ORF307 — Optimization

14. Duality II

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Ed Forum

- how exactly do we apply Lagrange multipliers in the context of linear programming to find the best lower bounds?

- can we interpret the primal as the dual problem and vice versa? Especially since they are solving for the same thing as stated in the Strong duality theorem.

Recap

Weak and strong duality

Optimal objective values

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem
If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

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$\longrightarrow$ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$

■

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$$0 \leq y^T(b - Ax) = b^T y - y^T A x = c^T x + b^T y \qquad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Exception to strong duality

Primal

minimize x

subject to $0 \cdot x \leq -1$

Optimal value is $p^\star = +\infty$

Dual

maximize y

subject to $0 \cdot y + 1 = 0$

$y \geq 0$

Optimal value is $d^\star = -\infty$

Exception to strong duality

Primal

minimize x

subject to $0 \cdot x \leq -1$

Dual

maximize y

subject to $0 \cdot y + 1 = 0$

$y \geq 0$

Optimal value is $p^\star = +\infty$

Optimal value is $d^\star = -\infty$

Both **primal** and **dual infeasible**

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1,1)$ and $(3,3)$ proven by **weak duality**
- $(3,1)$ and $(2,2)$ proven by **strong duality**

Example

Production problem

maximize $x_1 + 2x_2$

subject to $x_1 \leq 100$

$$2x_2 \leq 200$$

$$x_1 + x_2 \leq 150$$

$$x_1, x_2 \geq 0$$

Production problem

maximize	$x_1 + 2x_2$	$\longleftarrow$ Profits
subject to	$x_1 \leq 100$	
	$2x_2 \leq 200$	
	$x_1 + x_2 \leq 150$	
	$x_1, x_2 \geq 0$	

Production problem

maximize $x_1 + 2x_2$ ⟵ Profits

subject to

$$x_1 \leq 100$$
$$2x_2 \leq 200$$ ⟵ Resources
$$x_1 + x_2 \leq 150$$
$$x_1, x_2 \geq 0$$

Production problem

maximize $x_1 + 2x_2$ ⟵ Profits

subject to

$$x_1 \leq 100$$
$$2x_2 \leq 200$$
$$x_1 + x_2 \leq 150$$

⟵ Resources

$$x_1, x_2 \geq 0$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Dualize

1. Transform in inequality form

minimize $c^T x$

subject to $Ax \leq b$

Production problem

maximize $x_1 + 2x_2$ ⟵ Profits

subject to
$x_1 \leq 100$
$2x_2 \leq 200$ ⟵ Resources
$x_1 + x_2 \leq 150$
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$$b = (100, 200, 150, 0, 0)$$

Dualize

1. Transform in inequality form

minimize $c^T x$

subject to $Ax \leq b$

2. Derive dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Production problem

Dualized

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

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Production problem

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$c = (-1, -2)$

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$b = (100, 200, 150, 0, 0)$

Fill-in data

$$\begin{array}{ll} \text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 - y_4 = 1 \\ & 2y_2 + y_3 - y_5 = 2 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{array}$$

Production problem

Dualized

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$$c = (-1, -2)$$

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Fill-in data

$$\begin{array}{ll} \text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 - \cancel{y_4} = 1 \\ & 2y_2 + y_3 - \cancel{y_5} = 2 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{array}$$

→

Eliminate variables

$$\begin{array}{ll} \text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Production problem

The dual

$$
\begin{aligned}
\text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\
\text{subject to} \quad & y_1 + y_3 \geq 1 \\
& 2y_2 + y_3 \geq 2 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Production problem

The dual

$$\begin{array}{ll} \text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 \geq 1 \quad (x_1) \\ & 2y_2 + y_3 \geq 2 \quad (x_2) \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs $1\times$ resource 1 and 3): $y_1 + y_3 \geq 1$
  - Product 2 (price 2, needs $2\times$ resource 2 and $1\times$ resource 3): $2y_2 + y_3 \geq 2$

Today's agenda

More on duality

- Two-person zero-sum games
- Farkas lemma
- Complementary slackness
- KKT conditions

Two-person games

Rock paper scissors

Rules
At count to three declare one of: Rock, Paper, or Scissors

Winners
Identical selection is a draw, otherwise:
- Rock beats ("dulls") scissors
- Scissors beats ("cuts") paper
- Paper beats ("covers") rock

Extremely popular: world RPS society, USA RPS league, etc.

Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of m actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of n actions)

Two players make their choice independently

Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of m actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of n actions)

Two players make their choice independently

Rule

Player 1 pays A_{ij} to player 2

$A \in \mathbf{R}^{m \times n}$ is the **payoff matrix**

Rock, Paper, Scissors

$$A = \begin{array}{c} \\ \mathrm{R} \\ \mathrm{P} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{P} & \mathrm{S} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

Mixed (randomized) strategies

Deterministic strategies can be **systematically defeated**

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Deterministic strategies can be **systematically defeated**

Randomized strategies

- P1 chooses randomly according to distribution x:

  $x_i =$ probability that P1 selects action i

- P2 chooses randomly according to distribution y:

  $y_j =$ probability that P2 selects action j

Mixed (randomized) strategies

Deterministic strategies can be **systematically defeated**

Randomized strategies

- P1 chooses randomly according to distribution x:

  $x_i =$ probability that P1 selects action i

- P2 chooses randomly according to distribution y:

  $y_j =$ probability that P2 selects action j

Expected payoff (from P1 P2), if they use mixed-strategies x and y,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

Mixed strategies and probability simplex

Probability simplex in $\mathbf{R}^k$

$P_k = \{p \in \mathbf{R}^k \mid p \geq 0, \quad \mathbf{1}^T p = 1\}$

Mixed strategy

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $\longrightarrow$ $x \in P_m, \quad y \in P_n$

Optimal mixed strategies

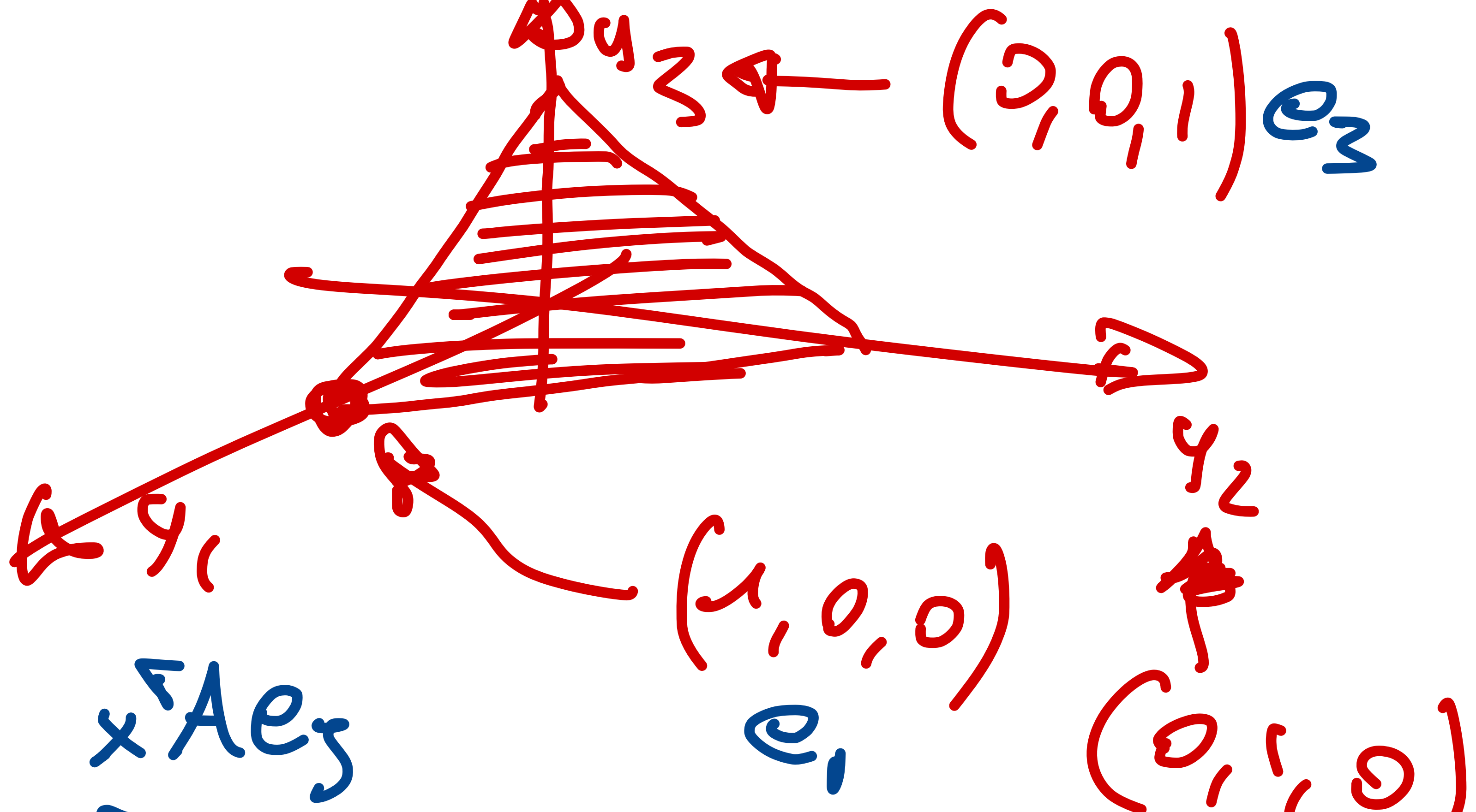
P1: optimal strategy $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & \max_{y \in P_n} x^T A y \\ \text{subject to} & x \in P_m \end{array}$$

P2: optimal strategy $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & \min_{x \in P_m} x^T A y \\ \text{subject to} & y \in P_n \end{array}$$

