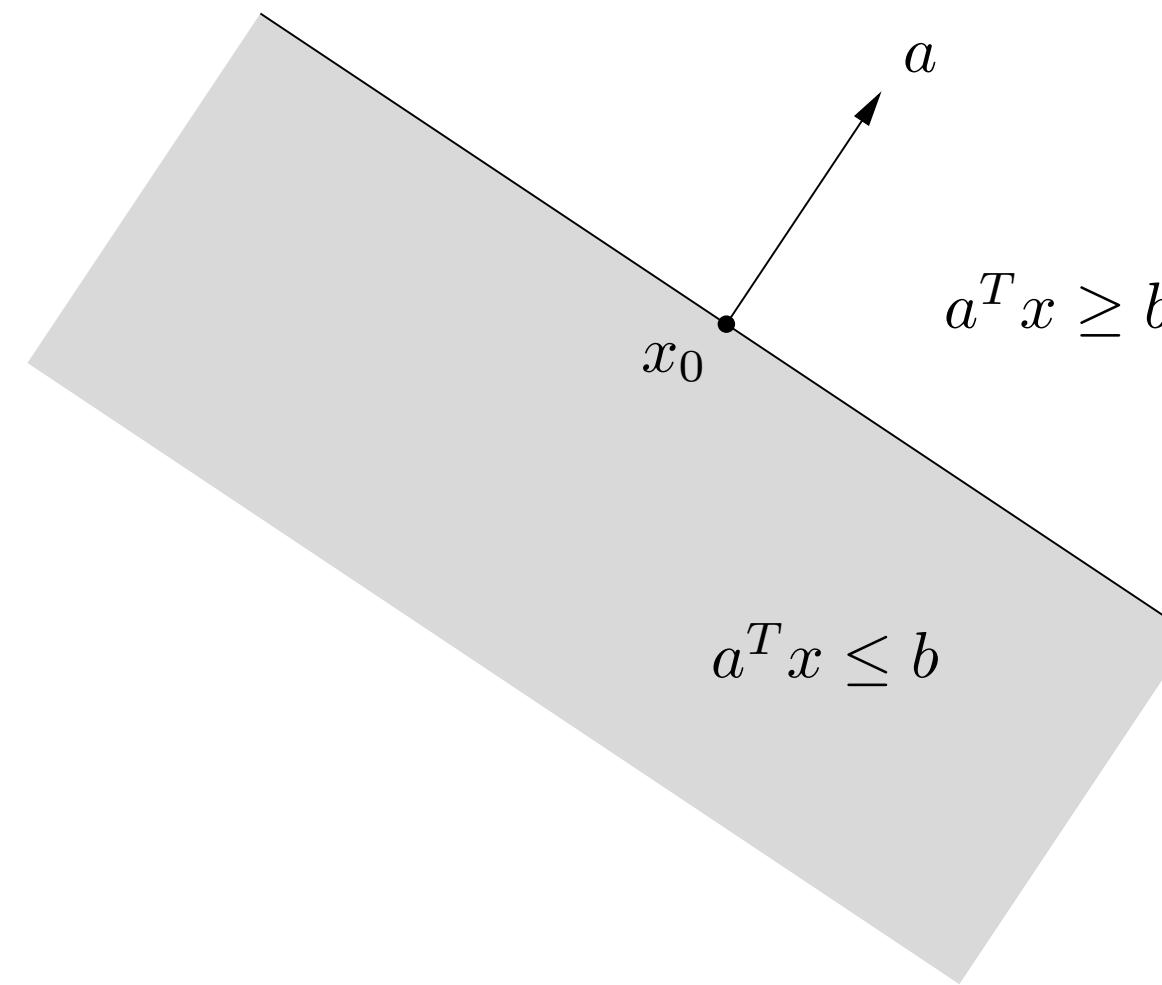
Hyperplane

$$\{x \mid a^T x = b\}$$



Halfspace

$$\{x \mid a^T x \leq b\}$$

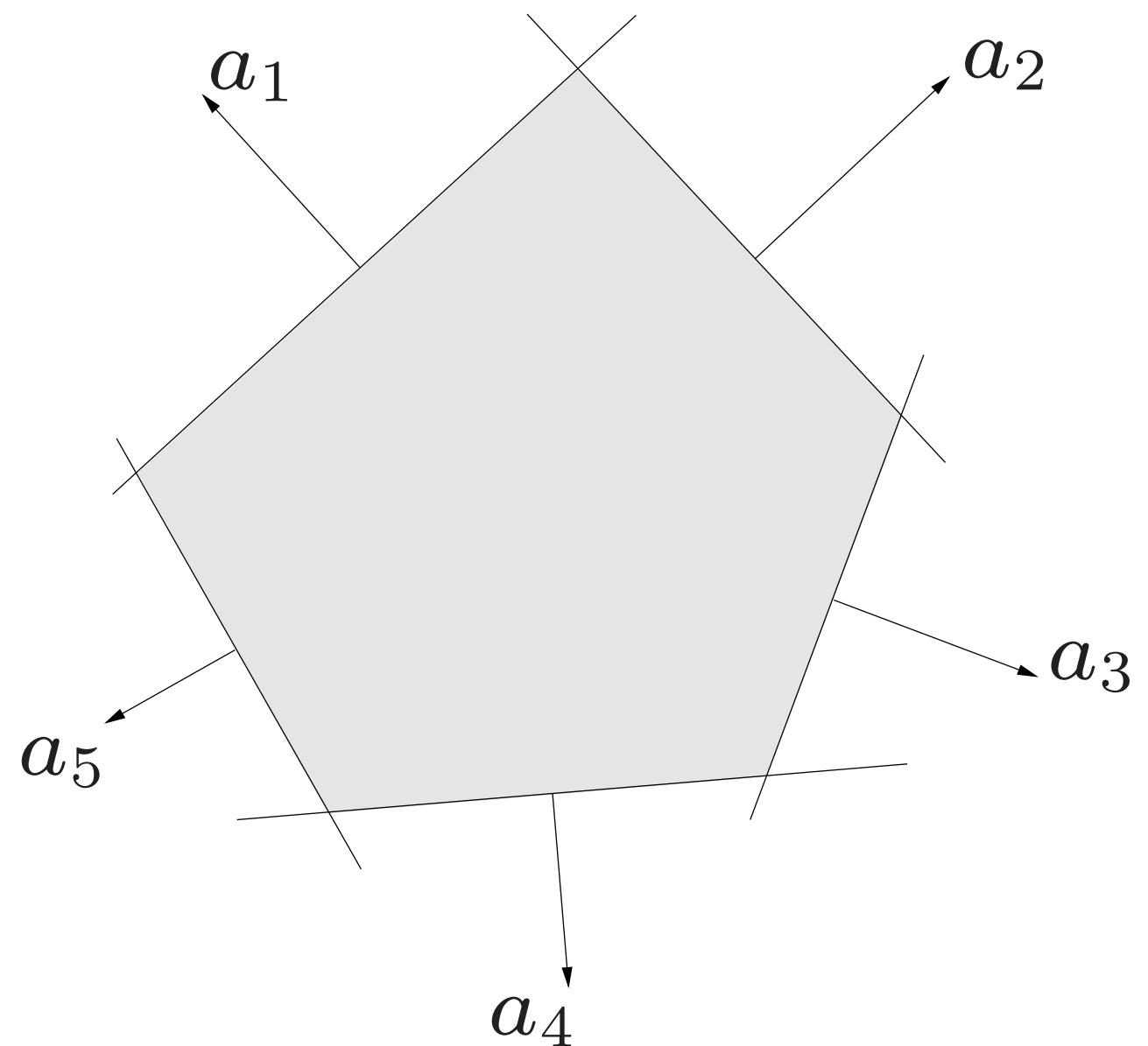


- x_0 is a specific point in the hyperplane
- For any x in the hyperplane defined by $a^T x = b$, $x - x_0 \perp a$
- The halfspace determined by $a^T x \leq b$ extends in the direction of $-a$

