ORF307 – Optimization

7. Linear optimization

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Ed Forum

- In this lecture, we learned about linearly constrained least squares problems. We also learned about the least norm problem, a special case where A=I and b = 0. I was wondering if there were other special cases of this problem that were worth knowing and their applications.

- Could you go over the Lagrangian function one more time -- I'm not sure I understand where the z comes in? Thank you!

Recap

Least squares with equality constraints

The (linearly) constrained least squares problem is

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|^2 \\ \text{subject to} & Cx = d \end{array}$$

equality constraints

objective function

Problem data

- $m \times n$ matrix A, m-vector b
- $p \times n$ matrix C, p-vector d

Definitions

x is *feasible* if $Cx = d$

$x^\star$ is a *solution* if

- $Cx^\star = d$
- $\|Ax^\star - b\|^2 \leq \|Ax - b\|^2$ for any x satisfying $Cx = d$

Interpretations

- Combine solving linear equations with least squares.
- Like a bi-objective least squares with ∞ weight on second objective, $\|Cx - d\|^2$.

Optimality conditions via calculus

minimize $f(x) = \|Ax - b\|^2$
subject to $Cx = d$

$\longrightarrow$

minimize $f(x) = \|Ax - b\|^2$
subject to $c_i^T x = d_i, \quad i = 1, \ldots, p$

Lagrangian function

$$L(x, z) = f(x) + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

Optimality conditions

$$\frac{\partial L}{\partial x_i}(x^\star, z) = 0, \quad i = 1, \ldots, n,$$

$$\frac{\partial L}{\partial z_i}(x^\star, z) = 0, \quad i = 1, \ldots, p$$

Optimality conditions via calculus

$$L(x,z) = x^T A^T A x - 2(A^T b)^T x + b^T b + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

Optimality conditions

$$\frac{\partial L}{\partial z_i}(x^\star, z) = c_i^T x - d_i = 0 \qquad \text{(we already knew)}$$

$$\frac{\partial L}{\partial x_i}(x^\star, z) = 2\sum_{j=1}^{n}(A^T A)_{ij} x_j^\star - 2(A^T b)_i + \sum_{j=1}^{p} z_j (c_j)_i = 0$$

Vector form

$$Cx = d$$

$$2A^T A x^\star - 2A^T b + C^T z = 0$$

Karush-Kuhn-Tucker (KKT) conditions

$$\begin{bmatrix} 2A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ z \end{bmatrix} = \begin{bmatrix} 2A^T b \\ d \end{bmatrix}$$

(square set of $n + p$ linear equations)

Note KKT equations are extension of normal equations to constrained least squares

Today's lecture

Linear optimization

- Some simple examples
- Linear optimization
- Special cases
- Standard form
- Software and solution methods

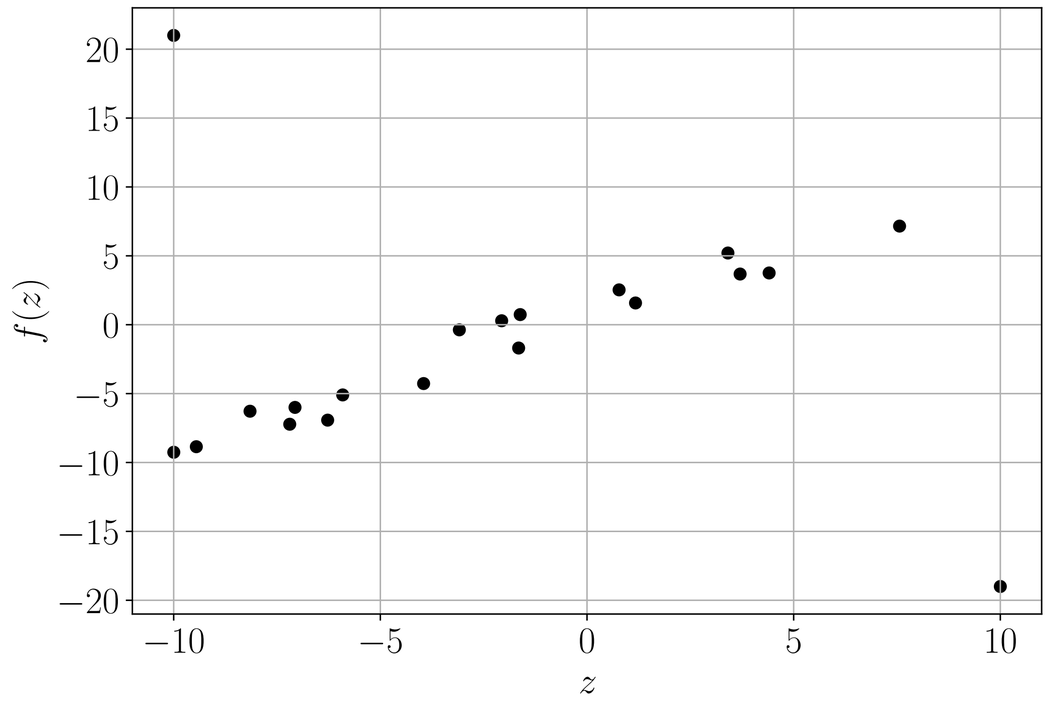
Some simple examples

Data-fitting example

Fit a linear function $f(z) = a + bz$ to m data points (z_i, f_i):

