ORF307 – Optimization

10. Applications of linear optimization

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Ed Forum

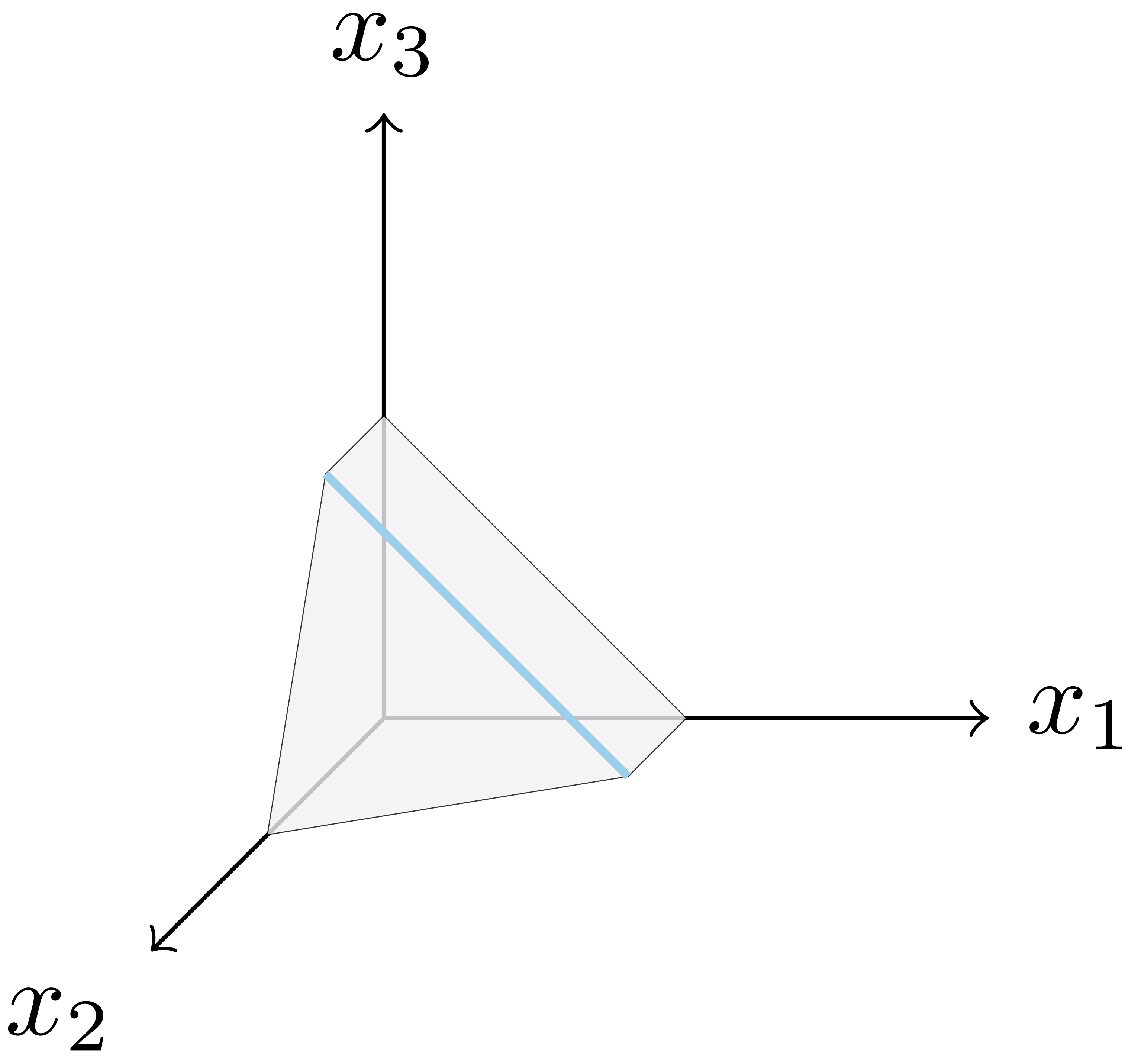
- **Midterm March 09**
  Time: 11:00am — 12:20pm
  Location: Same as lecture
  Topics: Up to last lecture (excluding equivalence theorem)
  Material allowed: Single sheet of paper. Double sided. Hand-written or typed.

- **Questions**
  - What is a basic feasible solution? how we solve for those?

Recap

Constructing a basic solution

Two equalities $(m = 2, n = 3)$



$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 + x_3 = 1 \\ & (1/2)x_1 + x_2 + (1/2)x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$n - m = 1$ inequalities have to be tight: $x_i = 0$

Set $x_1 = 0$ and solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow (x_2, x_3) = (0.5, 1)$$

Basic solutions

Standard form polyhedra

$P = \{x \mid Ax = b,\ x \geq 0\}$ with $A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
  - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
  - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

Basis matrix

Basis columns

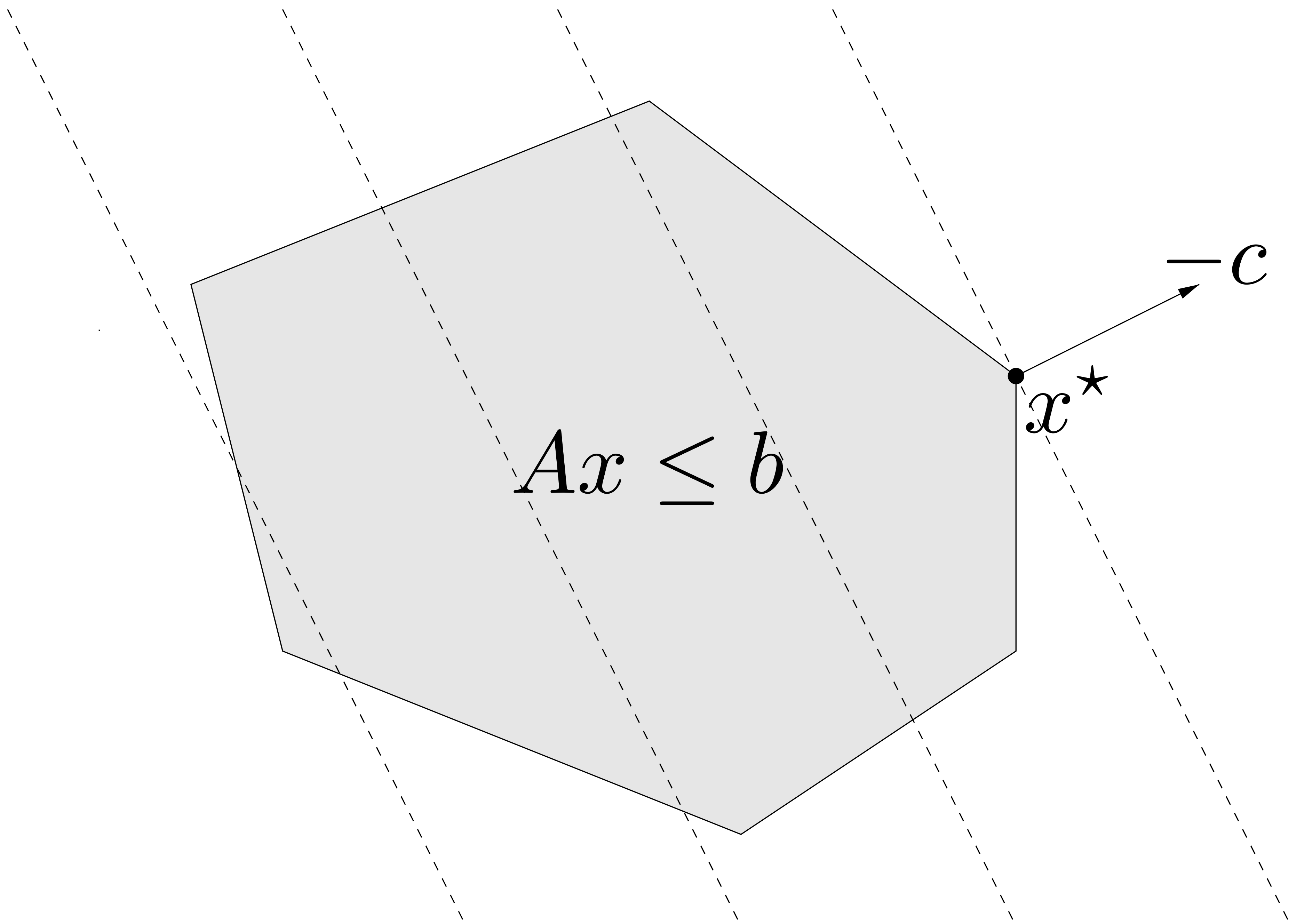
Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \quad \longrightarrow \quad \text{Solve } A_B x_B = b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

Optimality of extreme points

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$



If

- P has at least one extreme point
- There exists an optimal solution $x^\star$

Then, there exists an optimal solution that is an **extreme point** of P.

Solution method: restrict search to **extreme points.**

How to search among basic feasible solutions?