Polyhedron

Definition

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\} = \{x \mid Ax \leq b\}$$



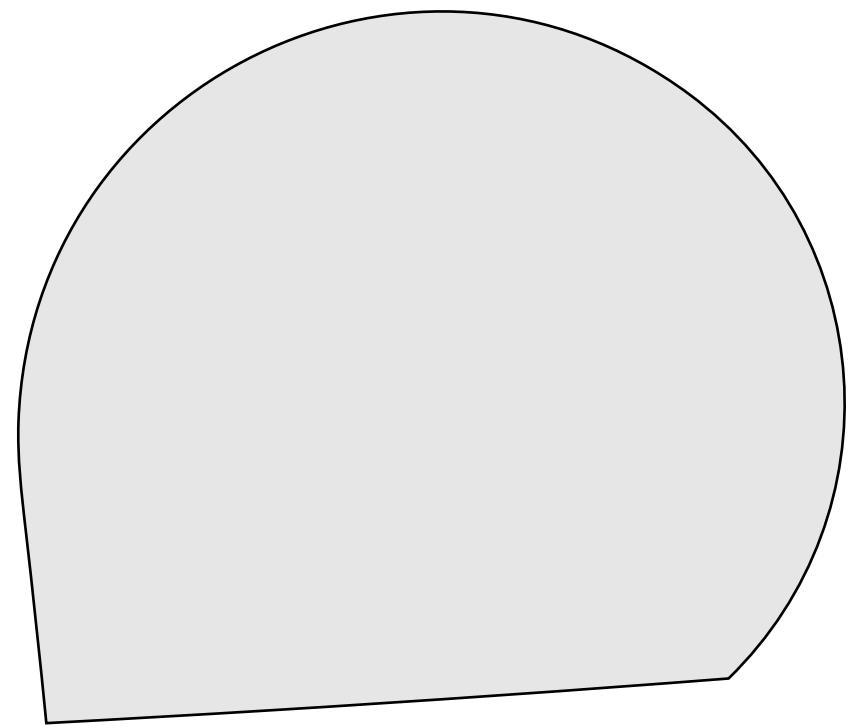
- Intersection of finite number of halfspaces
- Can include equalities

Convex set

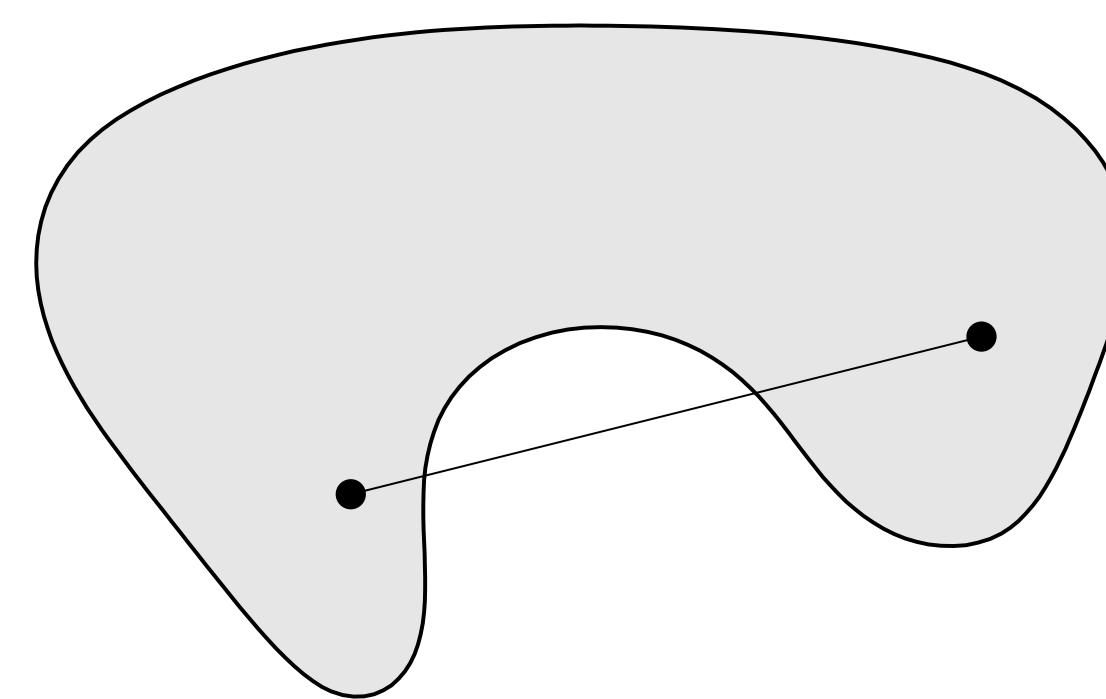
Definition

For any $x, y \in C$ and any $\alpha \in [0, 1]$

$$\alpha x + (1 - \alpha)y \in C$$



Convex



Not convex

Examples

- \mathbf{R}^n
- Hyperplanes
- Halfspaces
- Polyhedra

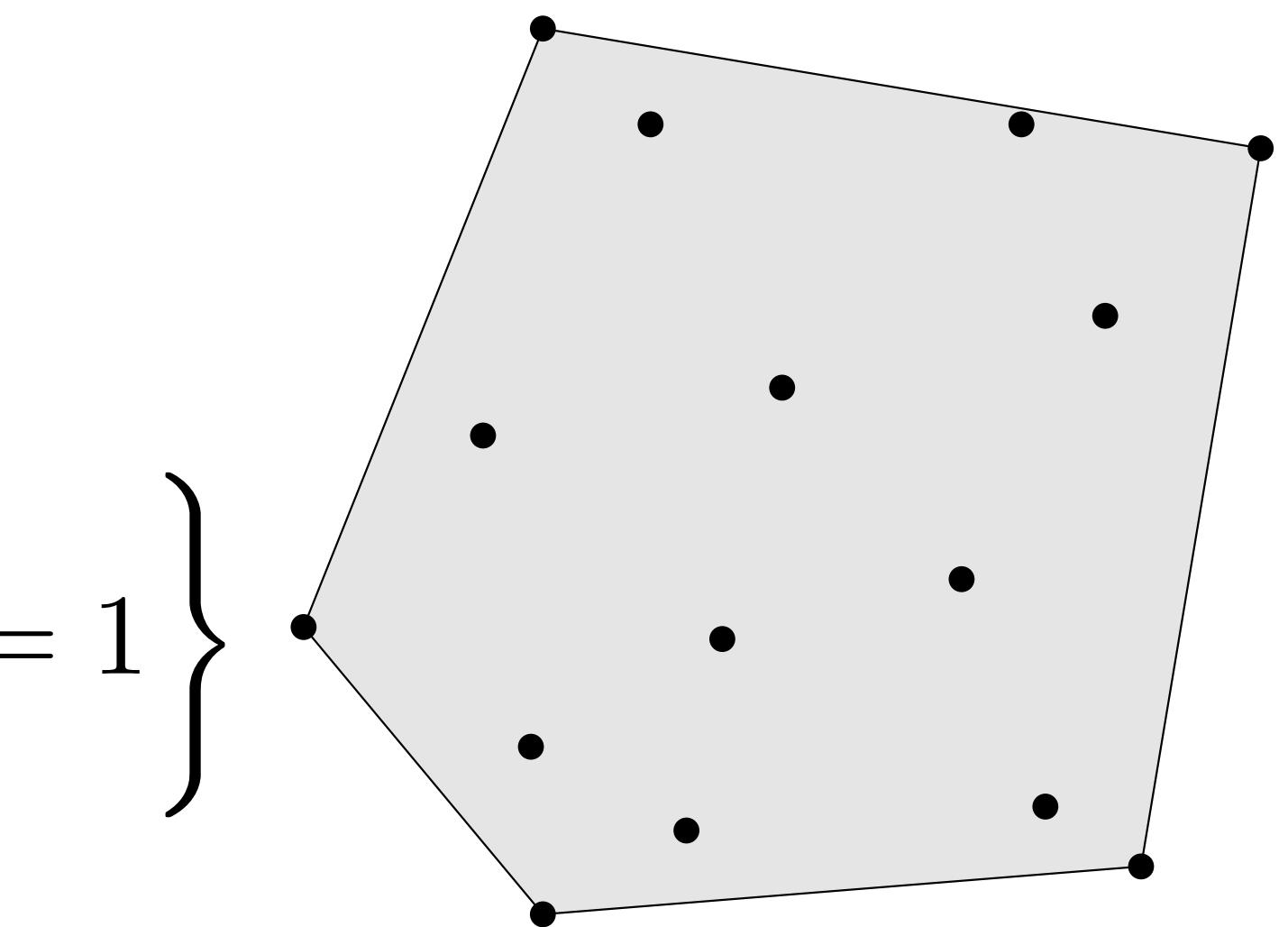
Convex combinations

Convex combination

$\alpha_1 x_1 + \cdots + \alpha_k x_k$ for any x_1, \dots, x_k and $\alpha_1, \dots, \alpha_k$ such that $\alpha_i \geq 0$, $\sum_{i=1}^k \alpha_i = 1$