Approximation problem $Ax \approx b$ where

$$\underbrace{\begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{b}$$



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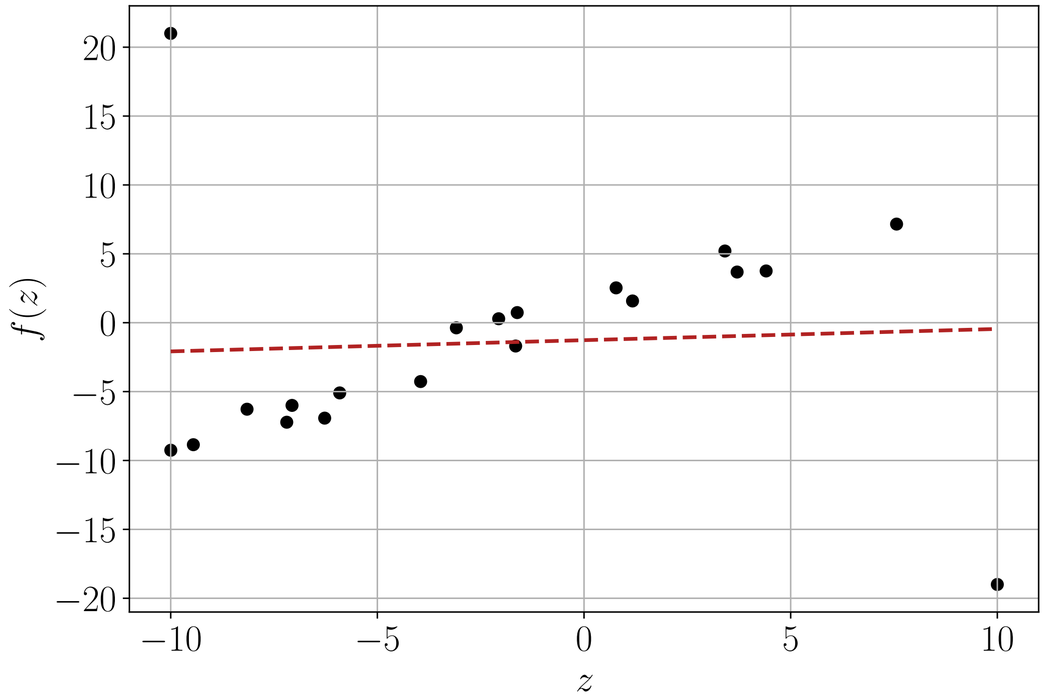


Least squares way:

minimize $\displaystyle\sum_{i=1}^{m} (Ax - b)_i^2 = \|Ax - b\|_2^2$

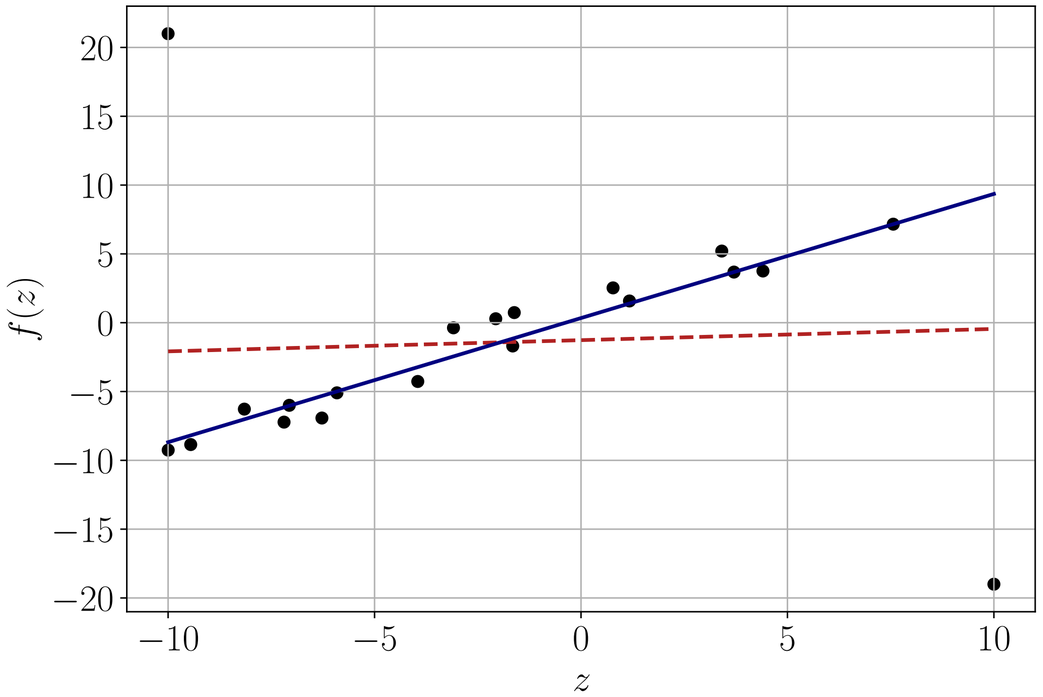
Good news: solution is in closed form $x^\star = (A^T A)^{-1} A^T b$
Bad news: solution is very sensitive to outliers!

