ORF307 – Optimization

15. Sensitivity analysis

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Ed forum

- Two-person games:

  - Isn't A a matrix with a single entry at (i, j)? Is that P1's payoff or P2's payoff?

    Expected payoff (from P1 P2), if they use mixed-strategies x and y,

    $$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

- Weak vs strong duality recap?

- How do you interpret dual variables (multipliers)? (This lecture!)

Recap

Optimal objective values

Primal

$$\text{minimize} \quad c^T x$$
$$\text{subject to} \quad Ax \leq b$$

$p^\star$ is the primal optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual

$$\text{maximize} \quad -b^T y$$
$$\text{subject to} \quad A^T y + c = 0$$
$$y \geq 0$$

$d^\star$ is the dual optimal value

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem $\longrightarrow$ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$ ■

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Optimal objective values

Primal

minimize $c^T x$

subject to $Ax \leq b$

$p^\star$ is the primal optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

$d^\star$ is the dual optimal value

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality
Corollaries

Unboundedness vs feasibility
- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition
If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Strong duality
Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}$$

$\longrightarrow$

- optimal basis B
- optimal solution $x^\star$ with $A_B x_B^\star = b$
- reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$

Define $y^\star$ such that $y^\star = -A_B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

$$-b^T y^\star = -b^T(-A_B^{-T} c_B) = c_B^T(A_B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

By weak duality theorem corollary, $y^\star$ is an optimal solution of the dual. Therefore, $d^\star = p^\star$.

Today's lecture
Sensitivity analysis and game theory

- Primal and dual simplex

- Adding variables and constraints

- Global sensitivity

- Local sensitivity

Primal and dual simplex

Optimality conditions

Primal problem

minimize $\quad c^T x$

subject to $\quad Ax = b$

$\qquad\qquad x \geq 0$

Dual problem

maximize $\quad -b^T y$

subject to $\quad A^T y + c \geq 0$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax = b$ and $x \geq 0$

- y is **dual feasible**: $A^T y + c \geq 0$

- The **duality gap** is zero: $c^T x + b^T y = 0$

Primal and dual basic feasible solutions

Primal problem

minimize $\quad c^T x$

subject to $\quad Ax = b$

$\qquad\qquad x \geq 0$

Dual problem

maximize $\quad -b^T y$

subject to $\quad A^T y + c \geq 0$

Given a **basis** matrix B

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

Reduced costs

Dual feasible: $A^T y + c \geq 0$. $\quad$ Set $y = -A_B^{-T} c_B$. Dual feasible if $\boxed{\bar{c} = c + A^T y} \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T} c_B = c_B x_B - c_B^T A_B^{-1} b = 0$

(by construction)
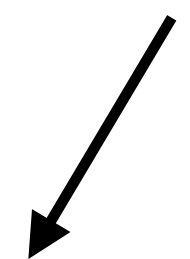
The primal (dual) simplex method

Primal problem

minimize $\quad c^T x$

subject to $\quad Ax = b$

$\qquad\qquad x \geq 0$

Dual problem

maximize $\quad -b^T y$

subject to $\quad A^T y + c \geq 0$

Primal simplex

- Primal feasibility
- Zero duality gap

↓

Dual feasibility

Dual simplex
(solve dual instead)

- Dual feasibility
- Zero duality gap

↓

Primal feasibility

Adding new constraints and variables

Adding new variables

minimize $c^T x$

subject to $Ax = b$

              $x \geq 0$

$\longrightarrow$

minimize $c^T x + c_{n+1} x_{n+1}$

subject to $Ax + A_{n+1} x_{n+1} = b$

              $x, x_{n+1} \geq 0$

Solution $x^\star, y^\star$

Is the solution $(x^\star, 0), y^\star$ **optimal** for the new problem?

