# ORF307 – Optimization

## 15. Sensitivity analysis

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# Ed Forum

- I was wondering what the word "certificate" and “tight” means
- When is a game not zero-sum?
- How does the minmax theorem apply when there is no optimal strategy/Nash Equilibrium for both players?

# Recap

# Optimal objective values

**Primal**

minimize $c^T x$
subject to $Ax \leq b$

**Dual**

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

# Weak duality

**Theorem**

If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

**Proof**

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y \qquad \blacksquare$$

**Remark**

- Any dual feasible $y$ gives a **lower bound** on the primal optimal value
- Any primal feasible $x$ gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

# Strong duality

**Theorem**
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

# Minmax theorem

**Theorem**

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

**Proof**

The optimal $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

The optimal $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \geq w\mathbf{1} \\ & \mathbf{1}^T y = 1 \\ & y \geq 0 \end{array}$$

The two LPs are **duals** and by **strong duality** the equality follows. ■

# Nash equilibrium

**Theorem**

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

**Consequence**

The pair of mixed strategies $(x^\star, y^\star)$ attains the **Nash equilibrium** of the two-person matrix game, i.e.,

$$x^T A y^\star \geq x^{\star T} A y^\star \geq x^{\star T} A y, \quad \forall x \in P_m, \ \forall y \in P_n$$

# Lagrangian and duality

**Primal**

minimize $c^T x$

subject to $Ax \leq b$

**Dual**

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

**Dual function**

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases}$$

⟶

**Lagrangian**

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

# Karush-Kuhn-Tucker conditions

## Optimality conditions for linear optimization

**Primal**

minimize $c^T x$

subject to $Ax \leq b$

**Dual**

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

**Primal feasibility** $Ax \leq b$

**Dual feasibility** $\nabla_x L(x, y) = A^T y + c = 0$ and $y \geq 0$

**Complementary slackness** $y_i (Ax - b)_i = 0, \quad i = 1, \ldots, m$

# Karush-Kuhn-Tucker conditions

## Solving linear optimization problems

**Primal**

minimize $c^T x$

subject to $Ax \leq b$

**Dual**

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

We can solve our optimization problem by solving a system of equations

$$\nabla_x L(x, y) = A^T y + c = 0$$
$$b - Ax \geq 0$$
$$y \geq 0$$
$$y^T (b - Ax) = 0$$

# Today's lecture

## Sensitivity analysis and game theory

- Primal and dual simplex
- Adding variables and constraints
- Global sensitivity
- Local sensitivity

# Primal and dual simplex

# Optimality conditions

**Primal problem**

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

**Dual problem**

maximize $-b^T y$
subject to $A^T y + c \geq 0$

$x$ and $y$ are **primal** and **dual** optimal if and only if

- $x$ is **primal feasible**: $Ax = b$ and $x \geq 0$
- $y$ is **dual feasible**: $A^T y + c \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

# Primal and dual basic feasible solutions

**Primal problem**

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

**Dual problem**

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

Given a **basis** matrix $B$

**Reduced costs**

**Primal feasible**: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

**Dual feasible**: $A^T y + c \geq 0$. Set $y = -A_B^{-T} c_B$. Dual feasible if $\bar{c} = c + A^T y \geq 0$

**Zero duality gap:** $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T} c_B = c_B x_B - c_B^T A_B^{-1} b = 0$

(by construction)

# The primal (dual) simplex method

**Primal problem**

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

**Dual problem**

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

**Primal simplex**

- Primal feasibility
- Zero duality gap

↓

Dual feasibility

**Dual simplex**
(solve dual instead)

- Dual feasibility
- Zero duality gap

↓

Primal feasibility

# Adding new constraints and variables

# Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Is the solution $(x^\star, 0), y^\star$ **optimal** for the new problem?