Data-fitting example

Fit a linear function $f(z) = a + bz$ to m data points (z_i, f_i):

Approximation problem $Ax \approx b$ where

$$\underbrace{\begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{b}$$



A different way:

minimize $\sum_{i=1}^{m} |Ax - b|_i = \|Ax - b\|_1$

Good news: solution is much more robust to outliers.
Bad news: there is no closed form solution.

Cheapest cat food problem

- Choose quantities $x_1, \ldots, x_n$ of n ingredients each with unit cost c_j.
- Each ingredient j has nutritional content a_{ij} for nutrient i.
- Require a minimum level b_i for each nutrient i.

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1 \ldots m \\ & x_j \geq 0, \quad j = 1 \ldots n \end{aligned}$$

[Photo of Phoebe, my cat]

Would you give her the optimal food ?

Linear optimization

Linear optimization

Linear Programming (LP)

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{n} c_i x_i \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} d_{ij} x_j = f_i, \quad i = 1, \ldots, p
\end{array}
$$

Ingredients

- n **decision variables** (or optimization variables): $x_1, \ldots, x_n$
- Constant **parameters** (or problem data) : c_j, a_{ij}, b_i, d_{ij}, f_i
- A linear **objective function**
- A collection of m **inequality constraints** and p **equality constraints**

Where does linear optimization appear?

Supply chain management

Assignment problems

Scheduling and routing problems

Finance

Optimal control problems

Network design and network operations

Many other domains…

A brief history of linear optimization

1940s :

- Foundations and applications in economics and logistics (Kantorovich, Koopmans)
- **1947** : Development of the **simplex method** by Dantzig

1950s—70s:

- Applications expand to engineering, OR, computer science…
- **1975** : Nobel prize in economics for Kantorovich and Koopmans

1980s:

- Development of polynomial time algorithms for LPs
- **1984** : Development of the **interior point method** by Karmarkar

—Today:

- Continued algorithm development. Expansion to very large problems.

Why linear optimization?

“Easy” to solve

- It is solvable in polynomial time, tractable in practice
- State-of-the-art software can solve LPs with tens of thousands of variables. We can solve LPs with millions of variables with specific structure.