Optimal mixed strategies



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Inner problem over deterministic strategies (**vertices**)

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Optimal strategies $x^\star$ and $y^\star$ can be computed using **linear optimization**

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

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The optimal $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \geq w\mathbf{1} \\ & \mathbf{1}^T y = 1 \\ & y \geq 0 \end{array}$$

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal $x^\star$ is the solution of

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The optimal $y^\star$ is the solution of

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The two LPs are **duals** and by **strong duality** the equality follows. ■

Nash equilibrium

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Consequence

The pair of mixed strategies $(x^\star, y^\star)$ attains the **Nash equilibrium** of the two-person matrix game, i.e.,

$$x^T A y^\star \geq x^{\star T} A y^\star \geq x^{\star T} A y, \quad \forall x \in P_m, \ \forall y \in P_n$$

Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

Optimal deterministic strategies

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

Optimal deterministic strategies

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

Optimal mixed strategies

$$x^\star = (0.37, 0.33, 0.3), \qquad y^\star = (0.4, 0, 0.13, 0.47)$$

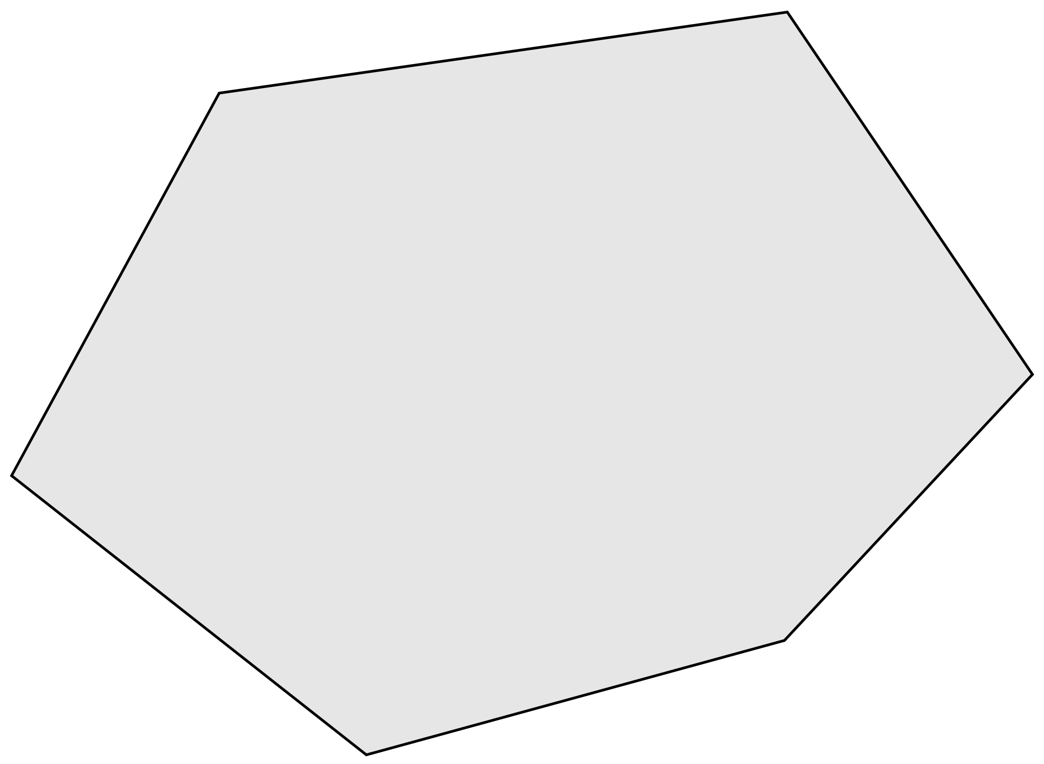
Expected payoff

$$x^{\star T} A y^\star = 0.2$$

Farkas lemma

Feasibility of polyhedra

$$P = \{x \mid Ax = b, \quad x \geq 0\}$$



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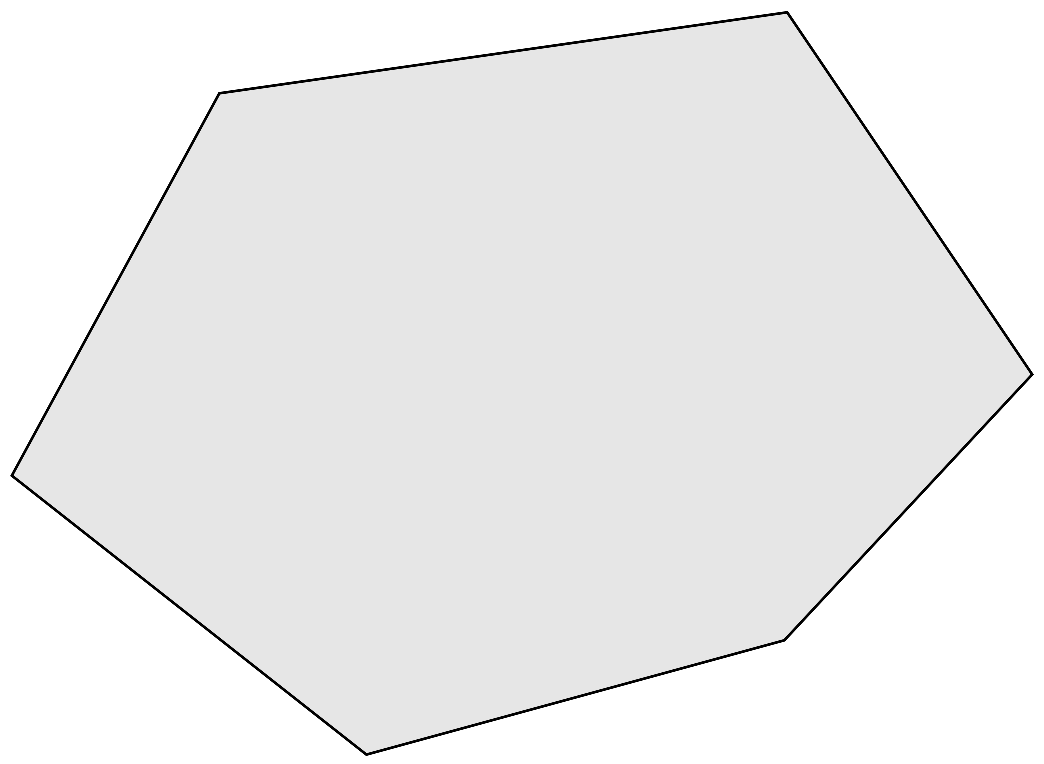
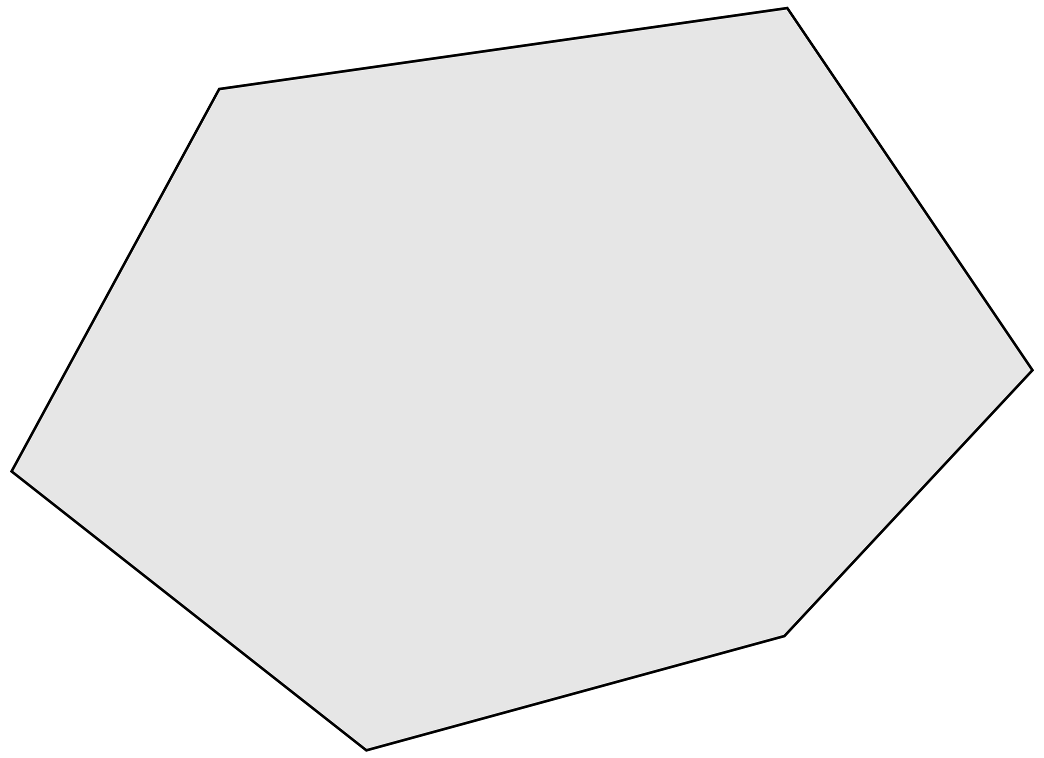


How to show that P is **feasible**?

Easy: we just need to provide an $x \in P$, i.e., a **certificate**

Feasibility of polyhedra

$$P = \{x \mid Ax = b, \quad x \geq 0\}$$



How to show that P is **feasible**?

Easy: we just need to provide an $x \in P$, i.e., a **certificate**

How to show that P is **infeasible**?