# Adding new variables

## Optimality conditions

$$\begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

$\longrightarrow$ Solution $(x^\star, 0)$ is still **primal feasible**

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

**Yes**

$(x^\star, 0)$ still **optimal** for new problem

**Otherwise**

Primal simplex

# Adding new variables

## Example

$$\begin{array}{lll} \text{minimize} & -60x_1 - 30x_2 - 20x_3 & \text{-profit} \\ \text{subject to} & 8x_1 + 6x_2 + x_3 \leq 48 & \text{material} \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20 & \text{production} \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 & \text{quality control} \\ & x \geq 0 & \end{array}$$

$$c = (-60, -30, -20, 0, 0, 0)$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

# Adding new variables

## Example: add new product?

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

minimize $c^T x + c_{n+1} x_{n+1}$

subject to $Ax + A_{n+1} x_{n+1} = b$

$x, x_{n+1} \geq 0$

**Previous solution**

$x^\star = (2, 0, 8, 24, 0, 0)$, $y^\star = (0, 10, 10)$, $c^T x^\star = -280$, basis $\{1, 3, 4\}$

$(x^\star, 0)$ **is still optimal**

$$A_{n+1}^T y^\star + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

**Shall we add a new product?**

# Adding new constraints

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

Solution $x^\star, y^\star$

$\longrightarrow$

minimize $c^T x$
subject to $Ax = b$
$a_{m+1}^T x = b_{m+1}$
$x \geq 0$

**Dual**

maximize $-b^T y$
subject to $A^T y + a_{m+1} y_{m+1} + c \geq 0$

Is the solution $x^\star, (y^\star, 0)$ **optimal** for the new problem?

# Adding new constraints

## Optimality conditions

maximize $\quad -b^T y$

subject to $\quad A^T y + a_{m+1} y_{m+1} + c \geq 0$ ⟶ Solution $(y^\star, 0)$ is still **dual feasible**

Is $x^\star$ still **primal feasible**?

$$\begin{aligned} &Ax = b \\ &a_{m+1}^T x = b_{m+1} \\ &x \geq 0 \end{aligned}$$

**Yes**

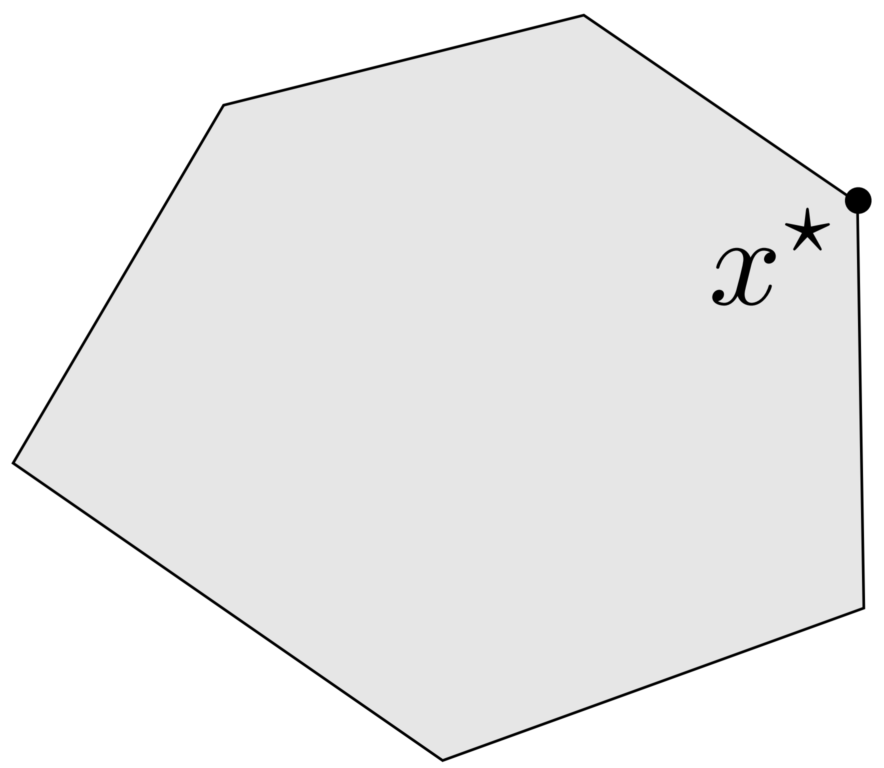
$x^\star$ still **optimal** for new problem