Extremely versatile

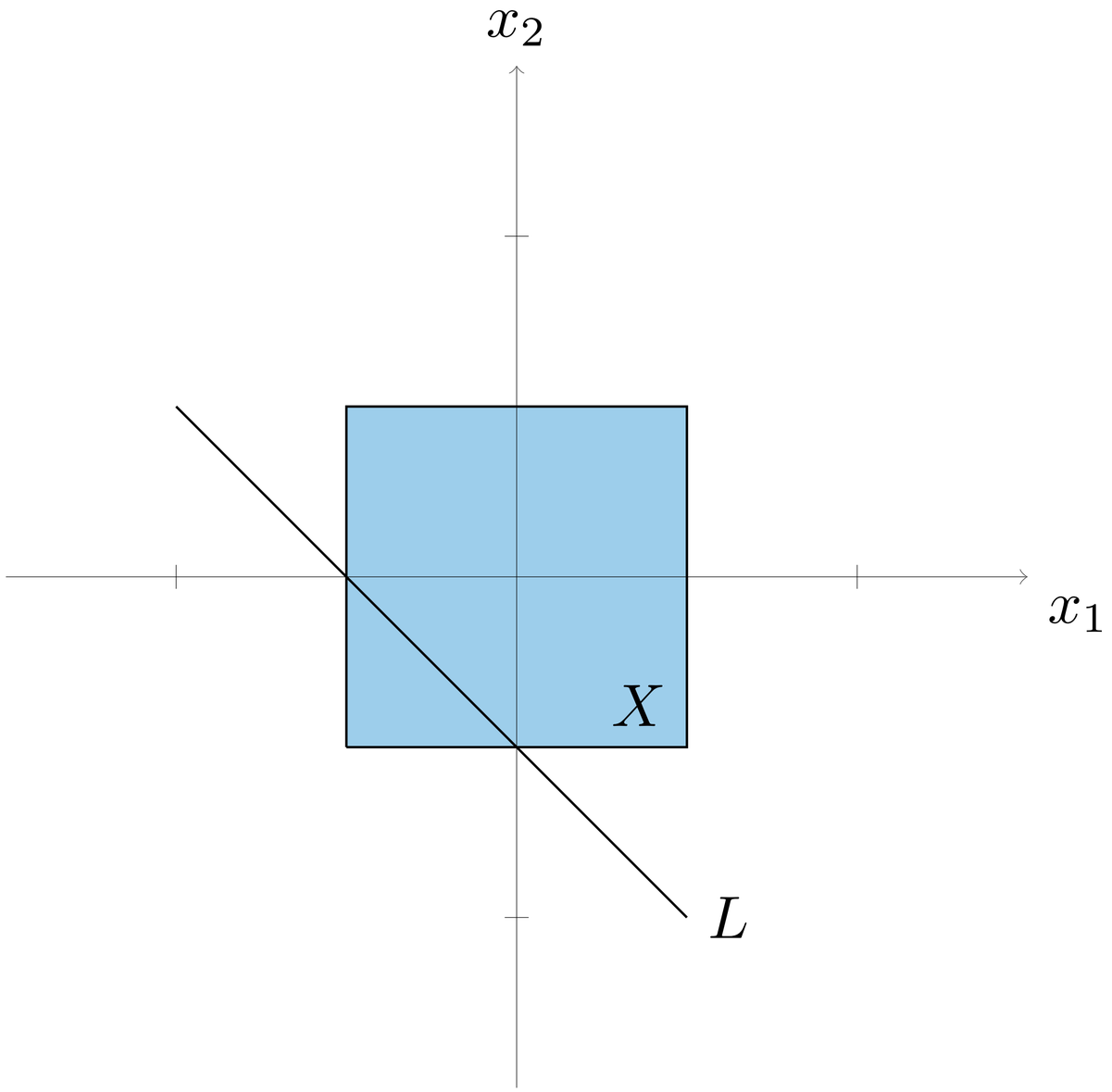
Can model many real-world problems, either exactly or approximately.

Fundamental

- The theory of linear optimization lays the foundation for most optimization theories
- Underpins solutions for more complicated problems, e.g. integer problems.

A simple example

Goal find point as far left as possible,
in the unit box X,
and restricted to the line L



A simple example

Goal find point as far left as possible,
in the unit box X,
and restricted to the line L