Idea

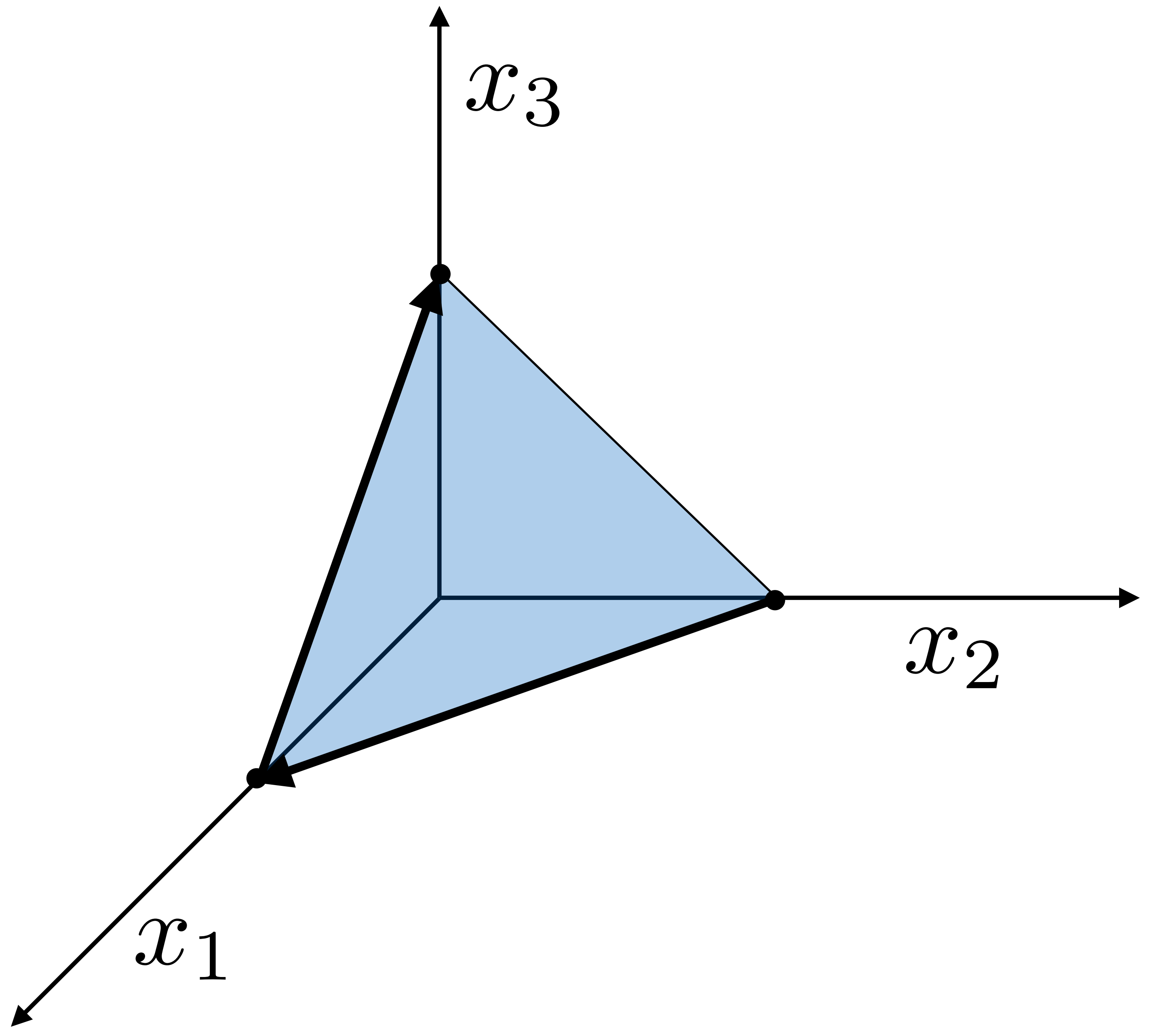
List all the basic feasible solutions, compare objective values and pick the best one.

Intractable!

If $n = 1000$ and $m = 100$, we have 10^{143} combinations!

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Today’s agenda

Applications of linear optimization

- Optimal control
- Character recognition
- Portfolio optimization

Optimal control

Optimal control problems

Linear dynamical system

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, 2, \ldots$$
$$y_t = Cx_t, \quad t = 1, 2, \ldots$$

- The n-vector x_t is the *state* at time t
- The m-vector u_t is the *input* at time t
- The p-vector y_t is the *output* at time t
- The $n \times n$ matrix A is the *dynamics matrix*
- The $n \times m$ matrix B is the *input matrix*
- The $p \times n$ matrix C is the *output matrix*

Simulation

- The sequence $x_1, x_2, \ldots$ is called *state trajectory*
- The sequence $y_1, y_2, \ldots$ is called *output trajectory*

- **Goal:** Given $x_1, u_1, u_2, \ldots$, find $x_2, x_3, \ldots$ and $y_2, y_3, \ldots$.
- Obtained by recursion. For $t = 1, 2, \ldots$, compute $x_{t+1} = Ax_t + Bu_t$ and $y_t = Cx_t$

Optimal control problem

Linear dynamical system

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, 2, \ldots$$
$$y_t = Cx_t, \quad t = 1, 2, \ldots$$

The problem

- The *initial state* $x_1 = x^{\mathrm{init}}$ is given
- **Goal.** Choose $u_1, u_2, \ldots, u_{T-1}$ to achieve some goals, e.g.,
  - Get to desired final state $x_T = x^{\mathrm{des}}$
  - Minimize the input effort (make $\|u_t\|$ small for all t)
  - Track desired output y_t^{des} (make $\|y_t - y_t^{\mathrm{des}}\|$ small for all t)

Least squares optimal control problem

$$\begin{array}{ll} \text{minimize} & \sum_{t=1}^{T} \|y_t - y_t^{\text{des}}\|^2 + \rho \sum_{t=1}^{T-1} \|u_t\|^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\ & y_t = Cx_t, \quad t = 1, \ldots, T \\ & x_1 = x^{\text{init}} \end{array}$$

Remarks

- The variables are $x_2, \ldots, x_T$, $y_2, \ldots, y_T$, and $u_t \ \ldots, u_{T-1}$
- Parameter $\rho > 0$ controls trade off between control ”energy” and tracking error
- It is a multi-objective and constrained least squares problem

1-norm optimal control problem

$$\begin{array}{ll}\text{minimize} & \sum_{t=1}^{T} \|y_t - y_t^{\text{des}}\|_1 + \rho \sum_{t=1}^{T-1} \|u_t\|_1 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\ & y_t = Cx_t, \quad t = 1, \ldots, T \\ & Dx_t \leq d, \quad t = 1, \ldots, T \\ & Eu_t \leq e, \quad t = 1, \ldots, T-1 \\ & x_1 = x^{\text{init}}\end{array}$$

Remarks

- $\|\cdot\|_1$ instead of $\|\cdot\|_2^2$
- Linear inequality constraints:

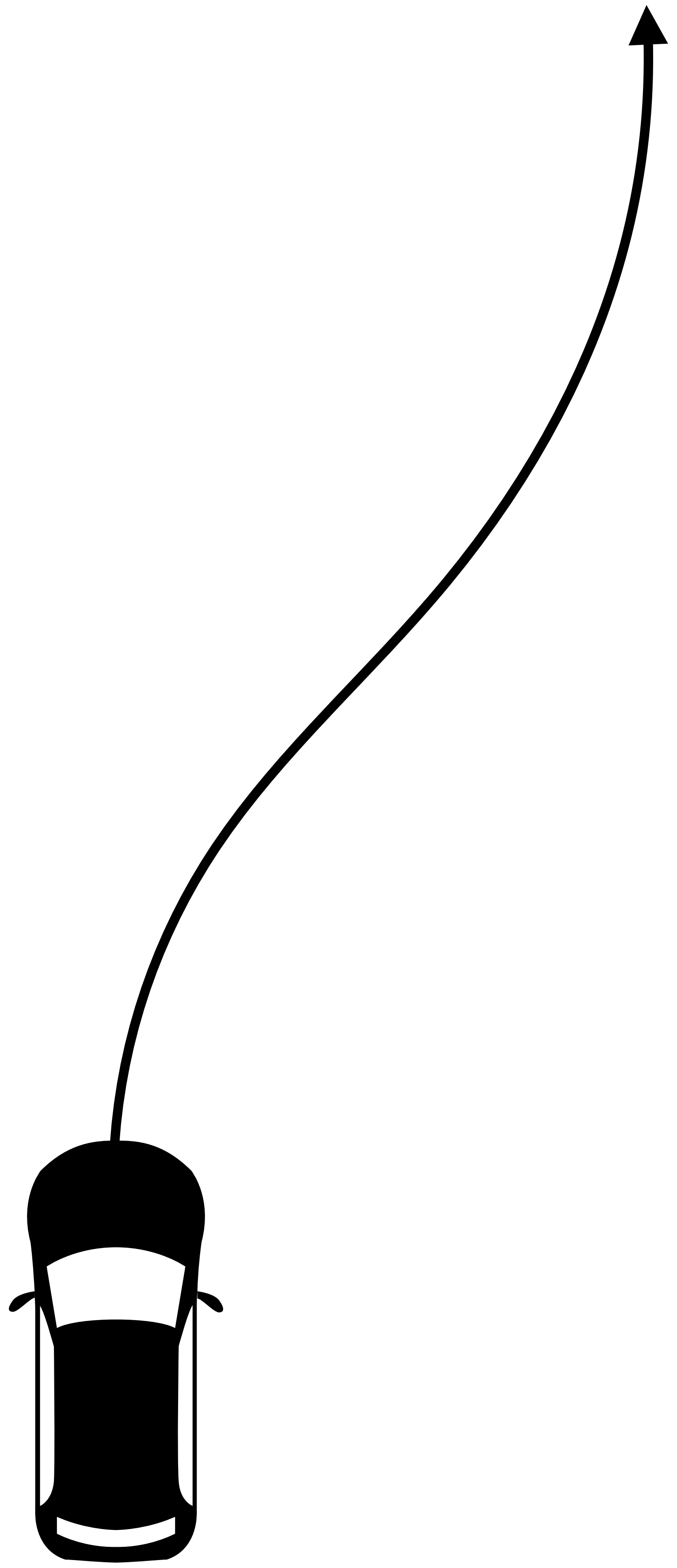
  $Dx_t \leq d$ for states and $Eu_t \leq e$ for inputs
- Is a linear optimization problem (with additional variables)