Convex hull

$$\text{conv } C = \left\{ \sum_{i=1}^k \alpha_i x_i \mid x_i \in C, \alpha_i \geq 0, i = 1, \dots, k, \mathbf{1}^T \alpha = 1 \right\}$$



Linear independence

a nonempty set of vectors $\{v_1, \dots, v_k\}$ is **linearly independent** if

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$$

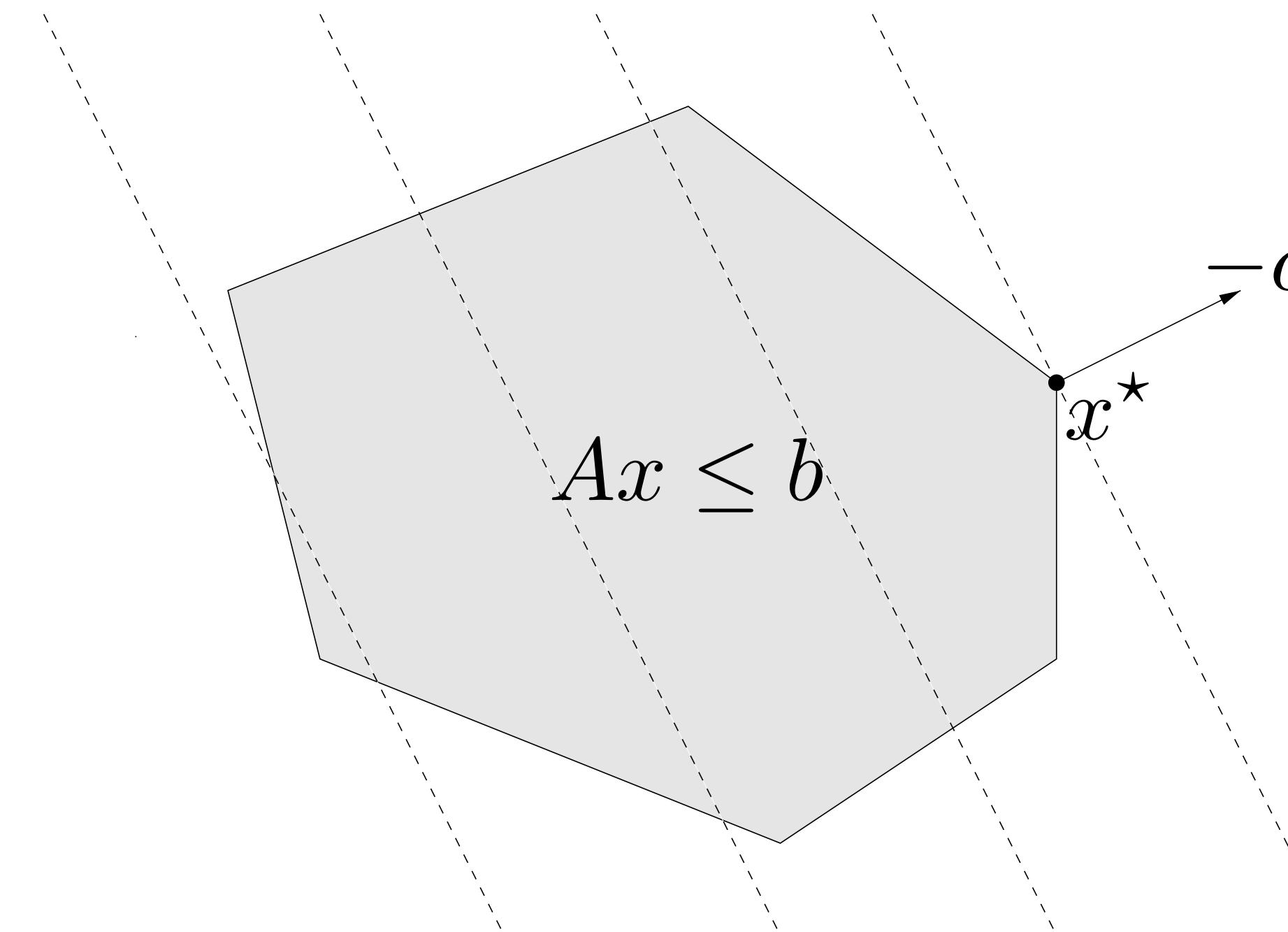
holds only for $\alpha_1 = \cdots = \alpha_k = 0$

Properties

- The coefficients α_k in a linear combination $x = \alpha_1 v_1 + \cdots + \alpha_k v_k$ are unique
- None of the vectors v_i is a linear combination of the other vectors

Geometrical interpretation of linear optimization

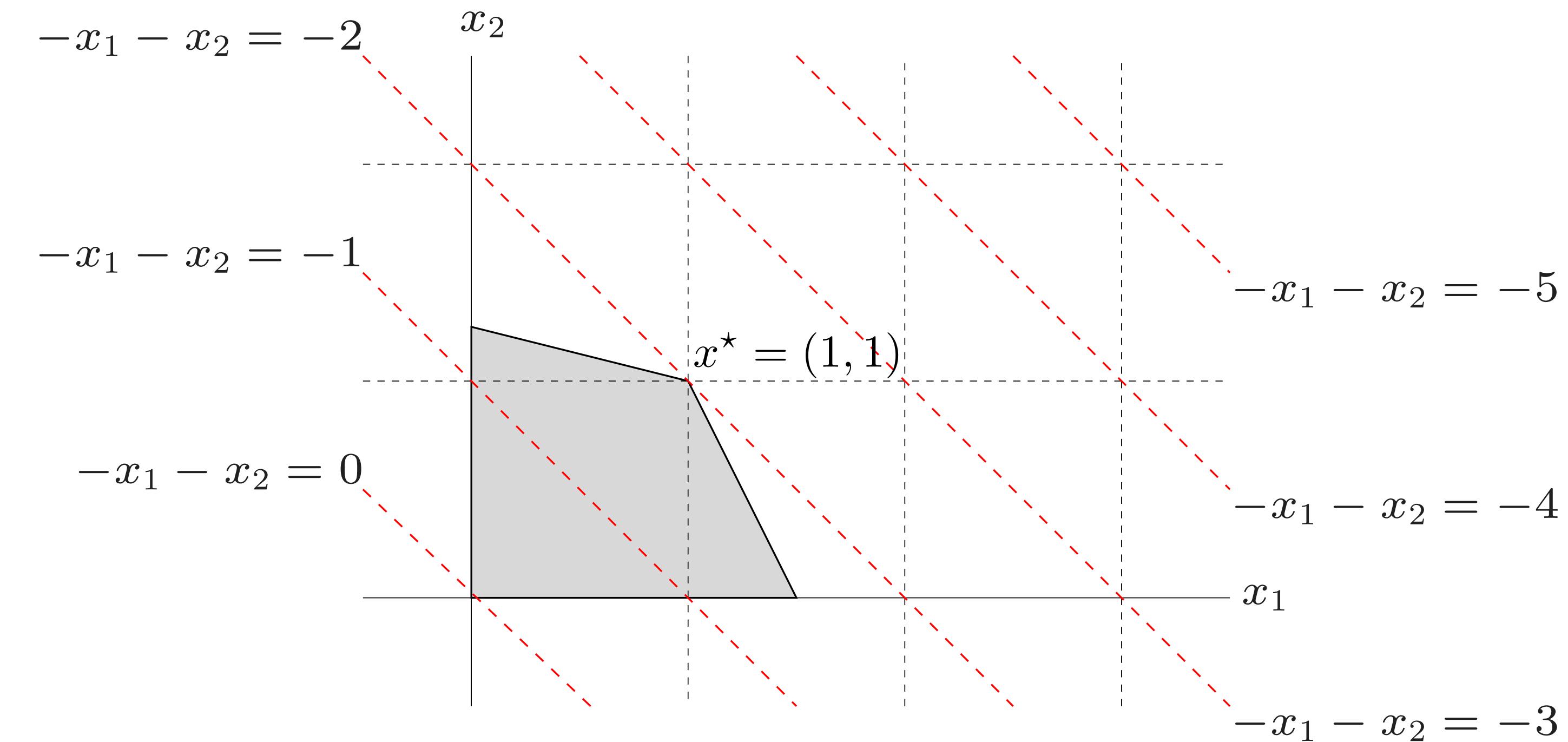
$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax \leq b \end{aligned}$$