Farkas lemma

Theorem

Given A and b, exactly one of the following statements is true:

1. There exists an x with $Ax = b$, $x \geq 0$
2. There exists a y with $A^T y \geq 0$, $b^T y < 0$

Farkas lemma

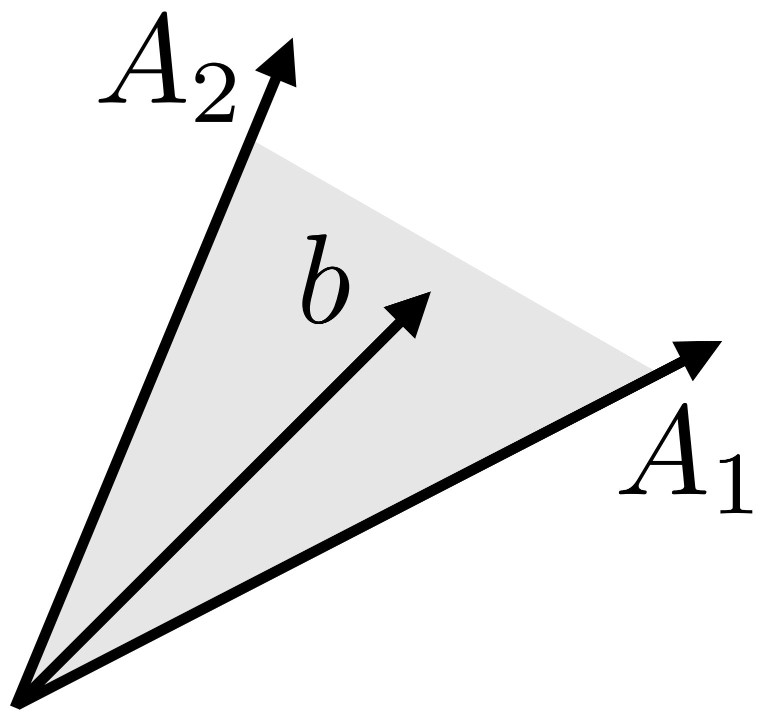
Geometric interpretation

1. First alternative

There exists an x with $Ax = b$, $x \geq 0$

$$b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0, \; i = 1, \ldots, n$$

b is in the cone generated by the columns of A



Farkas lemma

Geometric interpretation

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There exists an x with $Ax = b$, $x \geq 0$

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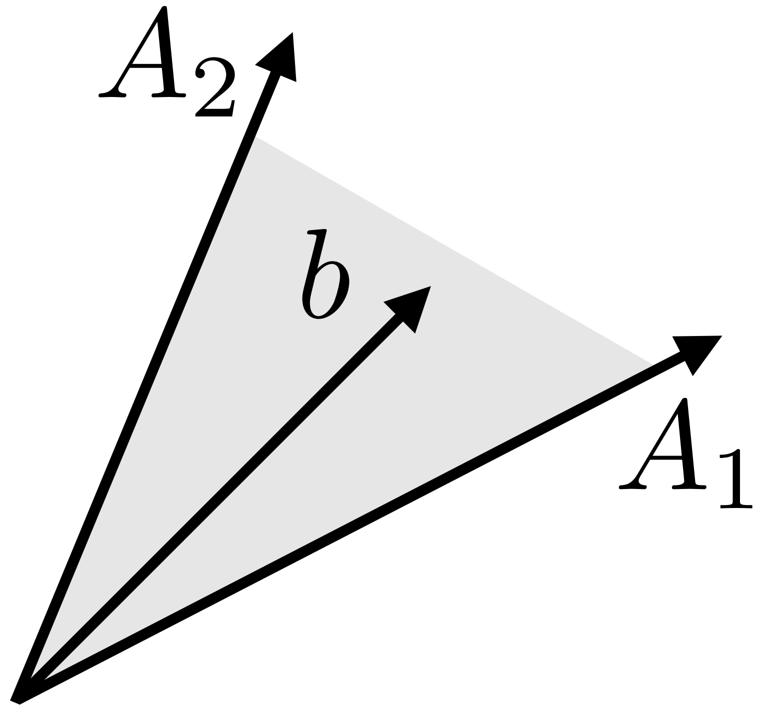
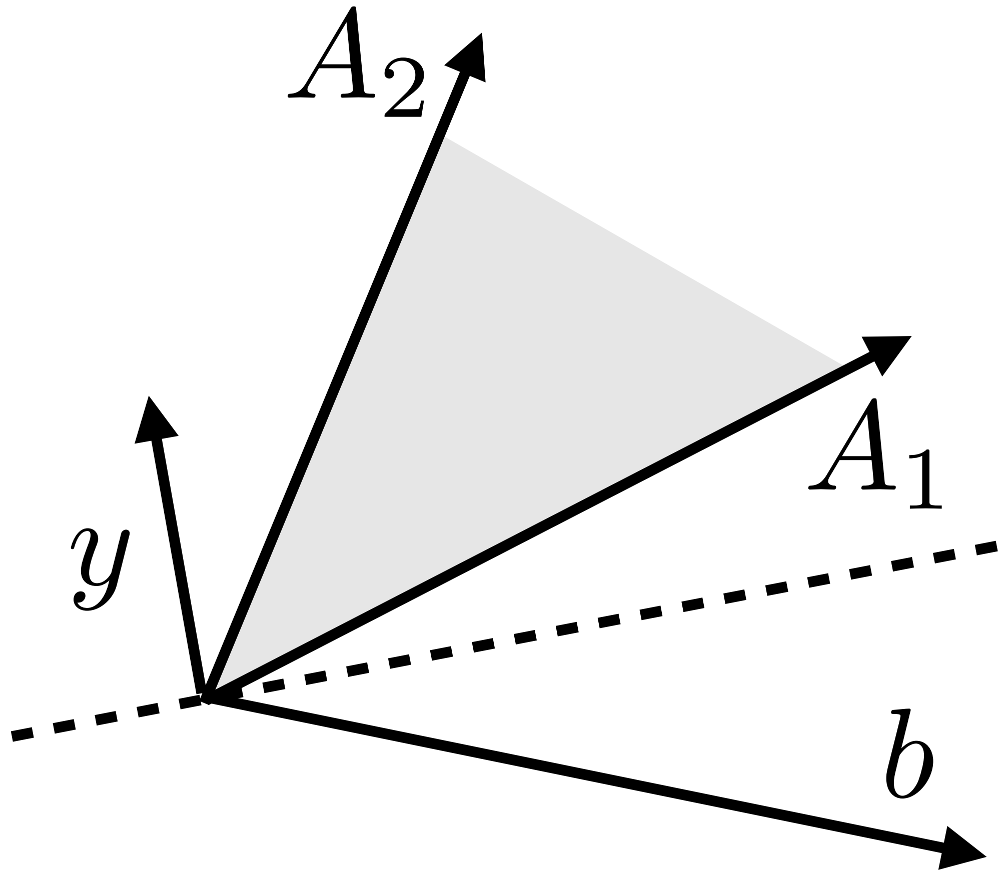