Vehicle example in a plane

Sample position and velocity at times $\tau = 0, h, 2h, \ldots$

Vehicle with mass m

- 2-vector p_t is the position at time ht
- 2-vector v_t is the velocity at time ht
- 2-vector u_t is the force applied at time ht
- $-\eta v_t$ is the friction force applied at ht



Small time interval h

$$\frac{p_{t+1} - p_t}{h} \approx v_t$$

$$m\frac{v_{t+1} - v_t}{h} \approx -hv_t + u_t$$

$$\longrightarrow$$

$$p_{t+1} = p_t + hv_t$$

$$v_{t+1} = (1 - h\eta/m)v_t + (h/m)u_t$$

Vehicle example in a plane

State

4-vector $x_t = (p_t, v_t)$

Dynamics

$$x_{t+1} = Ax_t + Bu_t$$
$$y_t = Cx_t$$

Laws of physics

$$p_{t+1} = p_t + hv_t$$
$$v_{t+1} = (1 - h\eta/m)v_t + (h/m)u_t$$

output = position

$$y_t = p_t$$

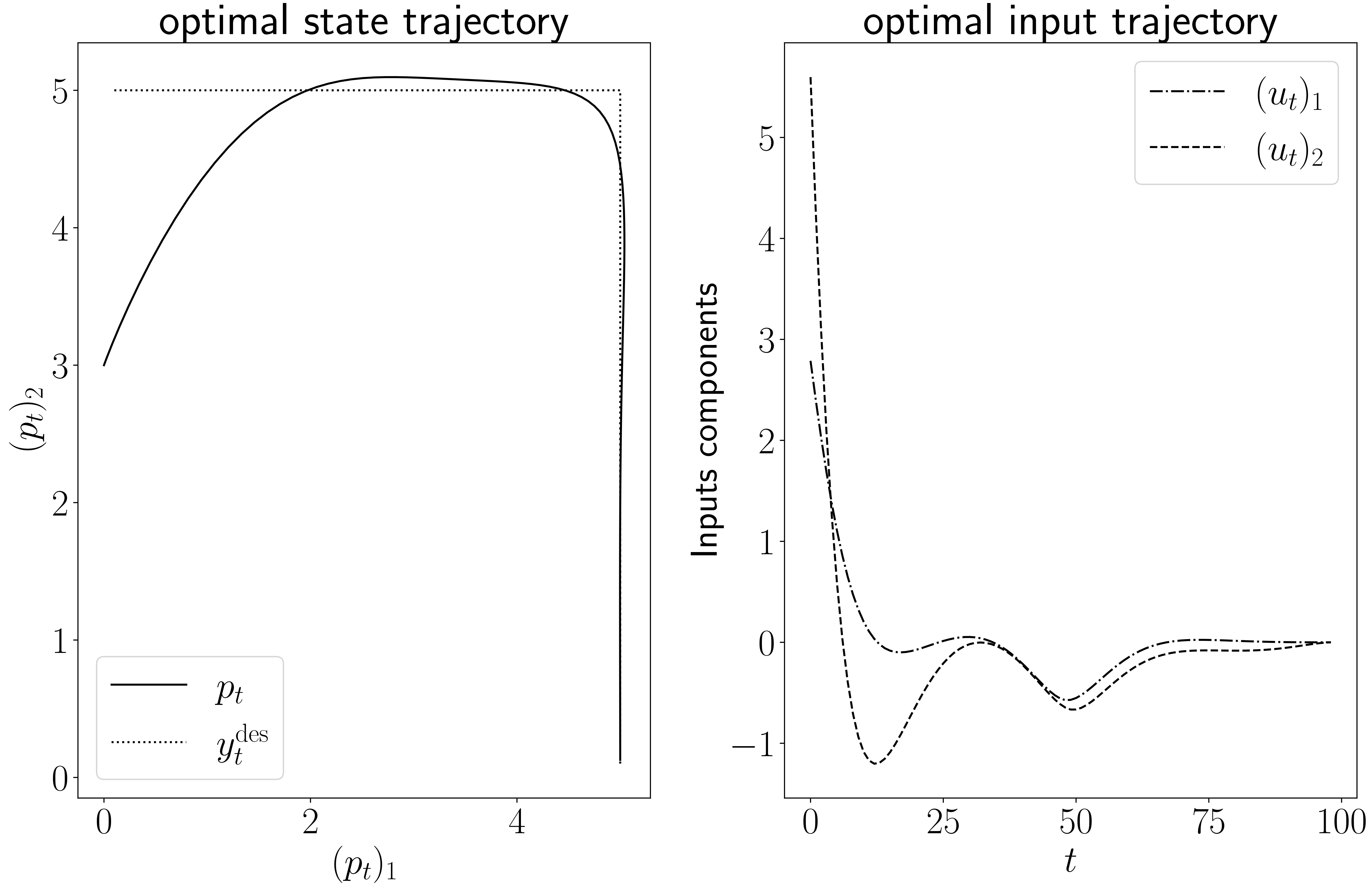
$$A = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 - h\eta/m & 0 \\ 0 & 0 & 0 & 1 - h\eta/m \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ h/m & 0 \\ 0 & h/m \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Vehicle example with output tracking