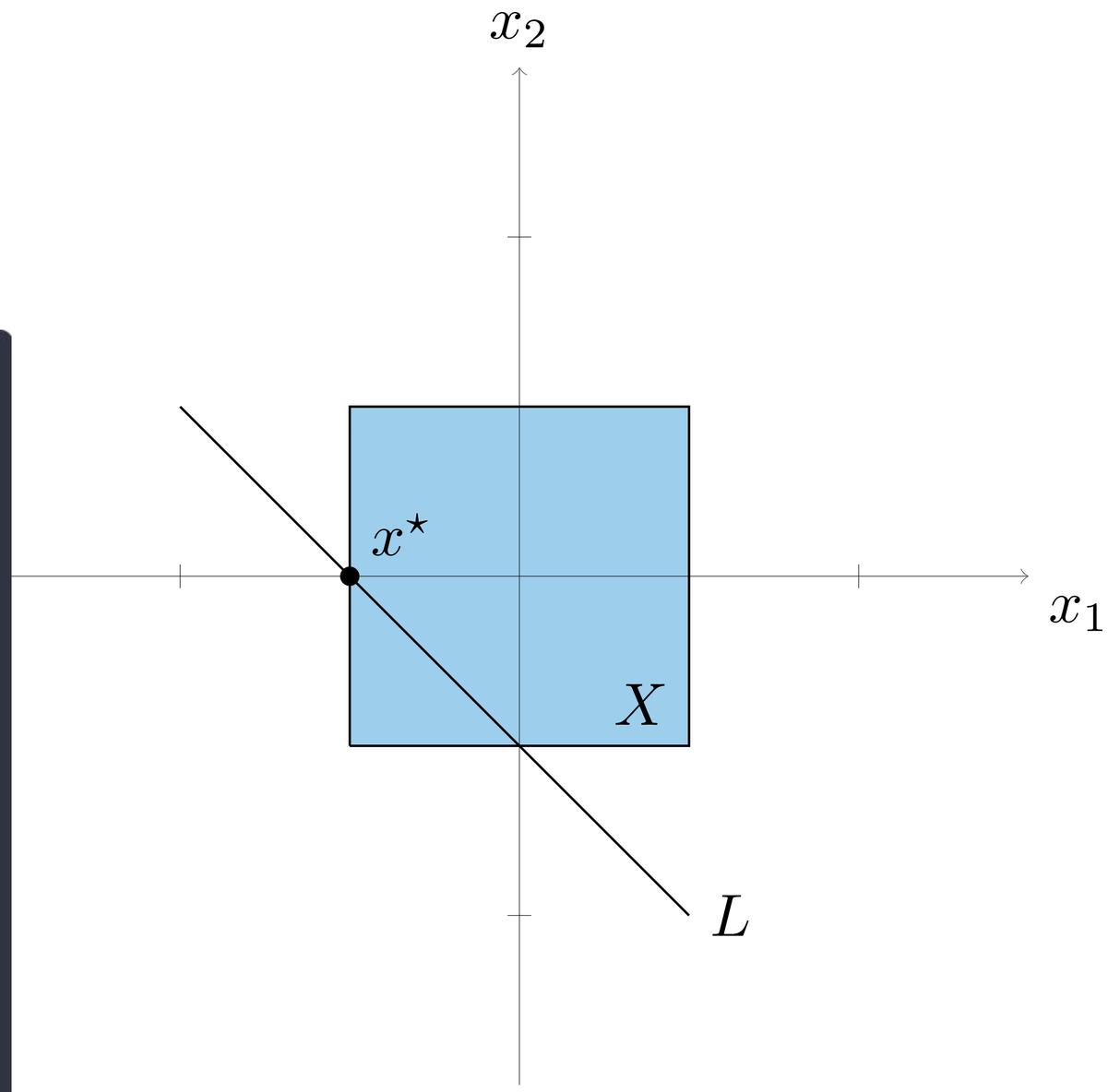
```
import cvxpy as cp

#make decision variable
x = cp.Variable(2)

#define objective
objective = x[0]

#define constraints
constraints = [-1 <= x[0], x[0] <= 1, #inequalities
               -1 <= x[1], x[1] <= 1, #inequalities
               x[0] + x[1] == -1]     #equalities

#make problem and solve
prob = cp.Problem(cp.Minimize(objective), constraints)
prob.solve()
```



Linear optimization

Using vectors

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{n} c_i x_i & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & i = 1, \ldots, m \\ & \sum_{j=1}^{n} d_{ij} x_j = f_i, & i = 1, \ldots, p \end{array} \longrightarrow \begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & a_i^T x \leq b_i, & i = 1, \ldots, m \\ & d_i^T x = f_i, & i = 1, \ldots, p \end{array}$$

$c,\ a_i,\ d_i$ are n-vectors

$$c = (c_1, \ldots, c_n)$$

$$a_i = (a_{i1}, \ldots, a_{in})$$

$$d_i = (d_{i1}, \ldots, d_{in})$$

Linear optimization

Using matrices

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{n} c_i x_i & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & i = 1, \ldots, m \\ & \sum_{j=1}^{n} d_{ij} x_j = f_i, & i = 1, \ldots, p \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

A is $m \times n$-matrix with elements a_{ij} and rows a_i^T

D is $p \times n$-matrix with elements d_{ij} and rows d_i^T

All (in)equalities are elementwise

Optimization terminology

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

x is **feasible** if it satisfies the constraints $Ax \leq b$ and $Dx = f$

The **feasible set** is the set of all feasible points

$x^\star$ is **optimal** if it is feasible and $c^T x^\star \leq c^T x$ for all feasible x

The **optimal value** is $p^\star = c^T x^\star$

Special cases

What can go wrong?

Problem might be “too hard”