Adding new variables
Optimality conditions

minimize $\quad c^T x + c_{n+1} x_{n+1}$

subject to $\quad Ax + A_{n+1} x_{n+1} = b \quad \longrightarrow \quad$ Solution $(x^\star, 0)$ is still **primal feasible**

$\qquad\qquad x, x_{n+1} \geq 0$

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Yes

$(x^\star, 0)$ still **optimal** for new problem

Otherwise

Primal simplex

17

Adding new variables

Example

$$\begin{array}{ll} \text{minimize} & -60x_1 - 30x_2 - 20x_3 \\ \text{subject to} & 8x_1 + 6x_2 + x_3 \leq 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\ & x \geq 0 \end{array}$$

-profit

material

production

quality control

$$c = (-60, -30, -20, 0, 0, 0)$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

$$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280, \quad \text{basis } \{1, 3, 4\}$$

Adding new variables

Example: add new product?

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

minimize $\quad c^T x + c_{n+1} x_{n+1}$

subject to $\quad Ax + A_{n+1} x_{n+1} = b$

$\quad\quad\quad\quad x, x_{n+1} \geq 0$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Previous solution

$$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280, \quad \text{basis } \{1, 3, 4\}$$

$(x^\star, 0)$ **is still optimal**

$$A_{n+1}^T y^\star + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

Shall we add a new product?

Adding new constraints

minimize $\quad c^T x$

subject to $\quad Ax = b$

$\qquad\qquad x \geq 0$

$\longrightarrow$

minimize $\quad c^T x$

subject to $\quad Ax = b$

$\qquad\qquad a_{m+1}^T x = b_{m+1}$

$\qquad\qquad x \geq 0$

Solution $x^\star, y^\star$

Dual

maximize $\quad -b^T y$

subject to $\quad A^T y + a_{m+1} y_{m+1} + c \geq 0$

Is the solution $x^\star, (y^\star, 0)$ **optimal** for the new problem?

Adding new constraints
Optimality conditions

maximize $\quad -b^T y$

subject to $\quad A^T y + a_{m+1} y_{m+1} + c \geq 0 \quad \longrightarrow \quad$ Solution $(y^\star, 0)$ is still **dual feasible**

Is $x^\star$ still **primal feasible**?