Least squares results

Parameters

$$T = 100, \quad h = 0.1, \quad \eta = 0.1, \quad m = 1$$

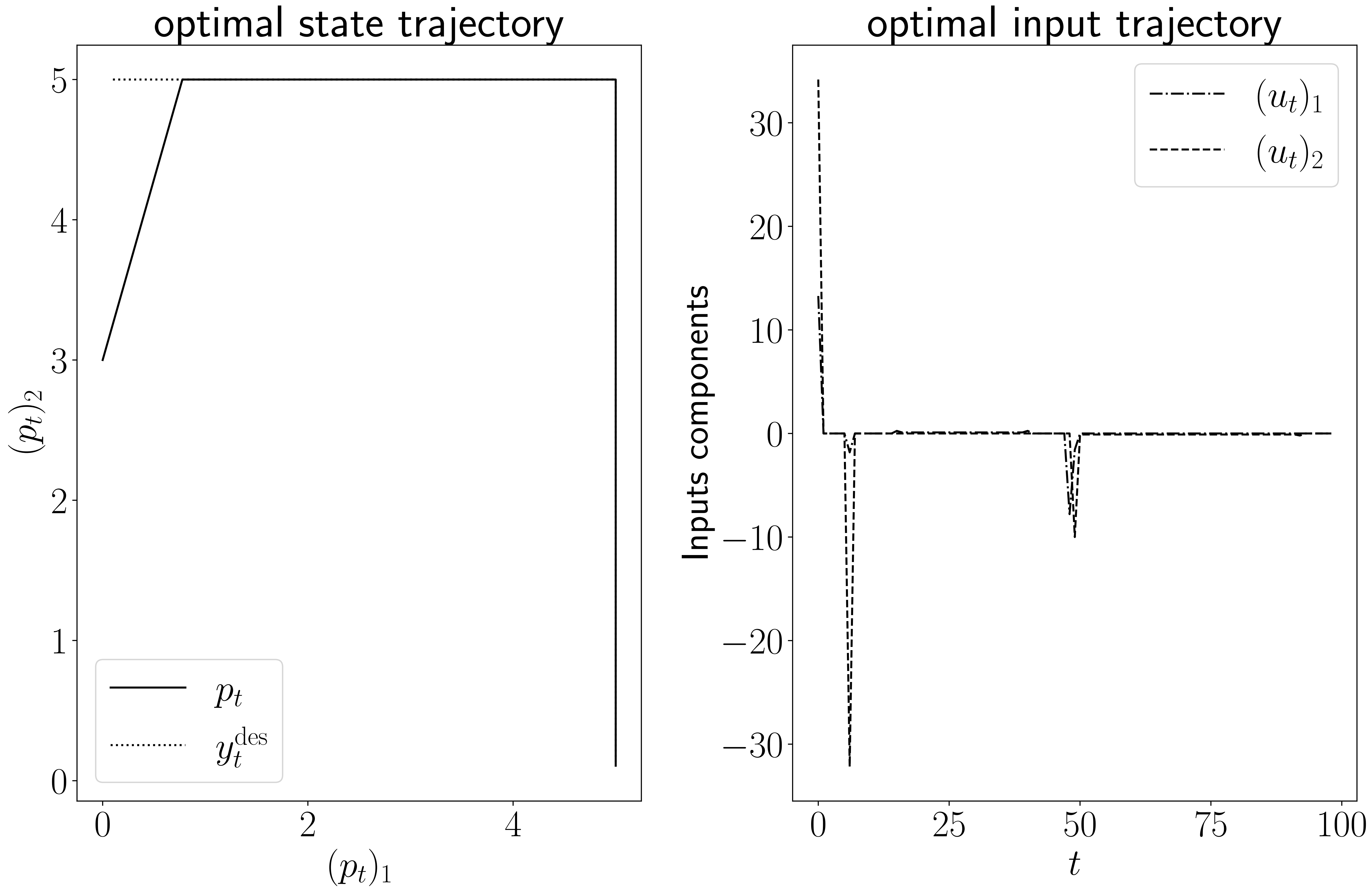


Vehicle example with output tracking

1-norm results

Parameters

$$T = 100, \quad h = 0.1, \quad \eta = 0.1, \quad m = 1$$



Vehicle example with output tracking

1-norm with constraints

Linear optimization can have more interesting constraints

$$
\begin{array}{ll}
\text{minimize} & \sum_{t=1}^{T} \|y_t - y_t^{\text{des}}\|_1 + \rho \sum_{t=1}^{T-1} \|u_t\|_1 \\
\text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \dots, T-1 \\
& y_t = Cx_t, \quad t = 1, \dots, T \\
& \|u_t\|_\infty \leq u^{\max}, \quad t = 1, \dots, T-1 \\
& \|u_t - u_{t-1}\|_1 \leq s^{\max}, \quad t = 1, \dots, T-1 \\
& x_1 = x^{\text{init}}
\end{array}
$$

max-input

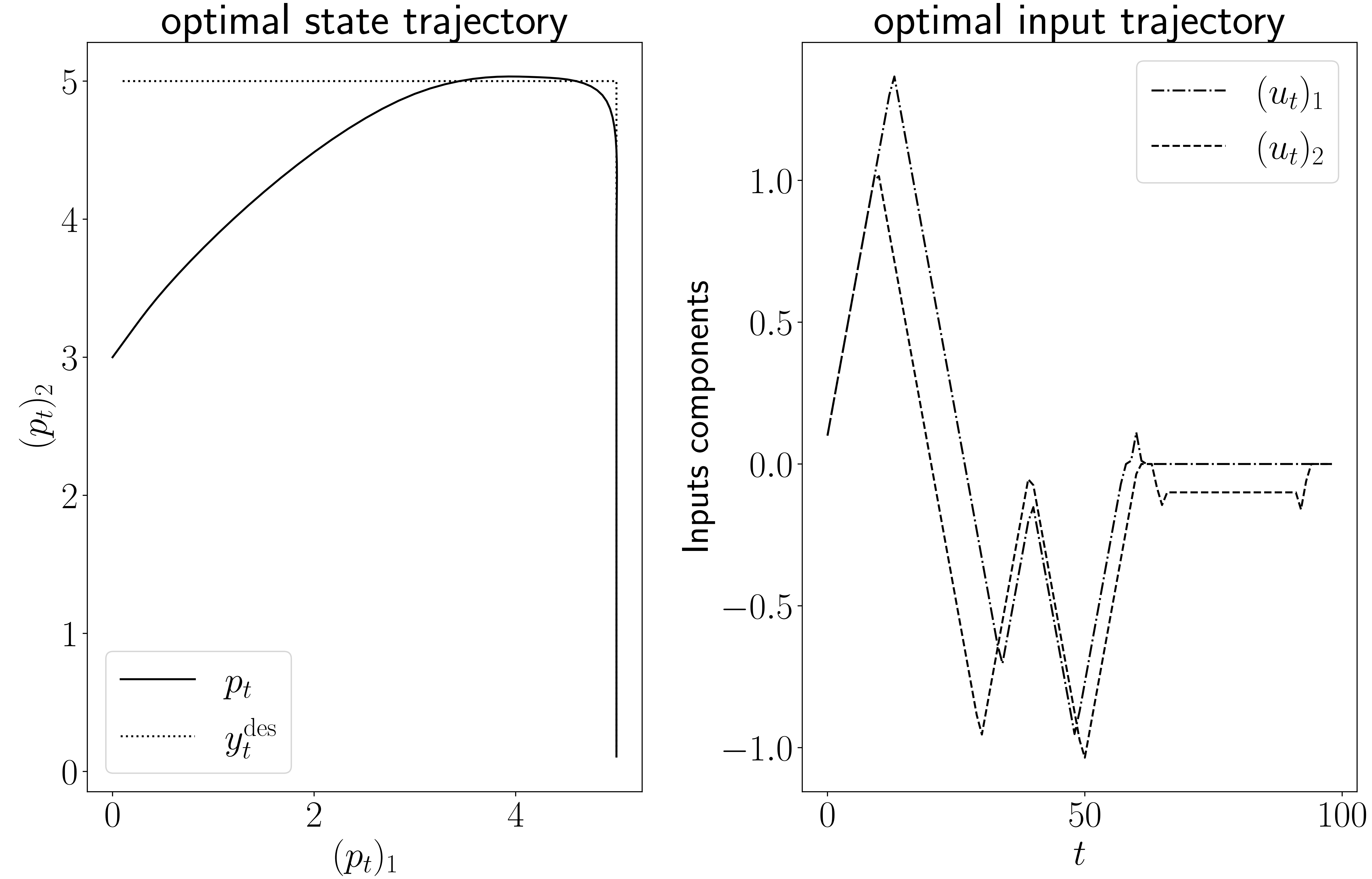
max-input variation

Vehicle example with output tracking

1-norm with constraints results

Parameters

$$u^{\max} = 10, \quad s^{\max} = 0.1$$

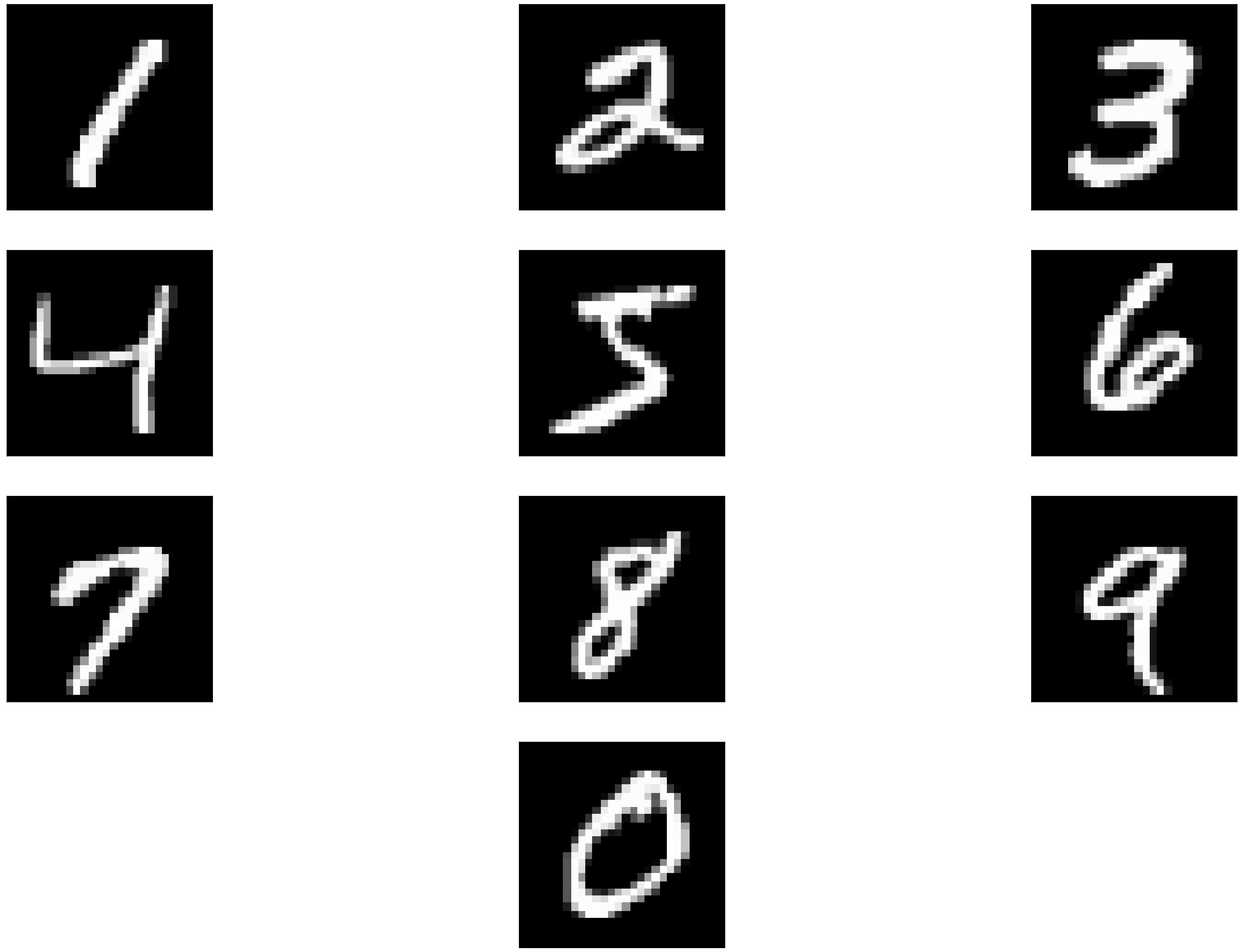


Character recognition

Character recognition

MNIST data set of handwritten numerals

- Each character is 28 x 28 pixels
- 60k example images
- 10k further testing images
- Each sample comes with a label 0 — 9