2. Second alternative

There exists a y with $A^T y \geq 0$, $b^T y < 0$

$$y^T A_i \geq 0, \quad i = 1, \ldots, m, \qquad y^T b < 0$$

The hyperplane $y^T z = 0$ separates b from $A_1, \ldots, A_n$



Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both true (easy)

$x \geq 0$, $Ax = b$ and $y^T A \geq 0$ $\longrightarrow$ $y^T b = y^T A x \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0

subject to $Ax = b$

$x \geq 0$

Dual

maximize $-b^T y$

subject to $A^T y \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0

subject to $Ax = b$

$x \geq 0$

Dual

maximize $-b^T y$

subject to $A^T y \geq 0$

↑

$y = 0$ always feasible

Strong duality holds

$$d^\star \neq -\infty, \quad p^\star = d^\star$$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0
subject to $Ax = b$
$x \geq 0$

Dual

maximize $-b^T y$
subject to $A^T y \geq 0$

Alternative 1: primal feasible $p^\star = d^\star = 0$

$b^T y \geq 0$ for all y such that $A^T y \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y \geq 0 \end{array}$$

Alternative 2: primal infeasible $p^\star = d^\star = +\infty$

There exists y such that $A^T y \geq 0$ and $b^T y < 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0
subject to $Ax = b$
$x \geq 0$

Dual

maximize $-b^T y$
subject to $A^T y \geq 0$

Alternative 2: primal infeasible $p^\star = d^\star = +\infty$

There exists y such that $A^T y \geq 0$ and $b^T y < 0$

y is an **infeasibility certificate**

Farkas lemma

Many variations

There exists x with $Ax = b,\ x \geq 0$

OR

There exists y with $A^T y \geq 0,\ b^T y < 0$

There exists x with $Ax \leq b,\ x \geq 0$

OR

There exists y with $A^T y \geq 0,\ b^T y < 0,\ y \geq 0$

There exists x with $Ax \leq b$

OR

There exists y with $A^T y = 0,\ b^T y < 0,\ y \geq 0$

Complementary slackness

Optimality conditions

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Optimality conditions

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax \leq b$
- y is **dual feasible:** $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Optimality conditions

Primal		**Dual**	
minimize	$c^T x$	maximize	$-b^T y$
subject to	$Ax \leq b$	subject to	$A^T y + c = 0$
			$y \geq 0$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax \leq b$
- y is **dual feasible:** $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Can we **relate** x and y (not only the objective)?