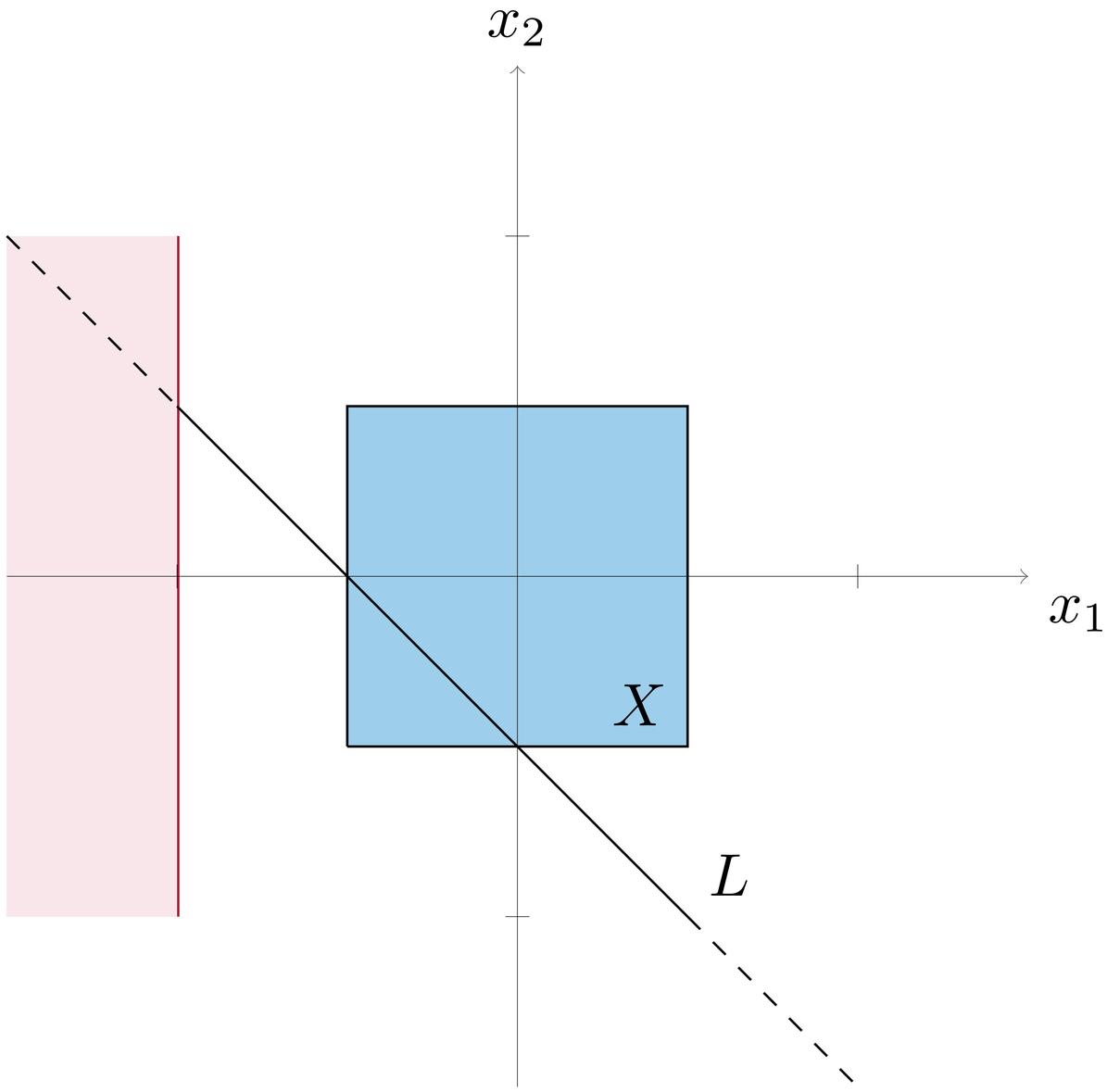
minimize x_1

subject to $-1 \leq x_1 \leq 1$

$-1 \leq x_2 \leq 1$

$x_1 + x_2 = -1$

$x_1 \leq -2$



Remarks

- The feasible set is empty.
- The problem is therefore **infeasible**.
- Define the optimal value as $p^\star = +\infty$.

What can go wrong?