Goal

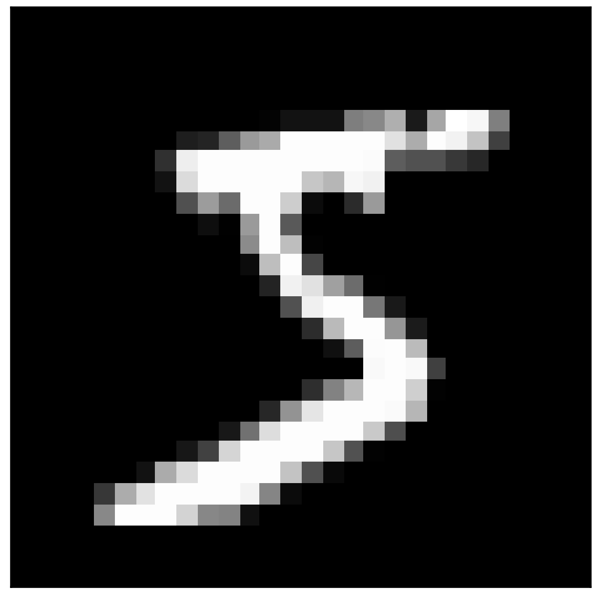
Use linear classification to identify handwritten numbers

Images representation

Monochrome images

Images represented as an $m \times n$ matrix X

$X =$

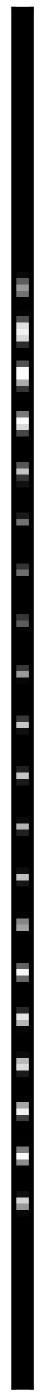


(in MNIST, $m = n = 28$)

Each value X_{ij} represents a pixel's intensity ($0 =$ black, and $255 =$ white)

We can represent an $m \times n$ matrix X by a single vector $x \in \mathbf{R}^{mn}$

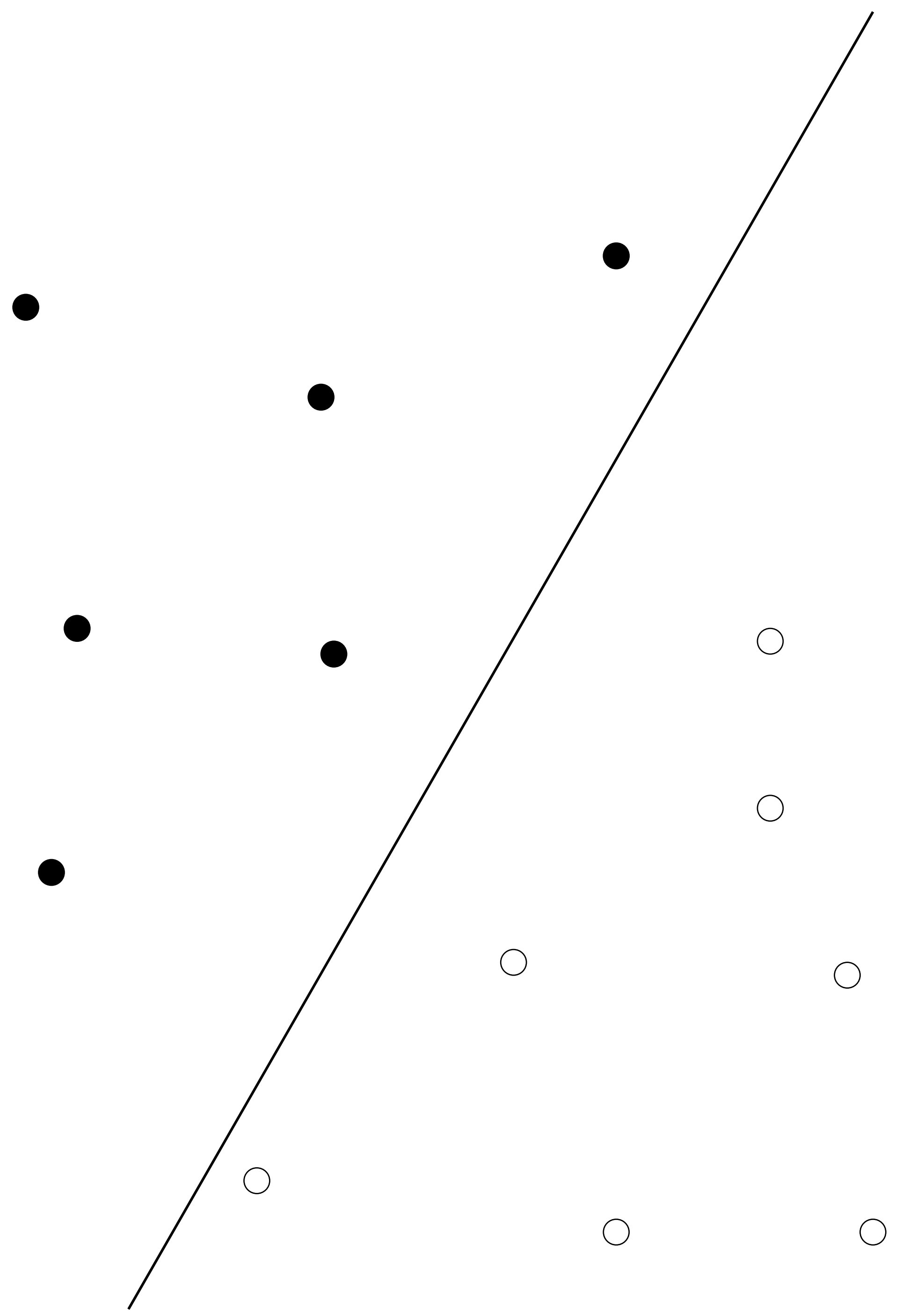
$x =$



$$X_{ij} = x_k, \qquad k = m(j-1) + i$$

Linear classification

Support vector machine (linear separation)



Given a set of points $\{v_1, \dots, v_N\}$ with binary labels $s_i \in \{-1, 1\}$
Find hyperplane that strictly separates the tho classes

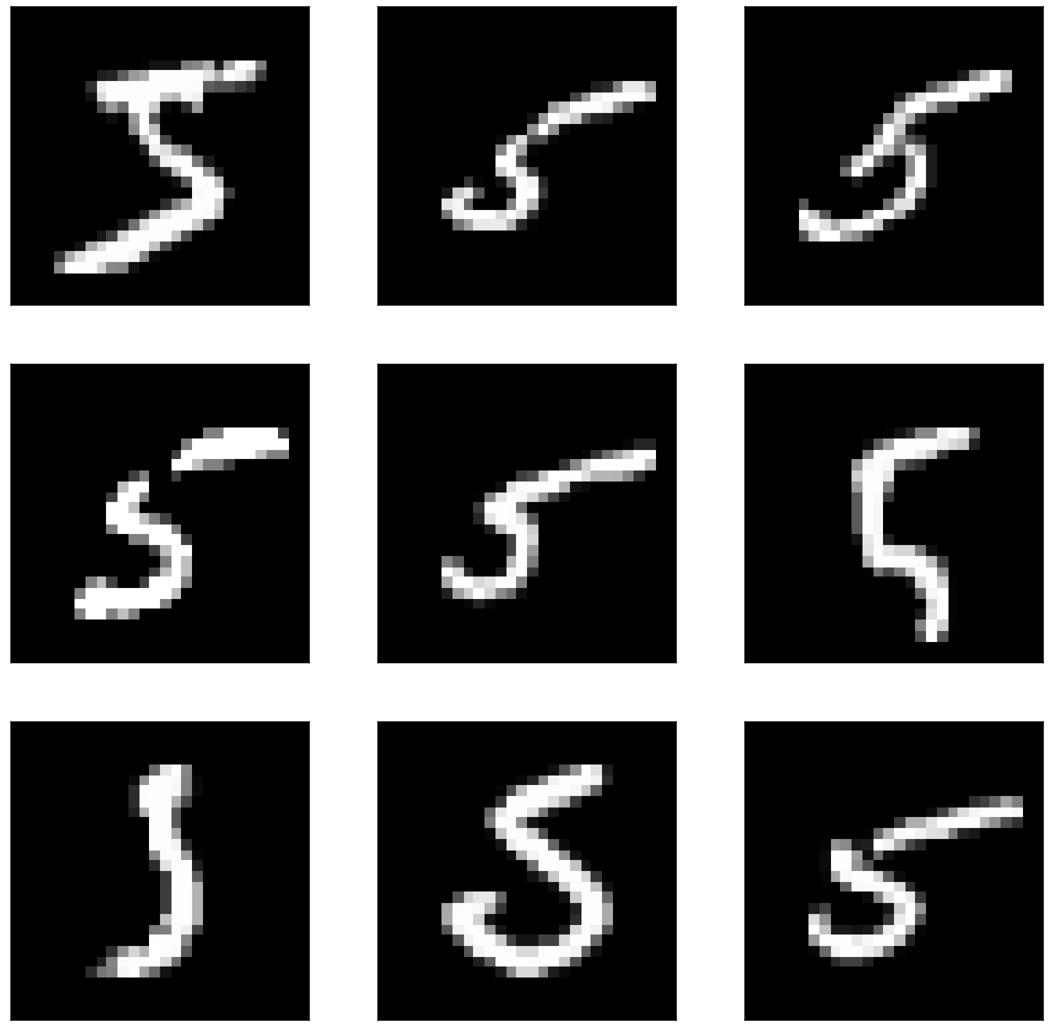
$$\begin{aligned} a^T v_i + b > 0 \quad \text{if} \quad s_i = 1 \\ a^T v_i + b < 0 \quad \text{if} \quad s_i = -1 \end{aligned} \longrightarrow \quad s_i(a^T v_i + b) \geq 1$$