Dashed lines (hyperplanes) are level sets $c^T x = \alpha$ for different α

Example of linear optimization

minimize $-x_1 - x_2$
subject to $2x_1 + x_2 \leq 3$
 $x_1 + 4x_2 \leq 5$
 $x_1 \geq 0, x_2 \geq 0$



Optimal solutions tend to be at a “corner” of the feasible set

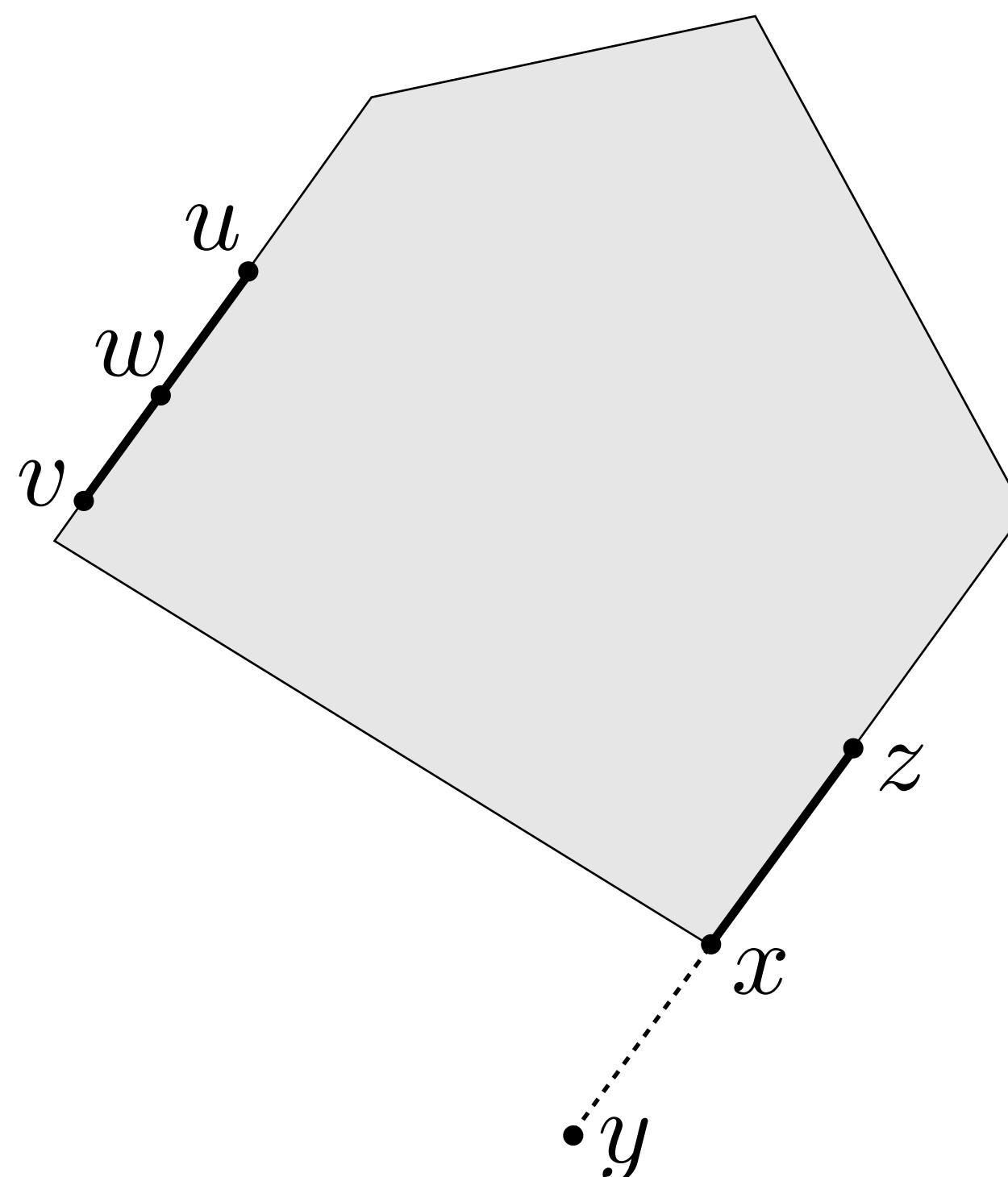
How do we formalize it?

Corners of linear optimization

Extreme points

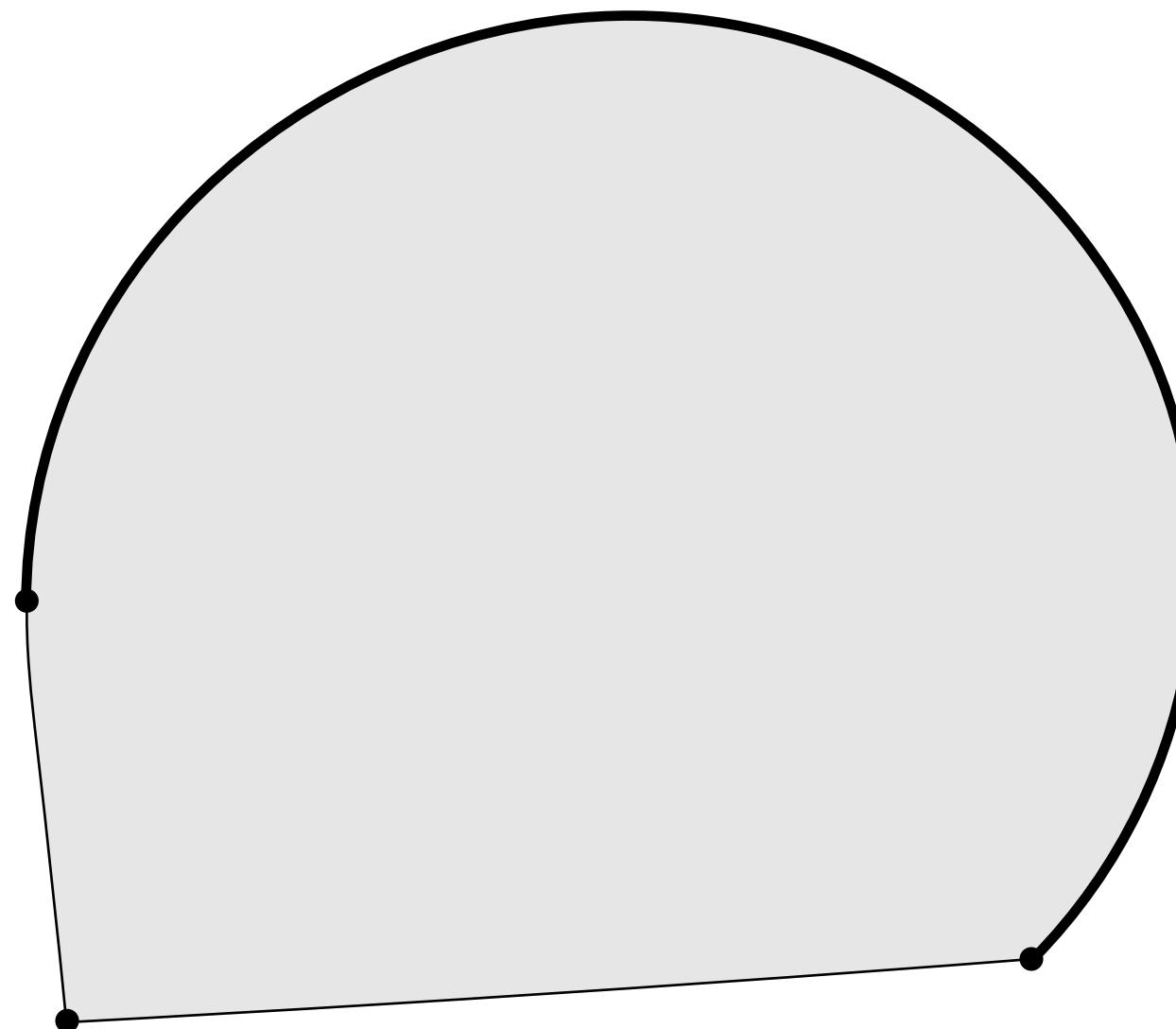
Definition

$x \in P$ is said to be an **extreme point** of P if
 $\exists y, z \in P$ ($y \neq x, z \neq x$) and $\alpha \in [0, 1]$ such that $x = \alpha y + (1 - \alpha)z$