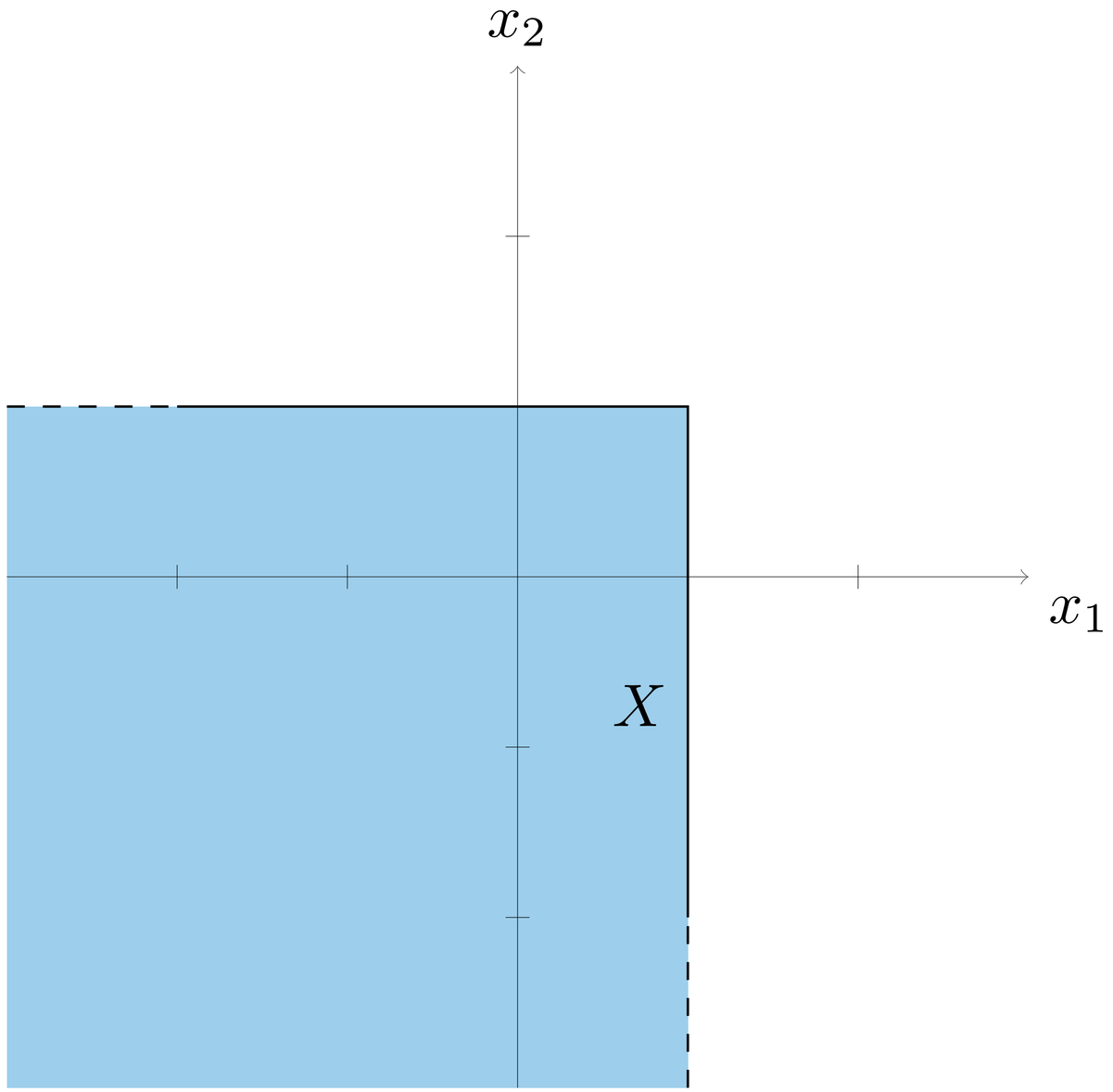
Problem might be “too easy”