Complementary slackness

Primal

minimize $\quad c^T x$

subject to $\quad Ax \le b \qquad (y)$

Dual

maximize $\quad -b^T y$

subject to $\quad A^T y + c = 0$

$\qquad y \ge 0$

Theorem

Primal,dual feasible x, y are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and y have a **complementary sparsity** pattern:

$$y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$$
$$a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$$

Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \le b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0 \rightarrow c = -A^T y$

$y \ge 0$

Proof

The duality gap at primal feasible x and dual feasible y can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

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Since all the elements of the sum are nonnegative, they must all be 0 ■

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For **feasible** x and y **complementary slackness** = **zero duality gap**

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible $x = (1, 1)$ is optimal

Example

minimize $\quad -4x_1 - 5x_2$

subject to $\quad \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at x $\longrightarrow$ $y = (0, y_2, 0, y_4)$

$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

and $\quad y_2 \geq 0, \quad y_4 \geq 0$

Example

minimize $\quad -4x_1 - 5x_2$

subject to
$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

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$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

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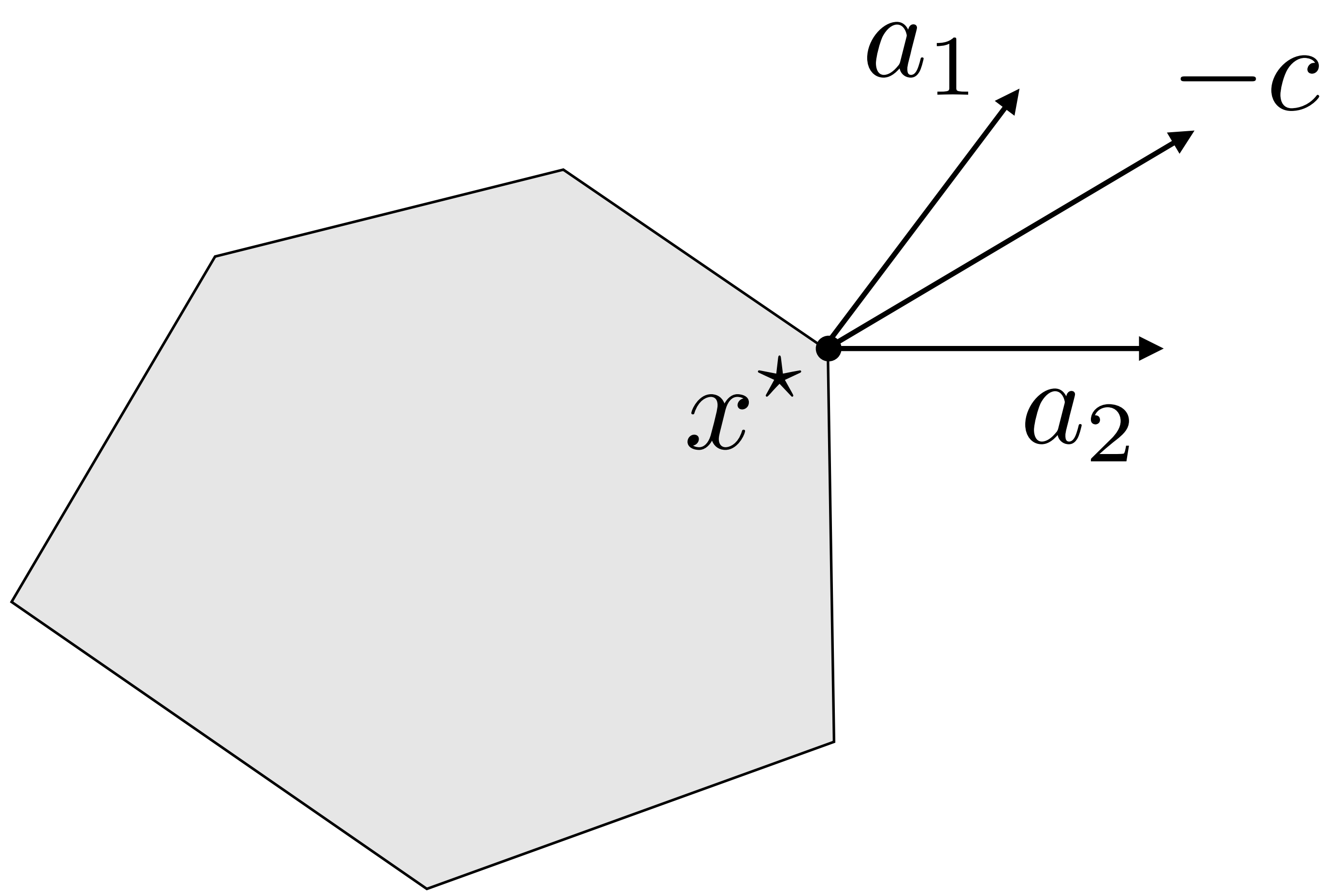
$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Complementary slackness is useful to recover $y^\star$ from $x^\star$