Extreme points

Convex sets



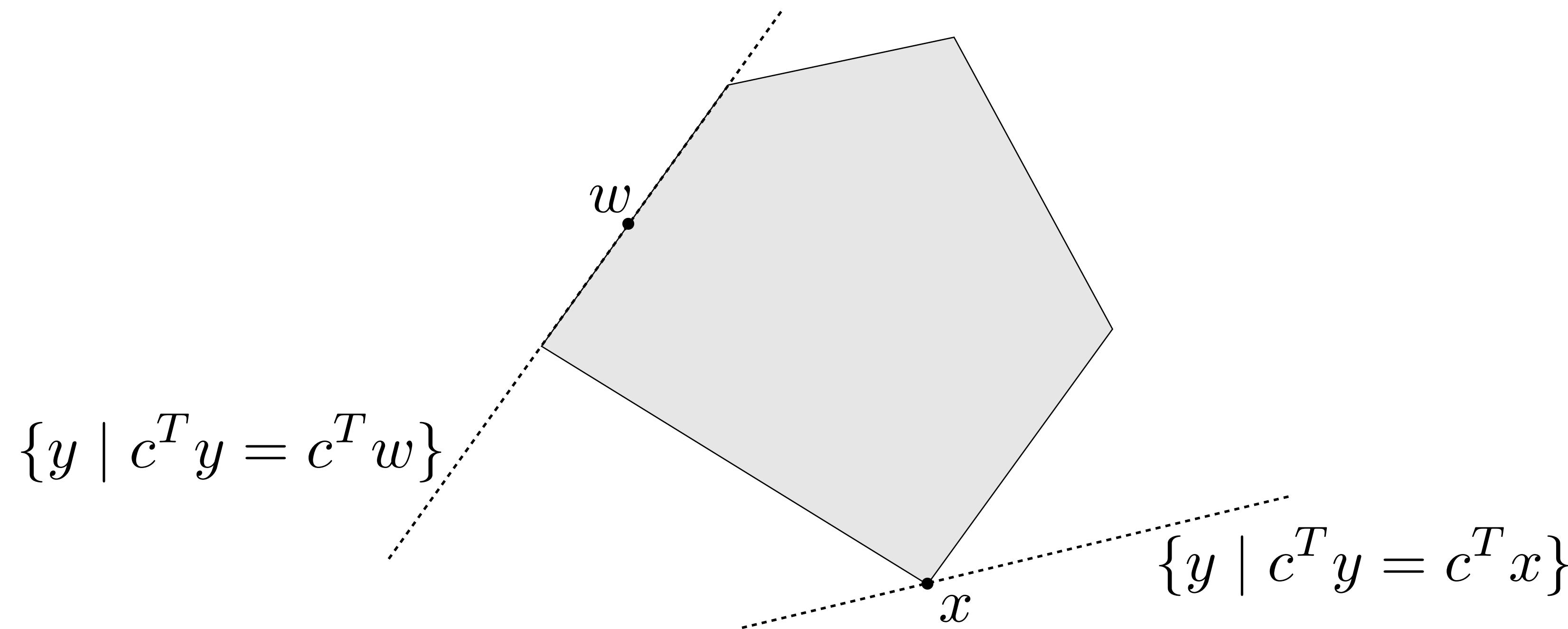
- Convex sets can have an infinite number of extreme points
- Polyhedra are convex sets with a finite number of extreme points

Vertices

Definition

$x \in P$ is a **vertex** if $\exists c$ such that x is the unique optimum of

$$\begin{array}{ll}\text{minimize} & c^T y \\ \text{subject to} & y \in P\end{array}$$



Basic feasible solution

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$$

Active constraints at \bar{x}

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \dots, m\} \mid a_i^T \bar{x} = b_i\}$$

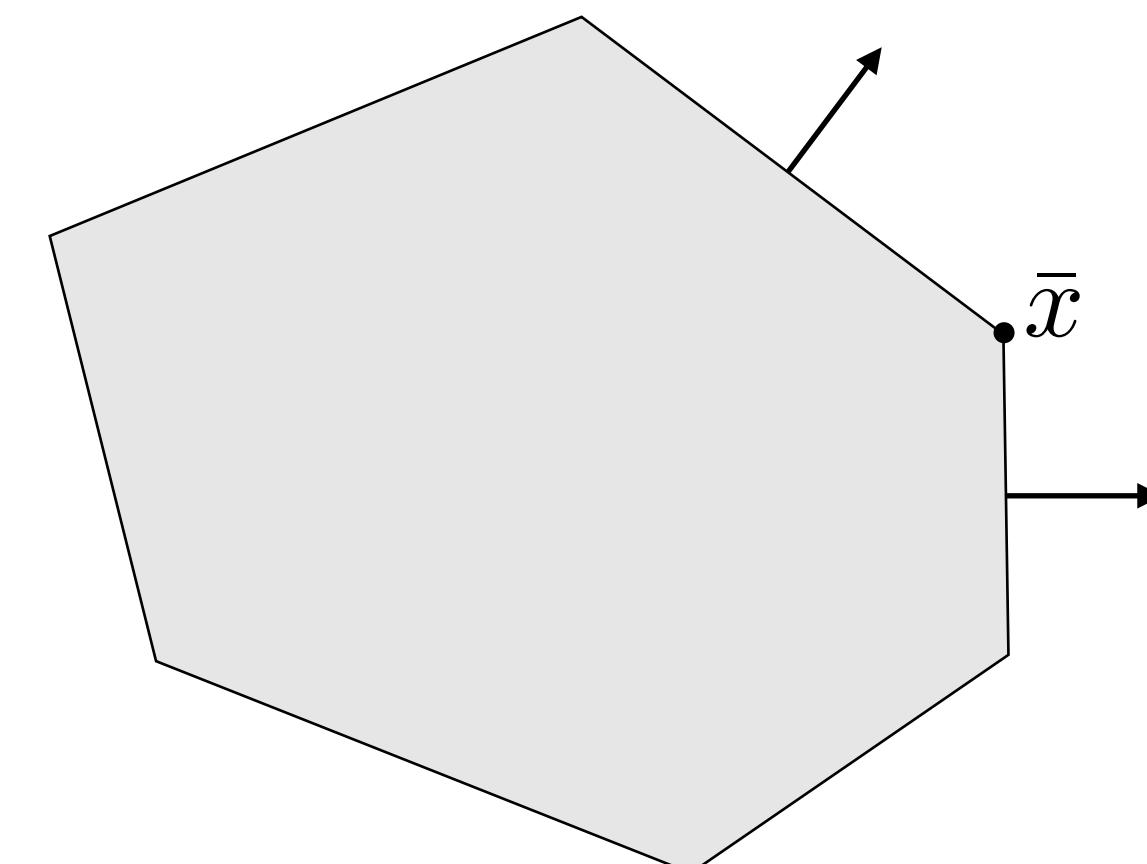
Index of all the constraints
satisfied as **equality**