**Otherwise**

Dual simplex

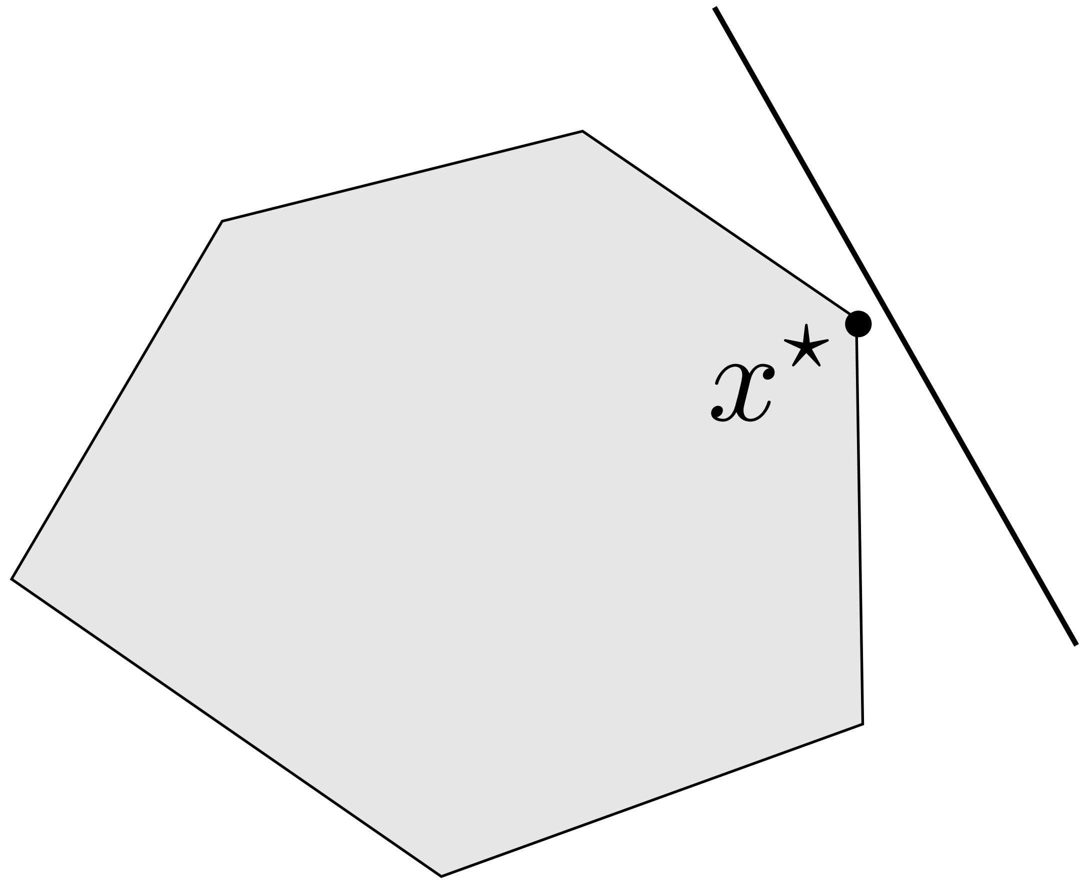
# Adding new constraints

## Example

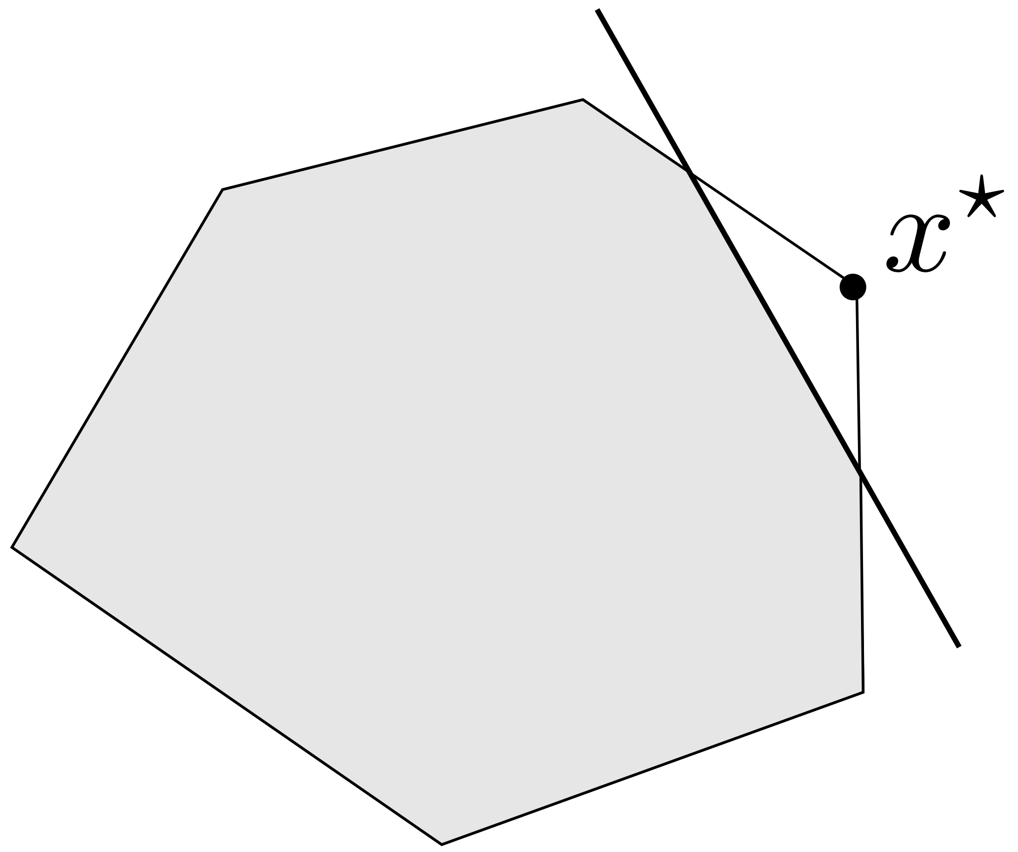



Add new constraint →

$x^\star$ still **feasible**




$x^\star$ **infeasible**



# Global sensitivity analysis

# Changes in problem data

**Goal:** extract information from $x^\star, y^\star$ about their sensitivity with respect to changes in problem data

**Modified LP**

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b + u \\
& x \geq 0
\end{array}
$$

**Optimal cost** $p^\star(u)$

# Global sensitivity

**Dual of modified LP**

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

**Global lower bound**

Given $y^\star$ a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T y^\star \\ &= p^\star(0) - u^T y^\star \end{aligned}$$

(from weak duality and dual feasibility)