$$Ax = b$$
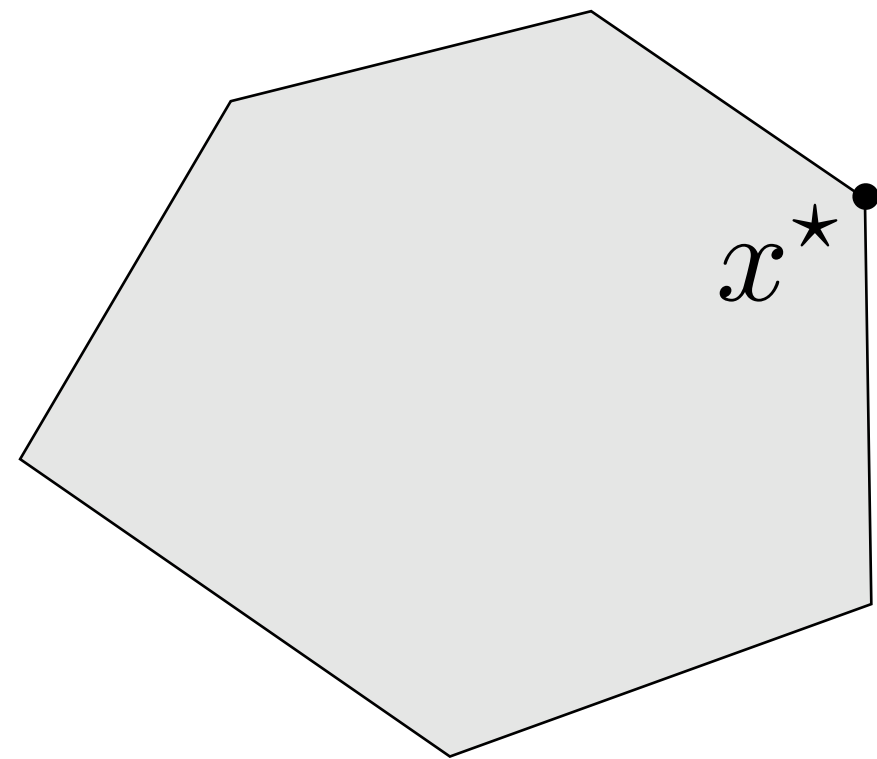$$a_{m+1}^T x = b_{m+1}$$
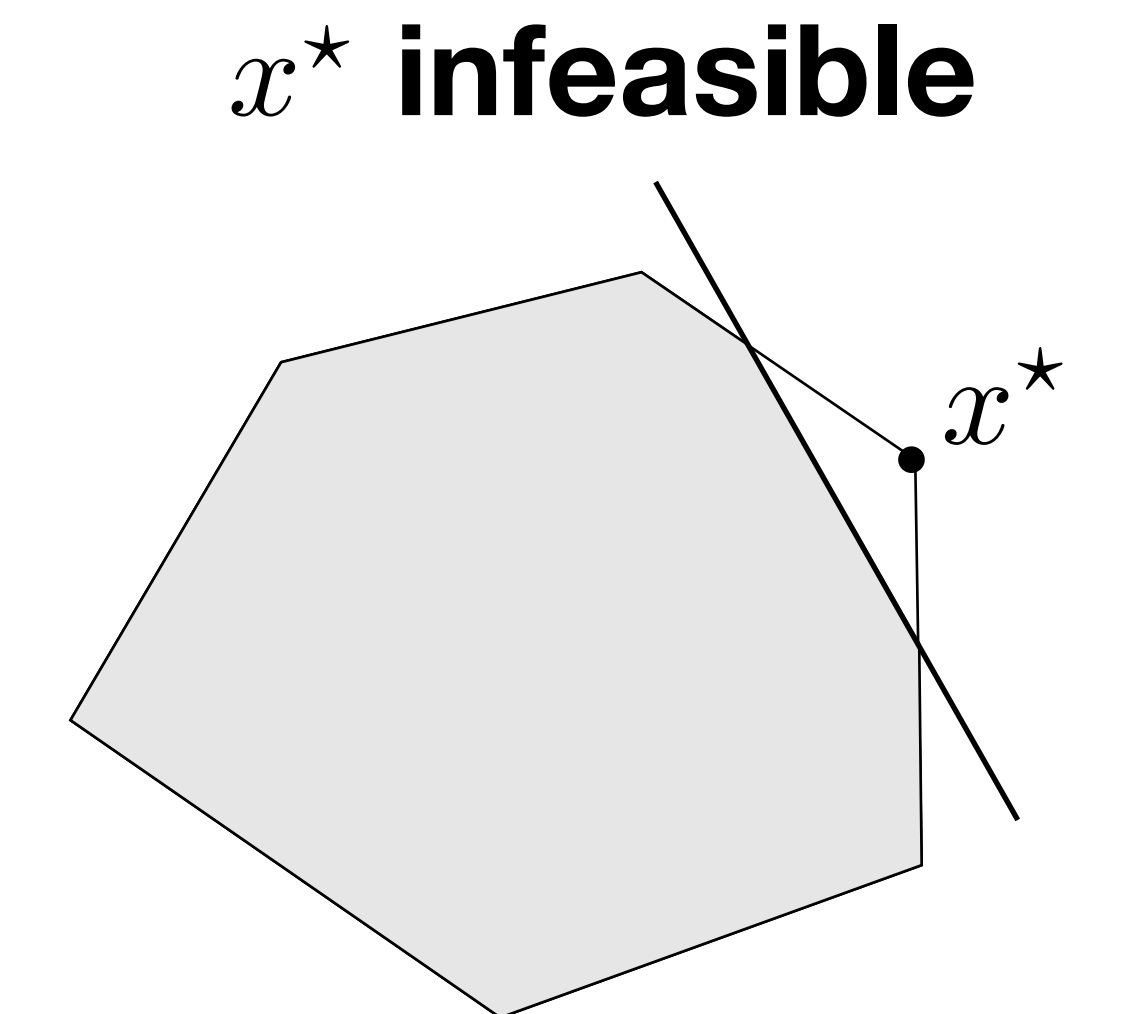$$x \geq 0$$