Minimize sum of the violations + regularization

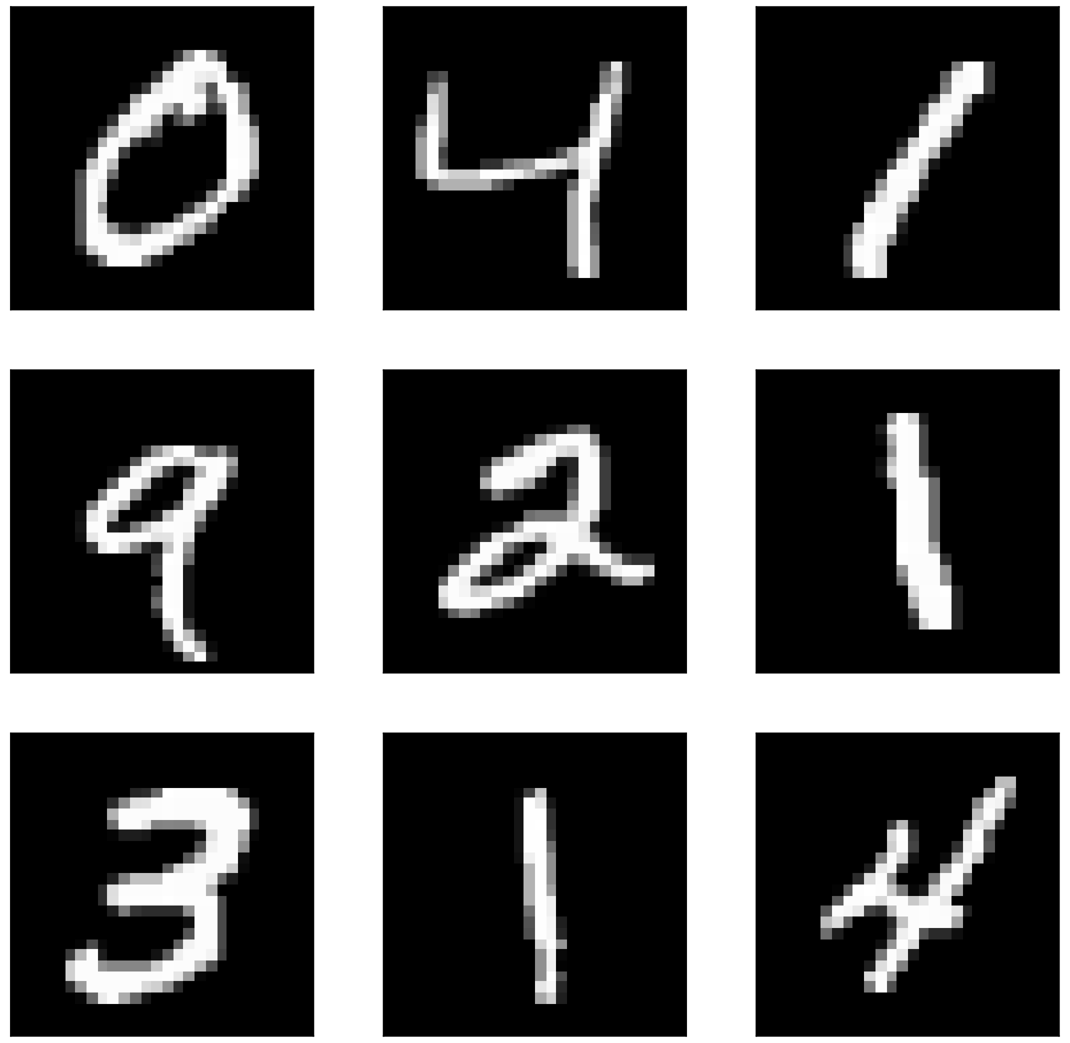
minimize $\sum_{i=1}^N \max\{0, 1 - s_i(a^T v_i + b)\} + \lambda \|a\|_1 \longleftarrow$ regularization