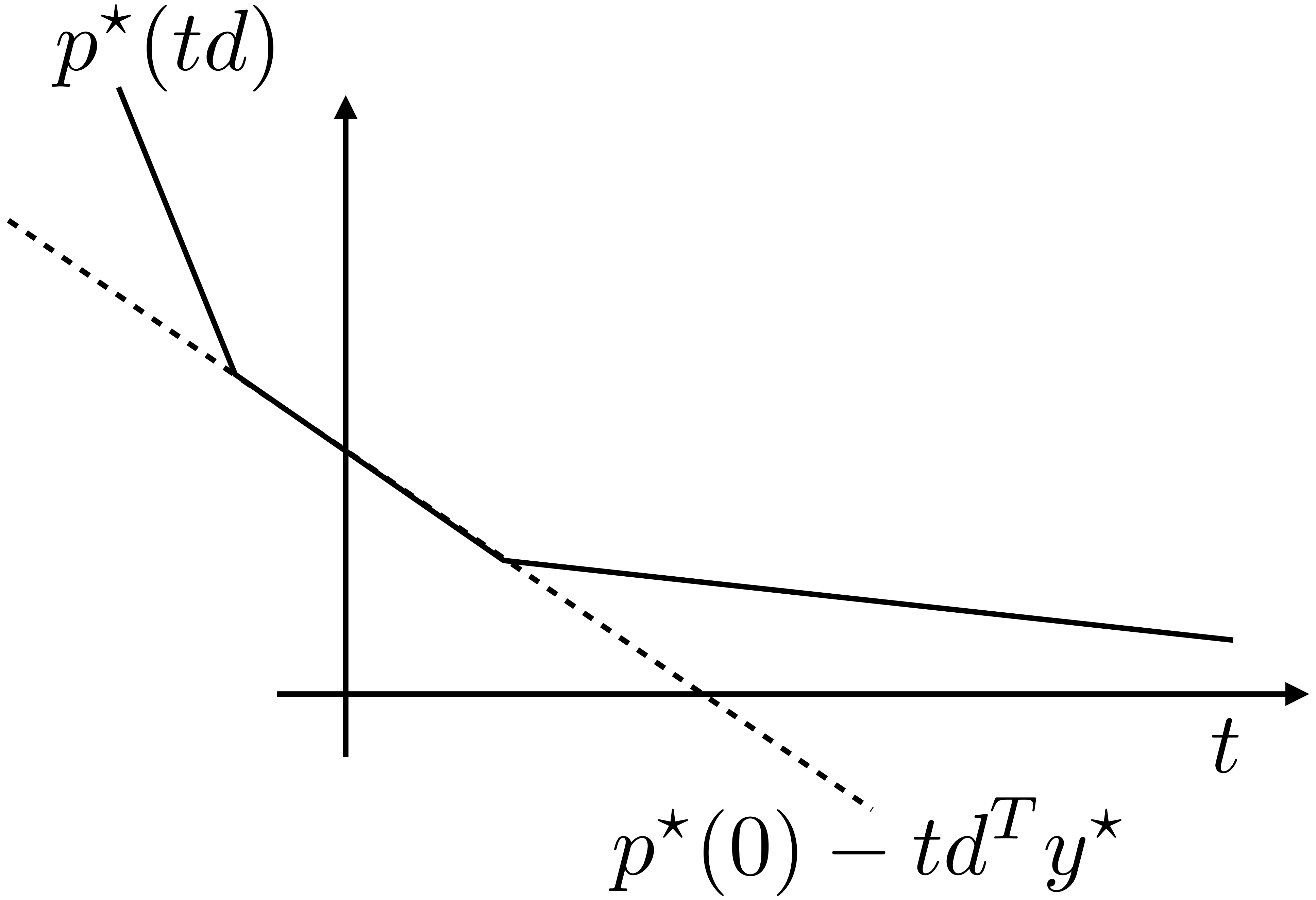
It holds for any $u$

# Global sensitivity

## Example

Take $u = td$ with $d \in \mathbf{R}^m$ fixed

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + td \\ & x \geq 0 \end{array}$$




$p^\star(td)$ is the optimal value as a function of $t$

**Sensitivity information** (assuming $d^T y^\star \geq 0$)

- $t < 0$ the optimal value increases
- $t > 0$ the optimal value decreases (not so much if $t$ is small)

# Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

**Assumption:** $p^\star(0)$ is finite

**Properties**

- $p^\star(u) > -\infty$ everywhere (from global lower bound)
- $p^\star(u)$ is piecewise-linear on its domain

# Optimal value function is piecewise linear

## Proof

**Dual feasible set**

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

$$D = \{y \mid A^T y + c \geq 0\}$$

**Assumption:** $p^\star(0)$ is finite

If $p^\star(u)$ finite

$$p^\star(u) = \max_{y \in D} -(b+u)^T y = \max_{k=1,\dots,r} -y_k^T u - b^T y_k$$

$y_1, \dots, y_r$ are the extreme points of $D$

# Local sensitivity analysis

# Local sensitivity

## $u$ in neighborhood of the origin

**Original LP**

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

→

**Optimal solution**

Primal $x_i^\star = 0, \quad i \notin B$

$x_B^\star = A_B^{-1} b$

Dual $y^\star = -A_B^{-T} c_B$

**Modified LP**

minimize $c^T x$

subject to $Ax = b + u$

$x \geq 0$

**Modified dual**

maximize $-(b+u)^T y$

subject to $A^T y + c \geq 0$