Yes

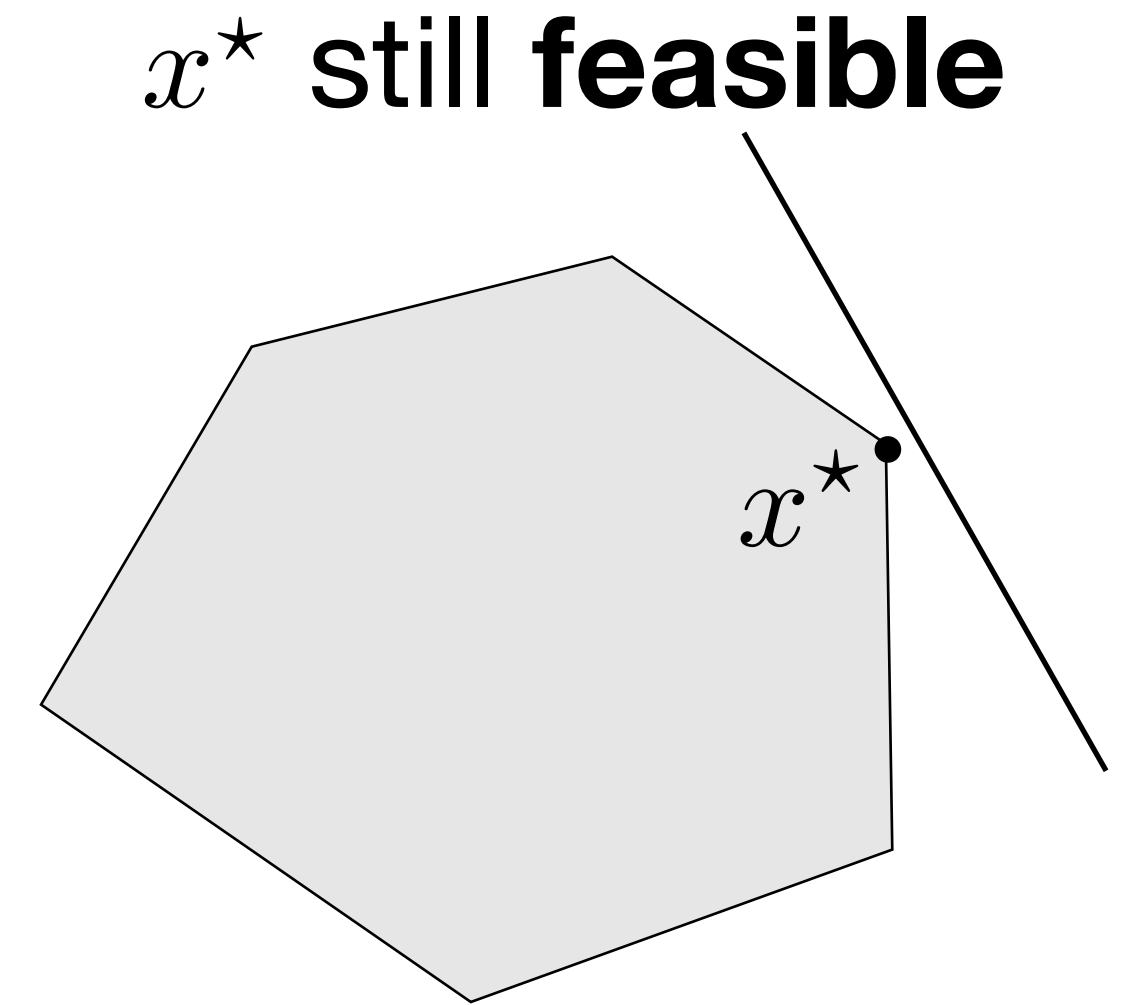$x^\star$ still **optimal** for new problem

Otherwise

Dual simplex

Adding new constraints
Example

$x^\star$ still **feasible**

Add new constraint

$x^\star$ **infeasible**

Global sensitivity analysis

Changes in problem data

Goal: extract information from $x^\star, y^\star$ about their sensitivity with respect to changes in problem data