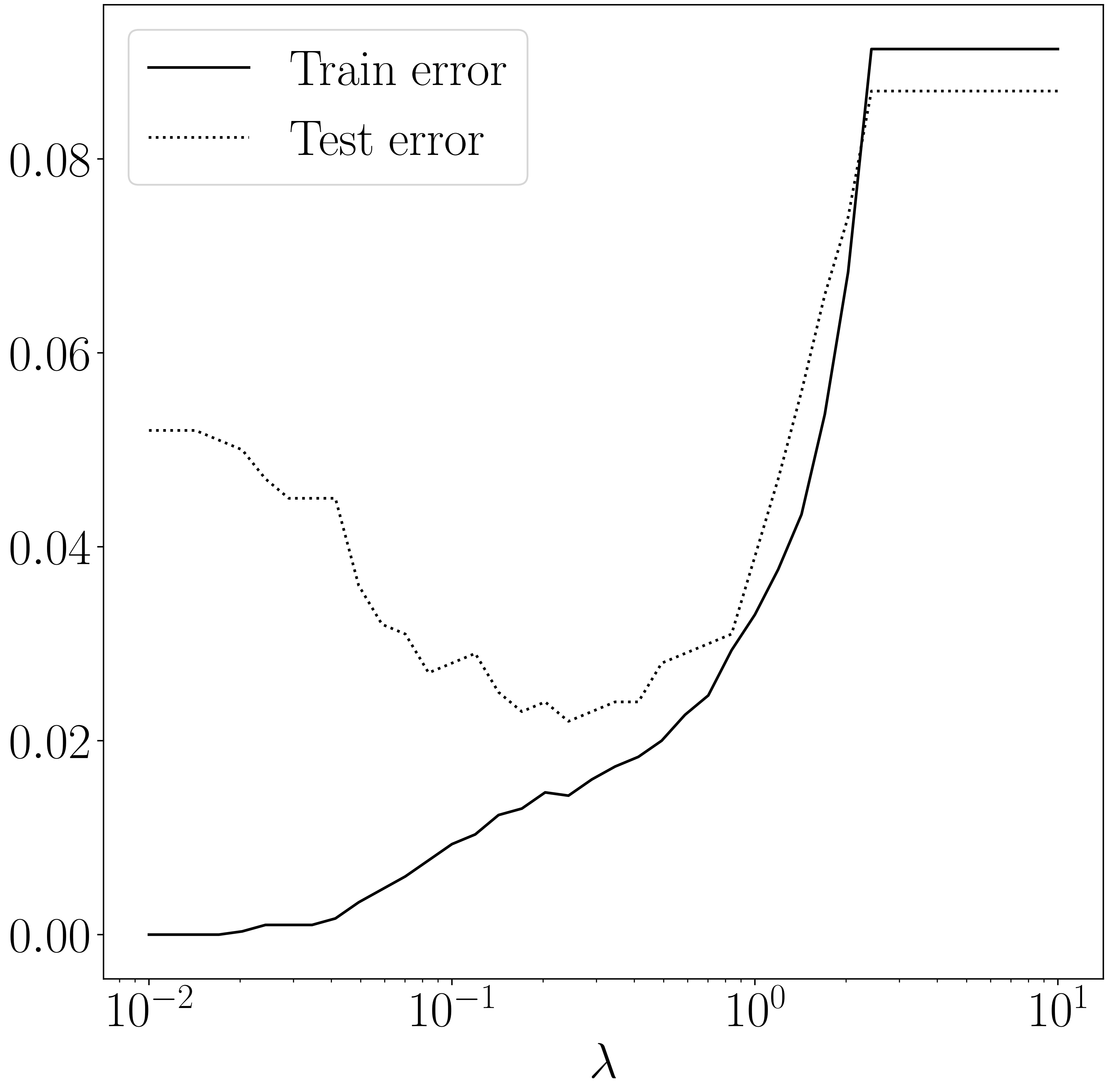
Learn to classify 5

5



Not 5

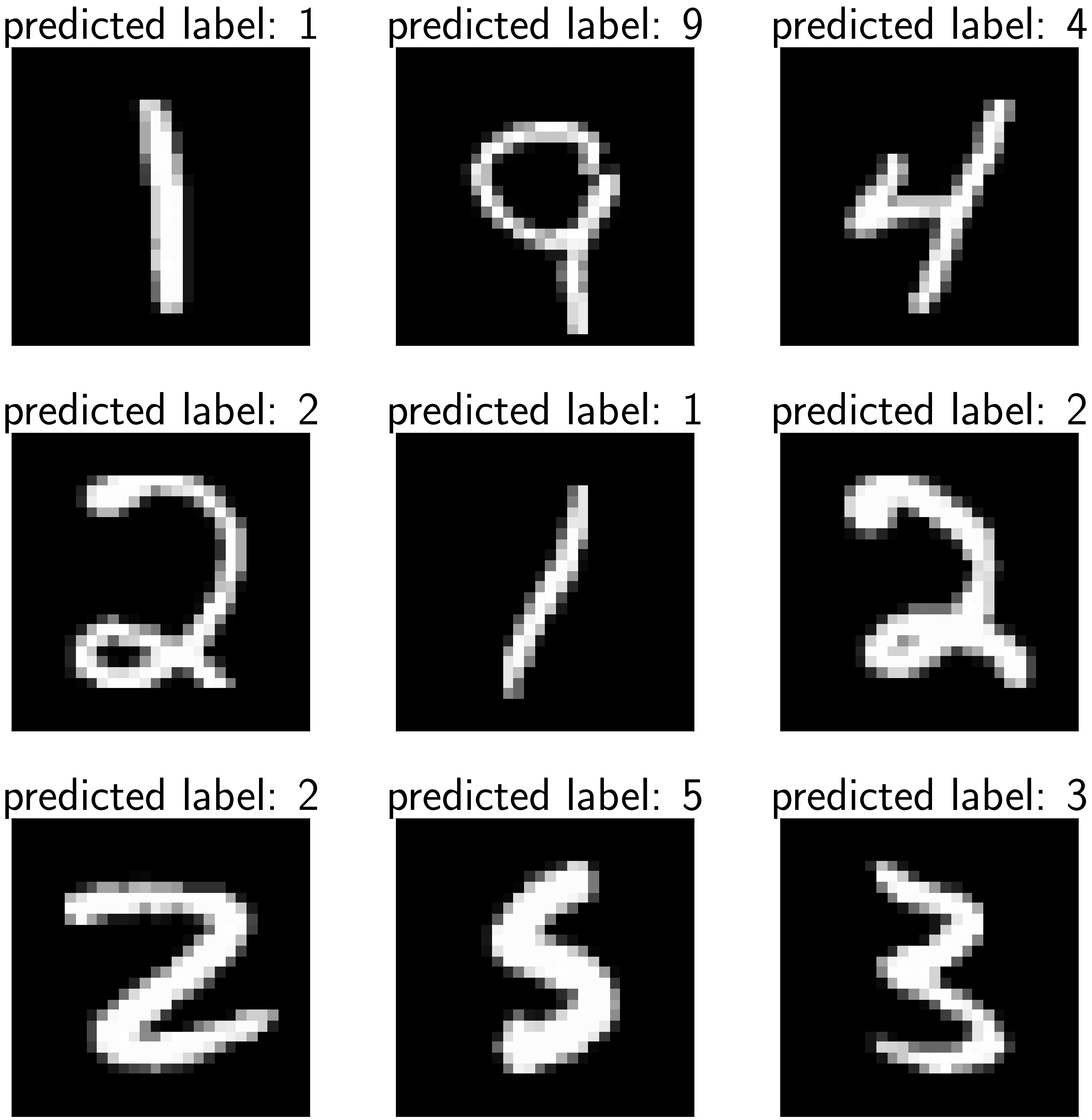




Multiclass classification

1. Train one classifier per label k (e.g., k vs anything else), obtaining (a_k, b_k)
2. Predict all results and take the maximum

$$\hat{y}^{(i)} = \text{argmax}_k \; a_k^T v^{(i)} + b_k$$



Portfolio optimization

Portfolio allocation weights

We want to invest V dollars in n different *assets* (stocks, bonds, ...) over periods $t = 1, \ldots, T$

Portfolio allocation weights

n-vector w gives the fraction of our total portfolio held in each asset

Properties

- Vw_j dollar value hold in asset j
- $\mathbf{1}^T w = 1$ (normalized)
- $w_j < 0$ means short positions (you borrow) (must be returned at time T)
- Example: $w = (-0.2, 0.0, 1.2)$

Short position of $0.2V$ on asset 1

Don't hold any of asset 2

Hold $1.2V$ in asset 3

Return over a period

Asset returns

$\tilde{r}_t$ is the (fractional) return of each asset over period t

example: $\tilde{r}_t = (0.01, -0.023, 0.02)$
(often expressed as percentage)

Portfolio return

$$r_t = \tilde{r}_t^T w$$

It is the (fractional) return for the entire portfolio over period t

Total portfolio value after a period

$$V_{t+1} = V_t + V_t \tilde{r}_t^T w = V_t(1 + r_t)$$

Portfolio optimization

How shall we choose the portfolio weight vector w?

Goals

High (average) return

Low risk

Data

- We know **realized asset returns** but not future ones
- **Optimization.** We choose w that would have worked well in the past
- **True goal.** Hope it will work well in the future (just like data fitting)

Linear optimization for portfolio objective

Average return

$$\begin{aligned}\mathbf{avg}(r) &= (1/T)\mathbf{1}^T(Rw)\\ &= (1/T)(R^T\mathbf{1})^T w = \mu^T w\end{aligned}$$

μ is the n-vector of
average returns per asset

1-norm risk approximation

$$\|r - \mathbf{avg}(r)\mathbf{1}\|_1/T$$

- No longer $\mathbf{std}(r)$ (divide by T instead of $\sqrt{T}$)
- Linear optimization representable
- Induces sparser fluctuations $|r_i - \mathbf{avg}(r)|$

Risk-return objective

$$-\mu^T w + \lambda\|Rw - (\mu^T w)\mathbf{1}\|_1/T$$

$$\uparrow$$

(tradeoff parameter)

Portfolio optimization

Minimize risk-return tradeoff

Chose n-vector w to solve

$$\begin{array}{ll} \text{minimize} & -\mu^T w + \lambda \|Rw - (\mu^T w)\mathbf{1}\|_1 / T \\ \text{subject to} & \mathbf{1}^T w = 1 \\ & w \geq 0 \end{array}$$

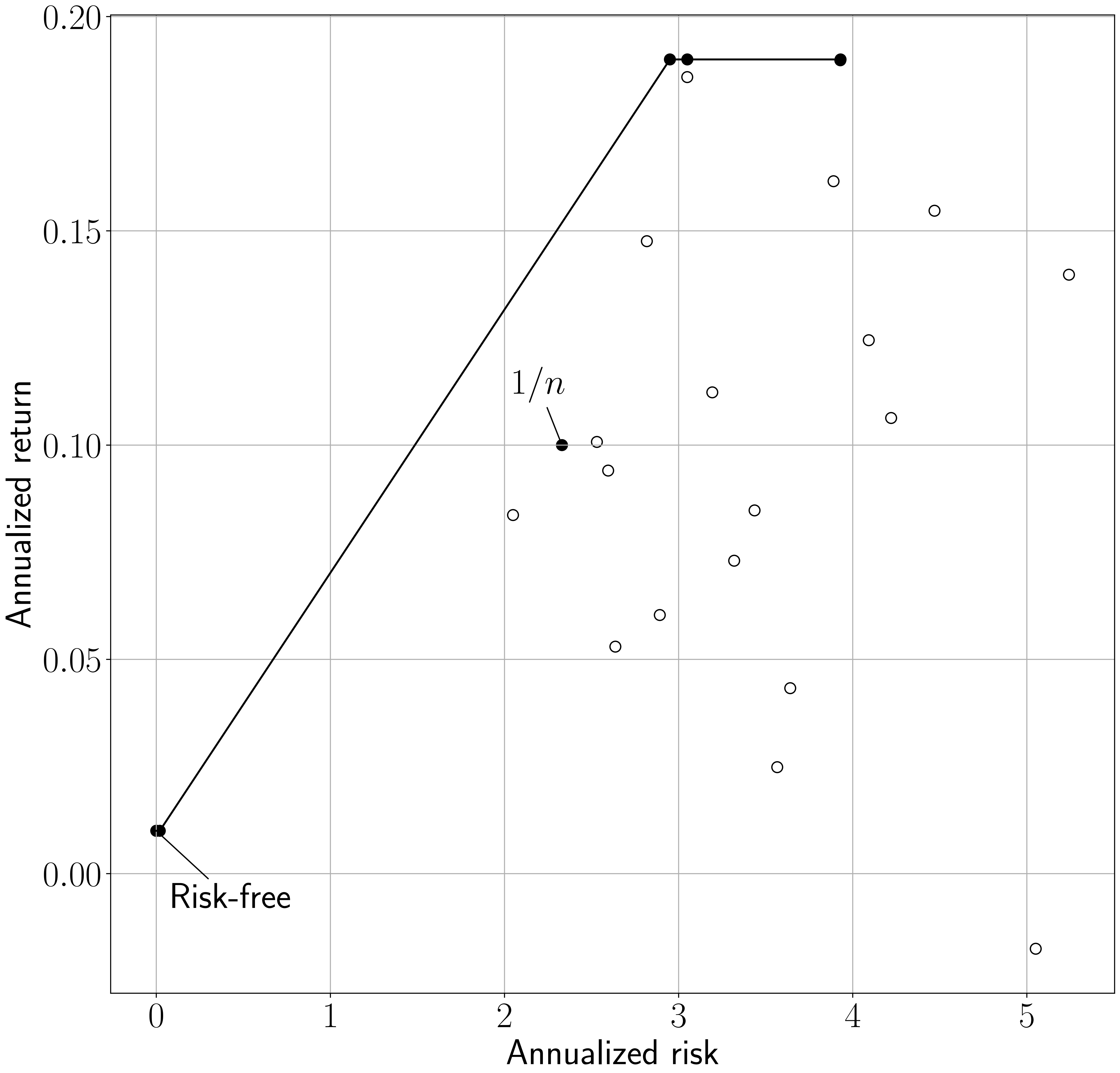
Remarks

- Can have inequality constraints (e.g., long-only)
- Tune λ to get desired Pareto-optimal point
- Gives the best allocation $w^\star$ given the past returns