**Optimal basis does not change**

**Modified optimal solution**

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1} u$$

$$y^\star(u) = y^\star$$

# Derivative of the optimal value function

**Modified optimal solution**

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1}u$$
$$y^\star(u) = y^\star$$

**Optimal value function**

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c_B^T A_B^{-1} u \\ &= p^\star(0) - y^{\star T} u \end{aligned}$$

(affine for small $u$)

**Local derivative**

$$\nabla p^\star(u) = -y^\star$$

($y^\star$ are the **shadow prices**)

# Sensitivity example

| | | |
|---|---|---|
| minimize | $-60x_1 - 30x_2 - 20x_3$ | -profit |
| subject to | $8x_1 + 6x_2 + x_3 \leq 48$ | material |
| | $4x_1 + 2x_2 + 1.5x_3 \leq 20$ | production |
| | $2x_1 + 1.5x_2 + 0.5x_3 \leq 8$ | quality control |
| | $x \geq 0$ | |

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

What does $y_3^\star = 10$ mean?

Let’s increase the quality control budget by $1$, i.e., $u = (0, 0, 1)$

$$p^\star(u) = p^\star(0) - y^{\star T} u = -280 - 10 = -290$$

# Sensitivity analysis

Today, we learned to:

- **Reuse** primal and dual solutions when variables or constraints are added
- **Analyze** value function as problem parameters change
- **Compute** local sensitivity to parameter perturbations

# References


- D. Bertsimas and J. Tsitsiklis: Introduction to Linear Optimization
    - Chapter 5: Sensitivity analysis

- R. Vanderbei: “Linear Programming
    - Chapter 7: Sensitivity and parametric analysis

# Next lecture

- Network optimization