Modified LP

$$
\begin{aligned}
& \text{minimize} && c^T x \\
& \text{subject to} && Ax = b + u \\
& && x \geq 0
\end{aligned}
$$

Optimal cost $p^\star(u)$

Global sensitivity

Dual of modified LP

$$\text{maximize} \quad -(b+u)^T y$$
$$\text{subject to} \quad A^T y + c \geq 0$$

Global lower bound

Given $y^\star$ a dual optimal solution for $u = 0$, then

$$p^\star(u) \geq -(b+u)^T y^\star$$

(from weak duality and dual feasibility)

$$= p^\star(0) - u^T y^\star$$

It holds for any u

Global sensitivity

Example

Take $u = td$ with $d \in \mathbf{R}^m$ fixed

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + td \\ & x \geq 0 \end{array}$$

$p^\star(td)$ is the optimal value as a function of t
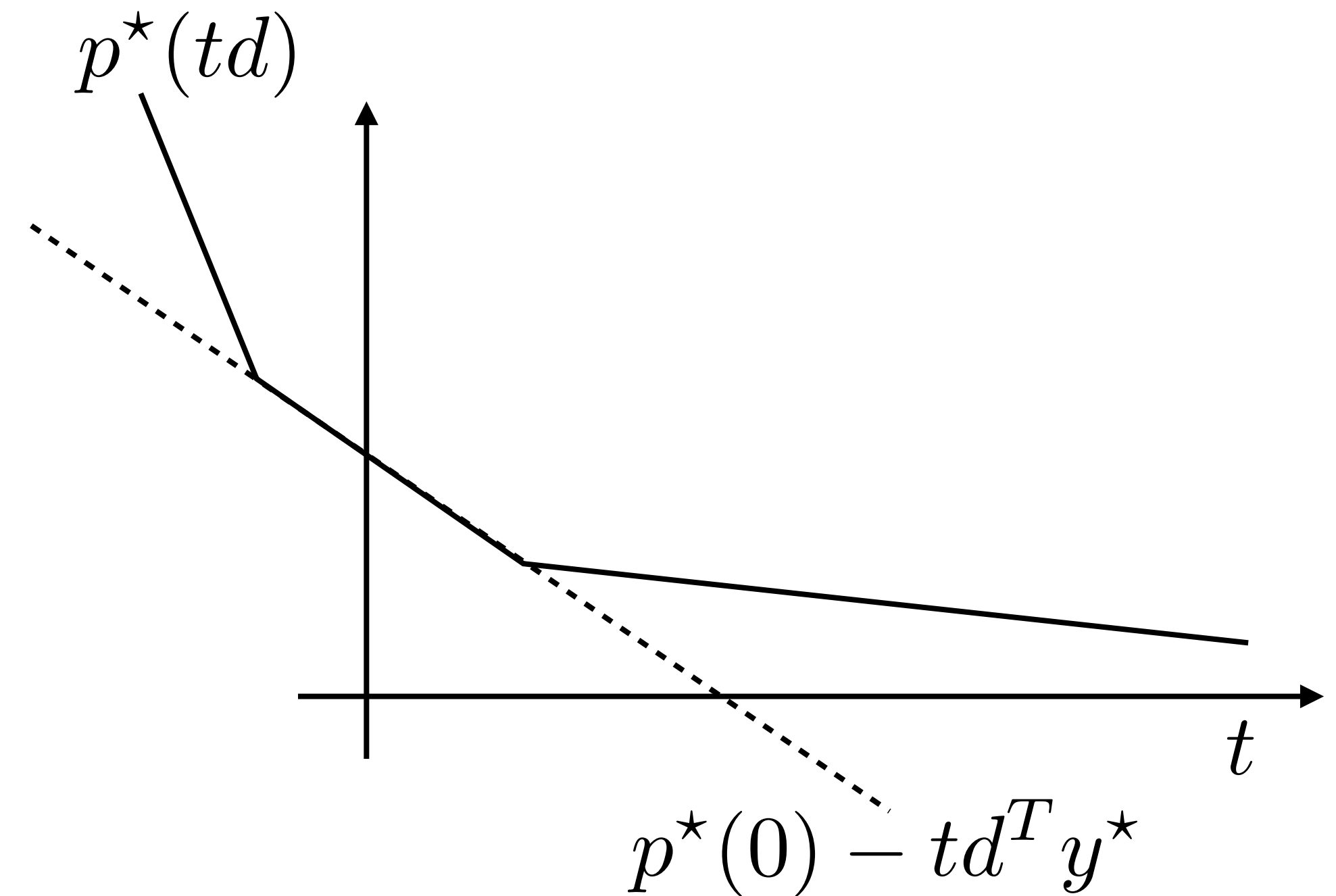
Sensitivity information (assuming $d^T y^\star \geq 0$)