Basic solution \bar{x}

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

Basic feasible solution \bar{x}

- $\bar{x} \in P$
- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

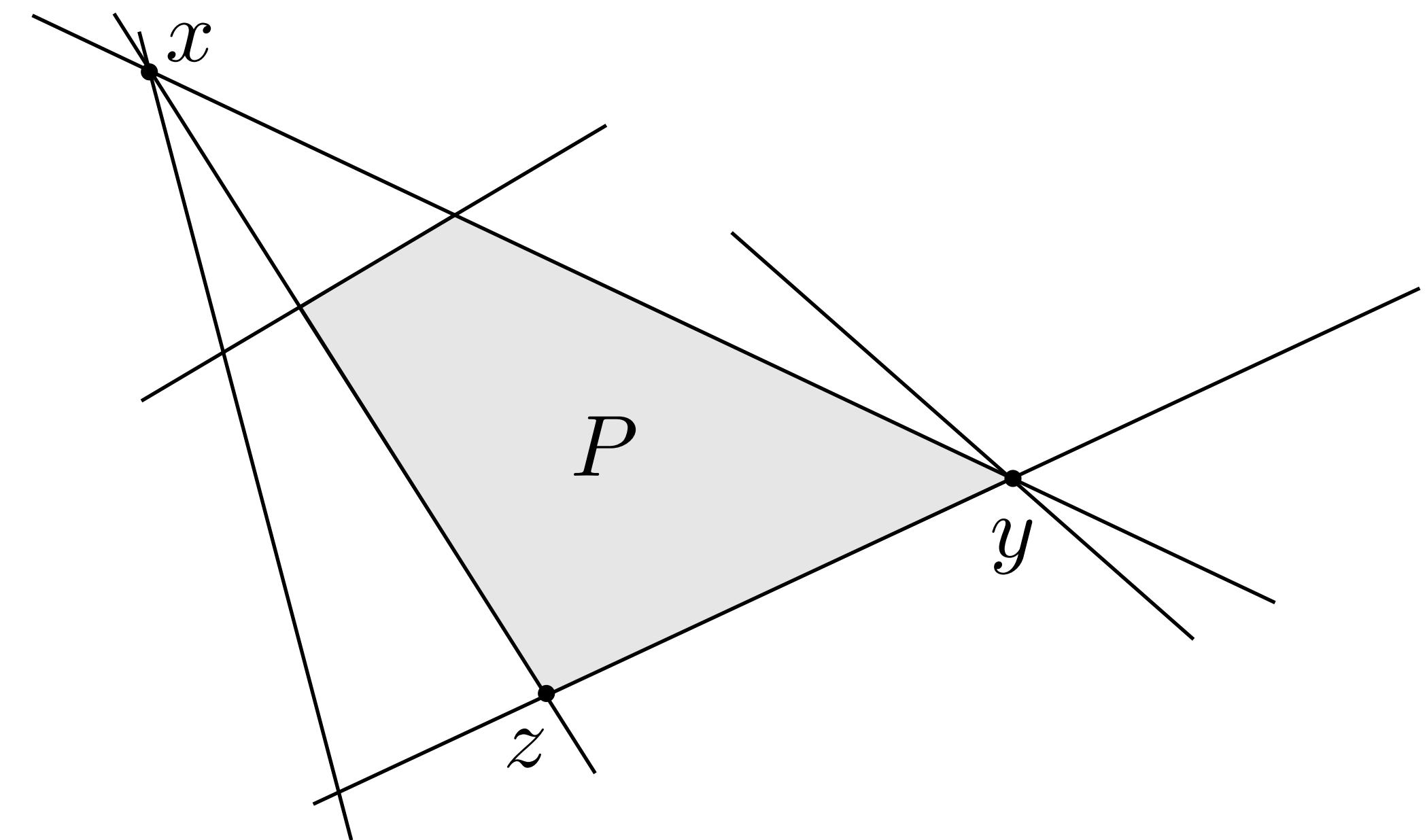


Degenerate basic feasible solutions

A solution \bar{x} is degenerate if $|\mathcal{I}(\bar{x})| > n$

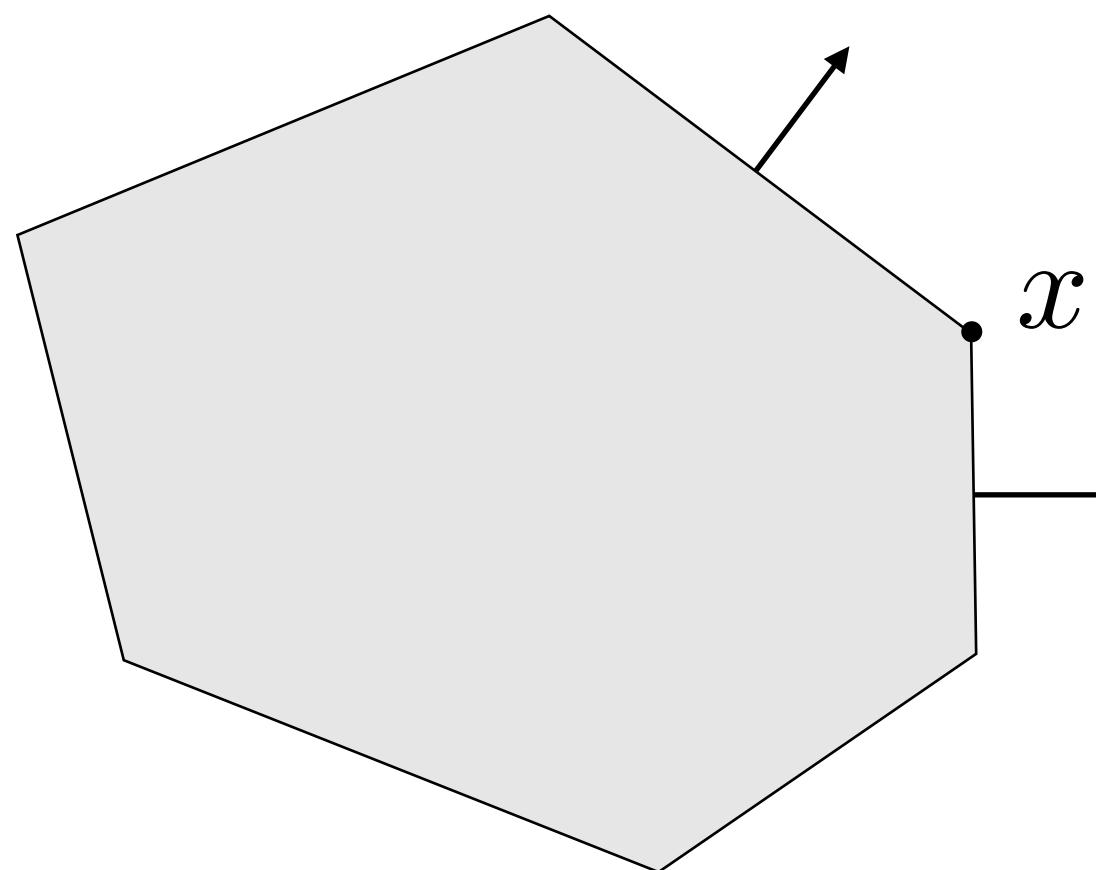
True or False?