Geometric interpretation

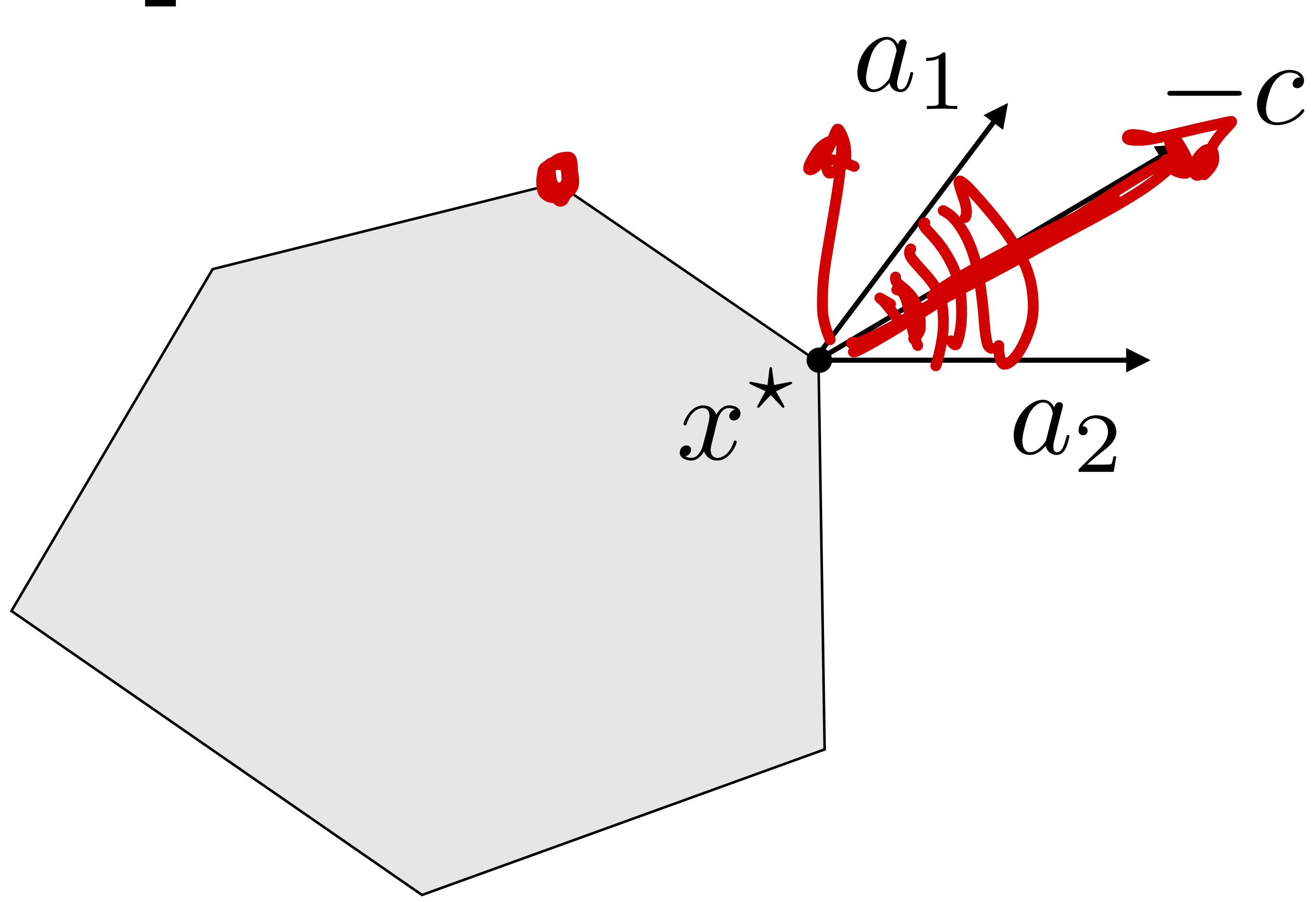
Example in $\mathbf{R}^2$



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Geometric interpretation

Example in R^2



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Optimal dual solution y satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$

KKT Conditions

Lagrangian and duality

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

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Lagrangian and duality

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$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

Dual function

$$\begin{aligned} g(y) &= \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right) \\ &= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x \\ &= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Lagrangian and duality

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$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$

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Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

Lagrangian and duality

Primal

minimize $\quad c^T x$

subject to $\quad Ax \leq b$

Dual

maximize $\quad -b^T y$

subject to $\quad A^T y + c = 0$

$\quad y \geq 0$

Dual function

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b) \right)$$

$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y \right)^T x$$

$$= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

Karush-Kuhn-Tucker conditions

Optimality conditions for linear optimization

Primal		Dual	
minimize	$c^T x$	maximize	$-b^T y$
subject to	$Ax \leq b$	subject to	$A^T y + c = 0$
			$y \geq 0$

Primal feasibility $Ax \leq b$

Dual feasibility $\nabla_x L(x, y) = A^T y + c = 0$ and $y \geq 0$

Complementary slackness $y_i(Ax - b)_i = 0, \quad i = 1, \ldots, m$

Karush-Kuhn-Tucker conditions

Solving linear optimization problems

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

We can solve our optimization problem by solving a system of equations

$$\nabla_x L(x, y) = A^T y + c = 0$$

$$b - Ax \geq 0$$

$$y \geq 0$$

$$y^T (b - Ax) = 0$$

Linear optimization duality

Today, we learned to:

- **Interpret** linear optimization duality using game theory
- **Prove** Farkas lemma using duality
- **Geometrically link** primal and dual solutions with complementary slackness
- **Derive** KKT optimality conditions

References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 11: Game Theory

Next lecture

- Sensitivity analysis