Example

20 assets over 2000 days (past)

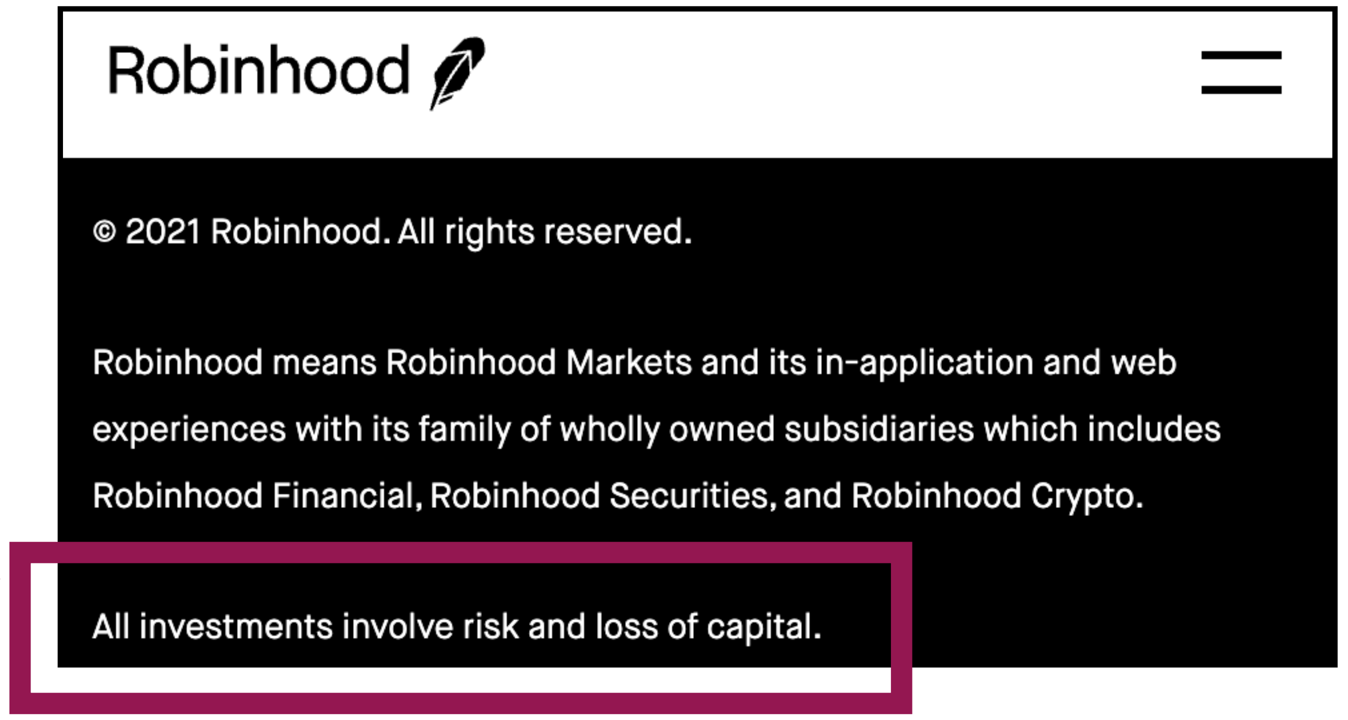
- Optimal portfolios on a **straight line**
- Line starts at risk-free portfolio ($\lambda = \infty$)
- $1/n$ much better than single portfolios



The big assumption

Future returns will look like past ones

- You are warned this is false, every time you invest
- It is often reasonable
- During crisis, market shifts, other big events not true



Robinhood

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All investments involve risk and loss of capital.

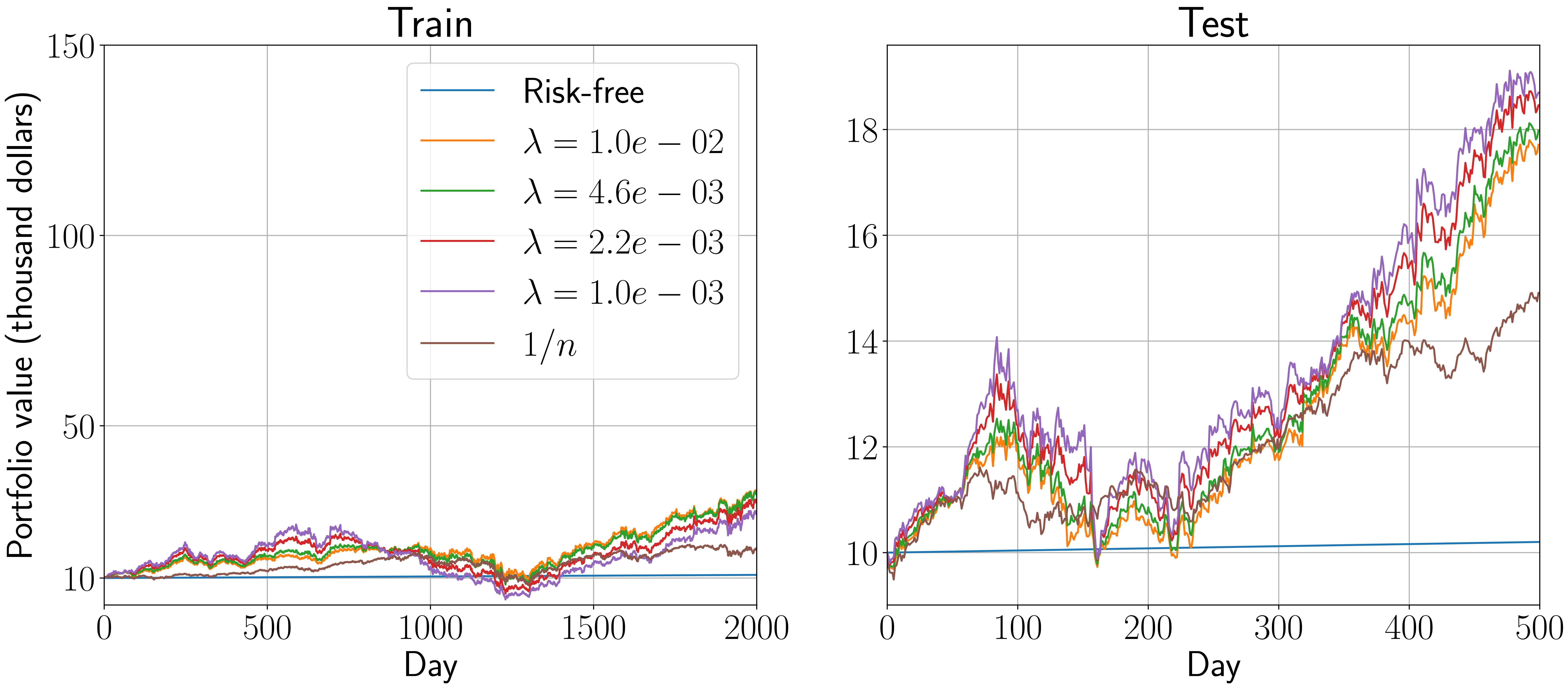
If assumption holds (even approximately), a good w on past returns leads to good future (unknown) returns

Example

- Pick w based on last 2 years of returns
- Use w during next 6 months

Total portfolio value

	Train return	Test return	Train risk	Test risk
Risk-free	0.01	0.01	0.00	0.00
$\lambda = 1.0e-02$	0.19	0.30	2.97	2.18
$\lambda = 4.6e-03$	0.19	0.31	3.05	2.21
$\lambda = 2.2e-03$	0.19	0.33	3.45	2.42
$\lambda = 1.0e-03$	0.19	0.34	3.93	2.73
$1/n$	0.10	0.21	2.33	1.51



Build your quantitative hedge fund

Rolling portfolio optimization

For each period t, find weight w_t using L past returns

$$r_{t-1}, \ldots, r_{t-L}$$

Variations

- Update w every K periods (monthly, quarterly, ...)
- Add secondary objective $\lambda \|w_t - w_{t-1}\|_1$ to discourage turnover, reduce transaction cost
- Add logic to detect when the future is likely to not look like the past
- Add "signals" that predict future return of assets (Twitter sentiment analysis)

Applications of linear optimization

Today, we learned to apply linear optimization in

- **Optimal control** problems with vehicle dynamics
- **Machine learning** problems for character recognition
- **Portfolio optimization** for investment strategies

References

- Github companion notebooks

Next steps

- Simplex method