Basic	Feasible	Degenerate
x		
y		
z		



Equivalence Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



Let $x \in P$

x is a vertex $\iff x$ is an extreme point $\iff x$ is a basic feasible solution

Constructing basic solutions

Standard form polyhedra

Definition

Standard form LP

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

Assumption

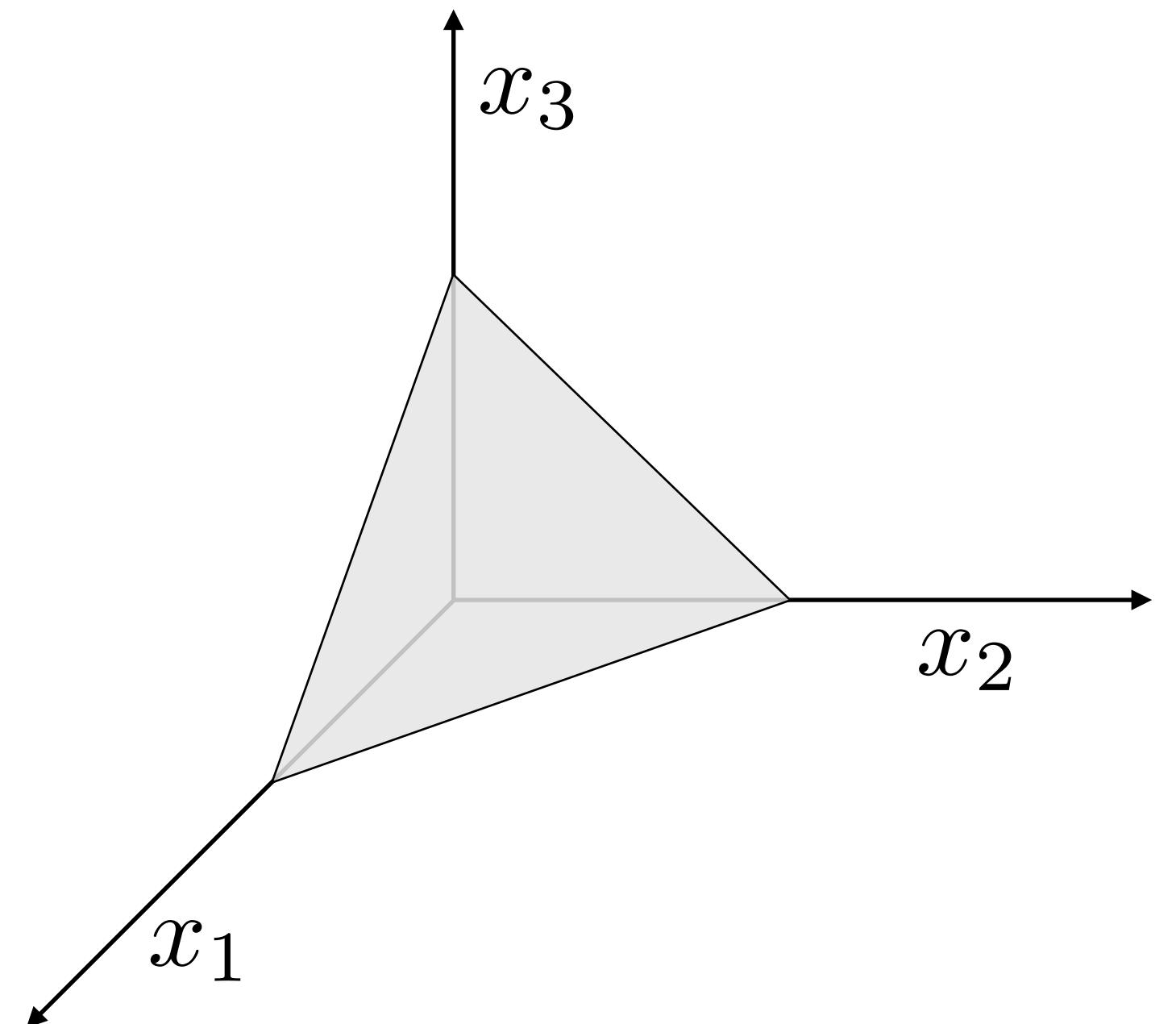
$A \in \mathbb{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P lives in $(n - m)$ -dimensional subspace

Standard form polyhedron

$$P = \{x \mid Ax = b, x \geq 0\}$$



Basic solutions

Standard form polyhedra

$$P = \{x \mid Ax = b, x \geq 0\} \quad \text{with} \quad A \in \mathbf{R}^{m \times n} \text{ has full row rank } m \leq n$$

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
 - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
 - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

Constructing basic solution

1. Choose any m independent columns of A : $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

Basis matrix Basis columns Basic variables

$$B = \begin{bmatrix} & & & \\ | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \quad \longrightarrow \quad x_B = B^{-1}b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

Quiz



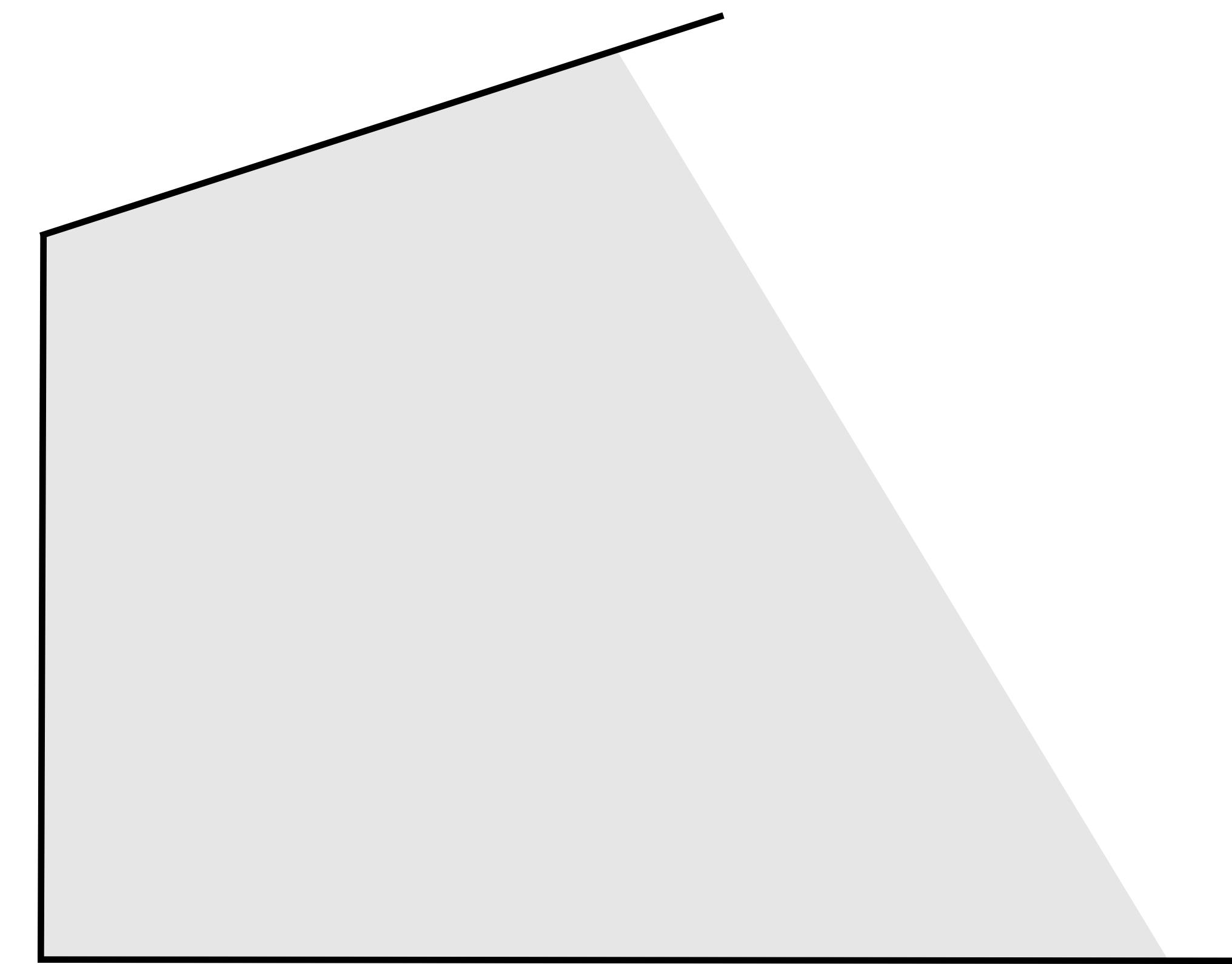
Existence and optimality of extreme points

Existence of extreme points

Example



No extreme points



Extreme points

Existence of extreme points

Characterization

A polyhedron P **contains a line** if
 $\exists x \in P$ and a nonzero vector d such that $x + \lambda d \in P, \forall \lambda \in \mathbf{R}$.