- $t < 0$ the optimal value increases
- $t > 0$ the optimal value decreases (not so much if t is small)

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u, \ x \geq 0\}$$

Assumption: $p^\star(0)$ is finite

Properties

- $p^\star(u) > -\infty$ everywhere (from global lower bound)

- $p^\star(u)$ is piecewise-linear on its domain

Optimal value function is piecewise linear
Proof

Dual feasible set

$$p^\star(u) = \min\{c^T x \mid Ax = b + u, \ x \geq 0\}$$

$$D = \{y \mid A^T y + c \geq 0\}$$

Assumption: $p^\star(0)$ is finite

If $p^\star(u)$ finite

$$p^\star(u) = \max_{y \in D} -(b + u)^T y = \max_{k=1,\ldots,r} -y_k^T u - b^T y_k$$

$y_1, \ldots, y_r$ are the extreme points of D

Local sensitivity analysis

Local sensitivity
u in neighborhood of the origin

Original LP

minimize $\quad c^T x$

subject to $\quad Ax = b$

$\qquad\qquad x \geq 0$

$\longrightarrow$

Optimal solution

Primal
$$x_i^\star = 0, \quad i \notin B$$
$$x_B^\star = A_B^{-1} b$$

Dual $\quad y^\star = -A_B^{-T} c_B$

Modified LP

minimize $\quad c^T x$

subject to $\quad Ax = b + u$

$\qquad\qquad x \geq 0$

Modified dual

maximize $\quad -(b + u)^T y$

subject to $\quad A^T y + c \geq 0$

Optimal basis does not change

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b + u) = x_B^\star + A_B^{-1} u$$
$$y^\star(u) = y^\star$$

Derivative of the optimal value function

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b + u) = x_B^\star + A_B^{-1}u$$
$$y^\star(u) = y^\star$$

Optimal value function

$$p^\star(u) = c^T x^\star(u)$$
$$= c^T x^\star + c_B^T A_B^{-1} u$$
$$= p^\star(0) - y^{\star T} u \qquad \text{(affine for small } u\text{)}$$

Local derivative

$$\nabla p^\star(u) = -y^\star \qquad (y^\star \text{ are the } \textbf{shadow prices})$$

Sensitivity example

$$\text{minimize} \quad -60x_1 - 30x_2 - 20x_3 \qquad \text{-profit}$$

$$\text{subject to} \quad 8x_1 + 6x_2 + x_3 \leq 48 \qquad \text{material}$$

$$4x_1 + 2x_2 + 1.5x_3 \leq 20 \qquad \text{production}$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \qquad \text{quality control}$$

$$x \geq 0$$

$$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280, \quad \text{basis } \{1, 3, 4\}$$

What does $y_3^\star = 10$ mean?

Let's increase the quality control budget by 1, i.e., $u = (0, 0, 1)$

$$p^\star(u) = p^\star(0) - y^{\star T} u = -280 - 10 = -290$$

32

Sensitivity analysis

Today, we learned to:

- **Reuse** primal and dual solutions when variables or constraints are added

- **Analyze** value function as problem parameters change

- **Compute** local sensitivity to parameter perturbations

References

- D. Bertsimas and J. Tsitsiklis: Introduction to Linear Optimization

  - Chapter 5: Sensitivity analysis

- R. Vanderbei: "Linear Programming

  - Chapter 7: Sensitivity and parametric analysis

Next lecture

- Network optimization