Given a polyhedron $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$, the following are **equivalent**

- P does not contain a line
- P has at least one extreme point
- n of the a_i vectors are linearly independent

Corollary

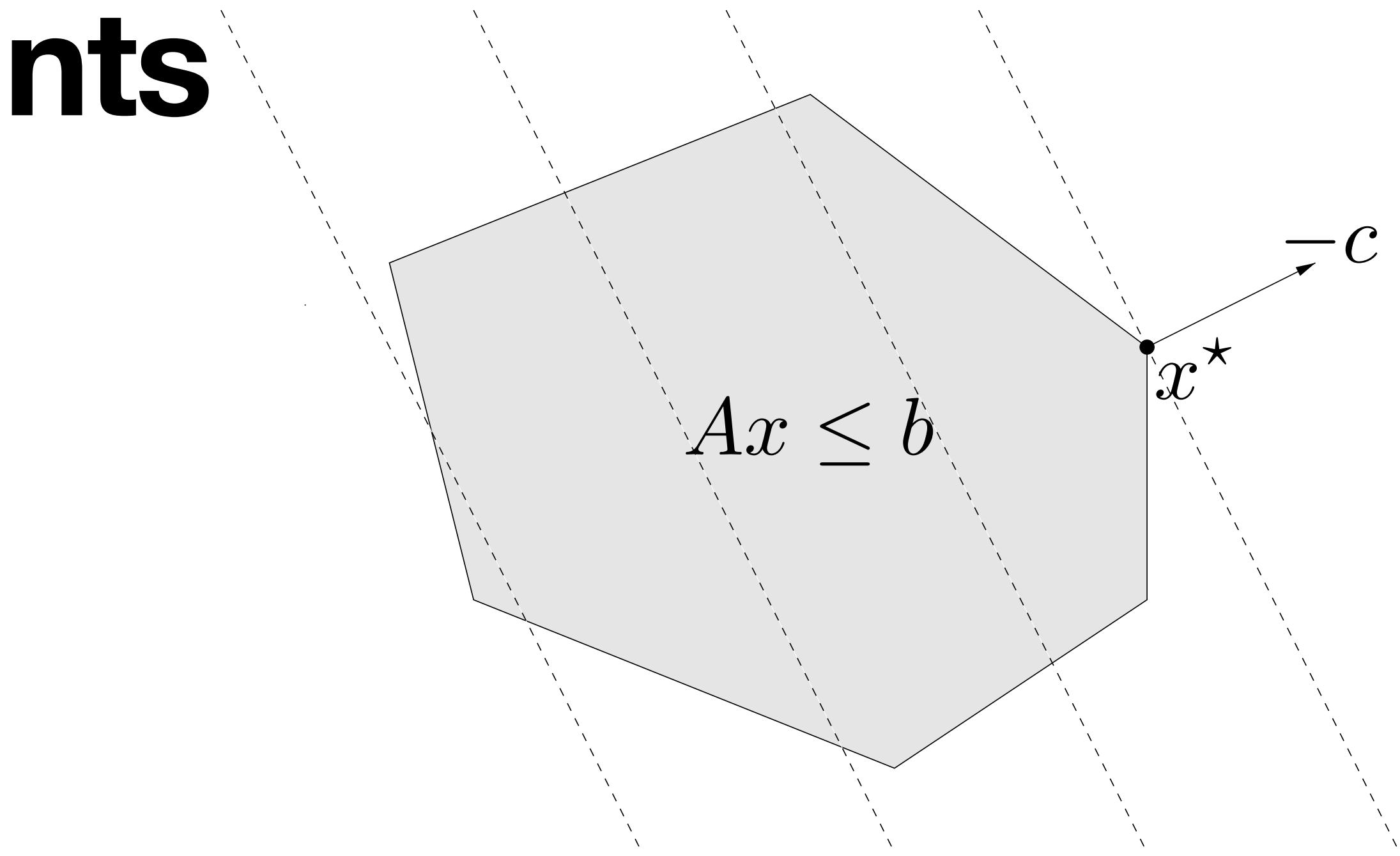
Every nonempty bounded polyhedron has
at least one basic feasible solution

Optimality of extreme points

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

- If
- P has at least one extreme point
 - There exists an optimal solution x^*

Then, there exists an optimal solution which is an **extreme point** of P



We only need to search between **extreme points**

Quiz



How to search among basic feasible solutions?

Idea

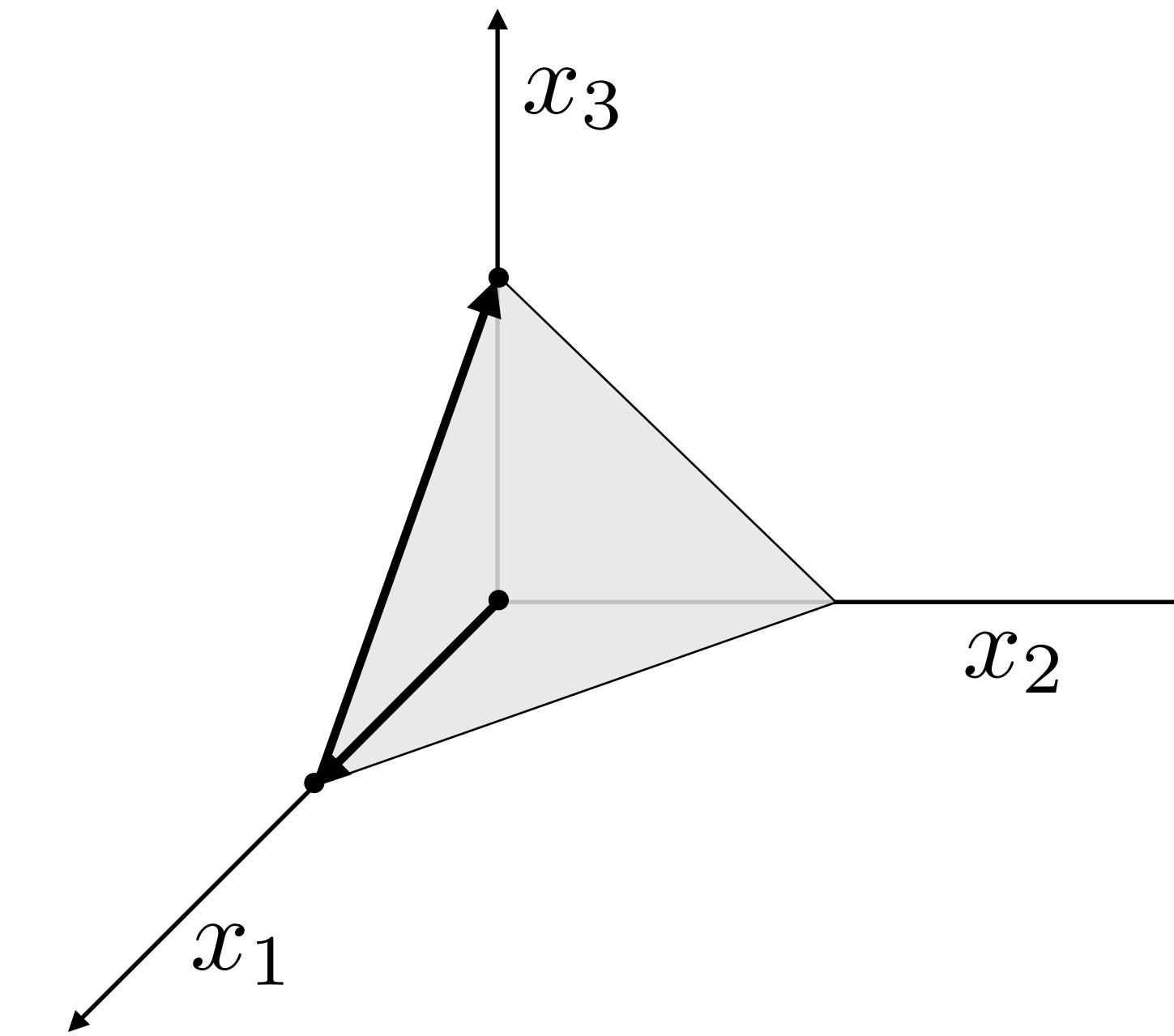
List all the basic feasible solutions, compare objective values and pick the best one.

Intractable!

If $n = 1000$ and $m = 100$, we have 10^{143} combinations!

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Geometry of linear optimization

Today, we learned to:

- **Apply geometric and algebraic properties** of polyhedra to characterize the “corners” of the feasible region.
- **Construct basic feasible solutions** by solving a linear system.
- **Recognize existence and optimality** of extreme points.

Next lecture

The simplex method

- Iterations
- Convergence
- Complexity