minimize x_1

subject to ~~$-1 \leq$~~ $x_1 \leq 1$

~~$-1 \leq$~~ $x_2 \leq 1$

~~$x_1 + x_2 = -1$~~



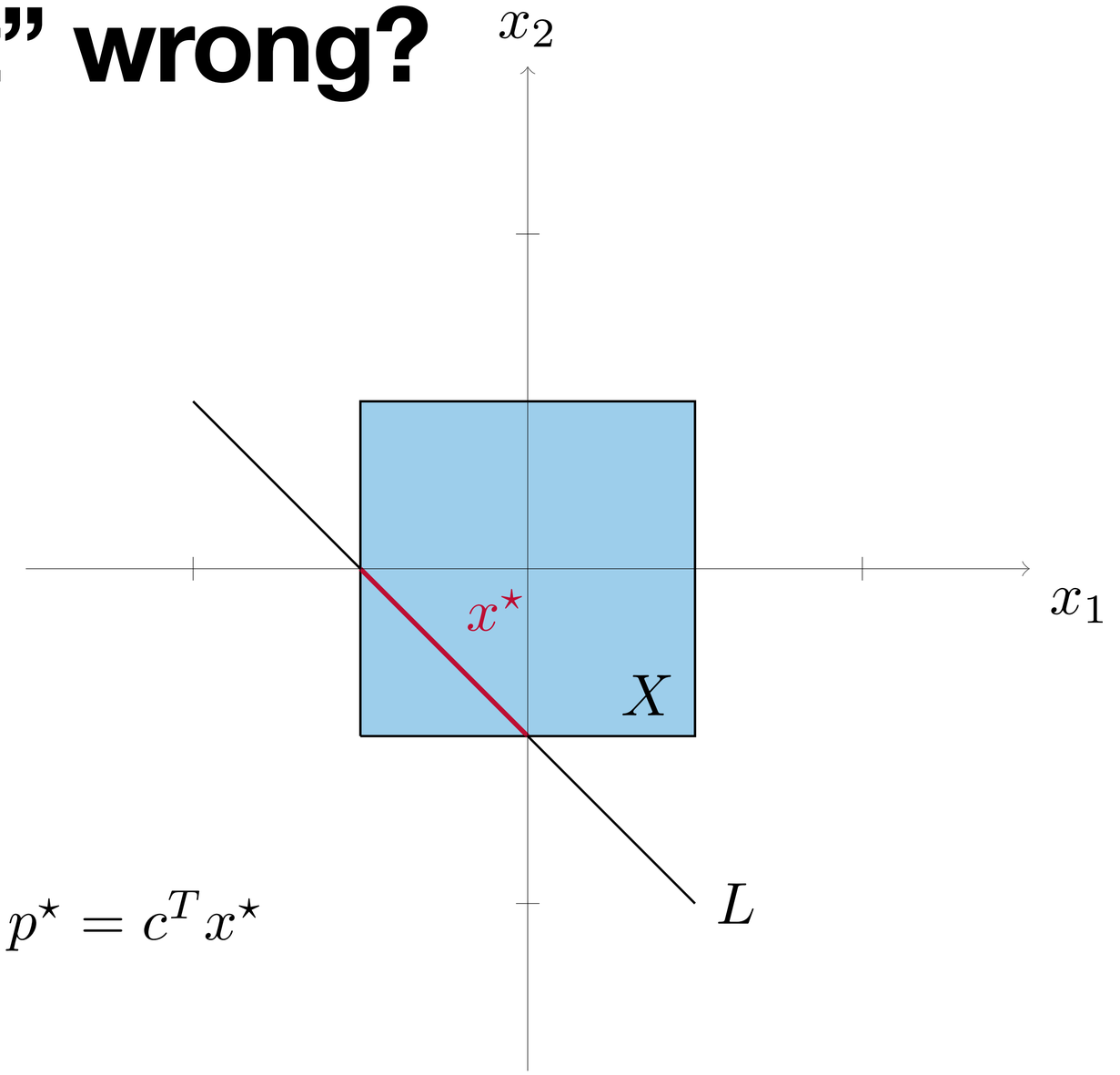
Remarks

- The value of $c^T x$ is **unbounded below** on the feasible set.
- Define the optimal value as $p^\star = -\infty$.

What can go “a little bit” wrong?

More than one optimizer

$$
\begin{array}{ll}
\text{minimize} & x_1 + x_1 \\
\text{subject to} & -1 \leq x_1 \leq 1 \\
& -1 \leq x_2 \leq 1 \\
& x_1 + x_2 = -1
\end{array}
$$



Remarks

- The optimal value is $p^\star = -1$
- There is more than one $x^\star$ that achieves $p^\star = c^T x^\star$
- The optimizer is **non-unique**

Feasibility problems

The constraints satisfiability problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

is a special case of ⟶

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

Remarks

- $p^\star = 0$ if constraints are feasible (consistent).
  Every feasible x is optimal
- $p^\star = \infty$ otherwise

Standard form

Standard form

Definition

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Minimization
- Equality constraints
- Nonnegative variables

- Matrix notation for **theory**
- Standard form for **algorithms**

Standard form

Transformation tricks

Change objective

If “maximize”, use $-c$ instead of c and change to “minimize”.

Eliminate inequality constraints

If $Ax \leq b$, define s and write $Ax + s = b,\ s \geq 0$.

If $Ax \geq b$, define s and write $Ax - s = b,\ s \geq 0$.

s are the **slack variables**

Change variable signs

If $x_i \leq 0$, define $y_i = -x_i$.

Eliminate “free” variables

If x_i unconstrained, define $x_i = x_i^+ - x_i^-$, with $x_i^+ \geq 0$ and $x_i^- \geq 0$.

Standard form

Transformation example

$$
\begin{aligned}
\text{minimize} \quad & 2x_1 + 4x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 3 \\
& 3x_1 + 2x_2 = 14 \\
& x_1 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{minimize} \quad & 2x_1 + 4x_2^+ - 4x_2^- \\
\text{subject to} \quad & x_1 + x_2^+ - x_2^- - x_3 = 3 \\
& 3x_1 + 2x_2^+ - 2x_2^- = 14 \\
& x_1, \; x_2^+, \; x_2^-, \; x_3 \geq 0.
\end{aligned}
$$

Software

Solvers for linear programs

Algorithms and **theory** are very mature:

- Simplex methods, interior-point methods, first order methods etc

Software is widely available:

- Can solve problems up to several million variables
- Widely used in industry and academic research

Examples

- Commercial solvers : Mosek, CPLEX, Gurobi, Matlab (linprog)
- Free solvers : GLPK, CLP, SCS, OSQP

Modelling tools for linear programs

Modelling tools simplify the formulation of LPs (and other problems)

- Accept optimization problem in common notation ($\max, \|\cdot\|_1, \ldots$)
- Recognize problems that can be converted to LPs
- Automatically convert to input format required by a specific LP solver

Examples

- AMPL, GAMS
- CVX, YALMIP (Matlab)
- CVXPY, Pyomo (Python)
- JuMP.jl, Convex.jl (Julia)

Simple example revisited

Goal find point as far left as possible,
in the unit box X,
and restricted to the line L

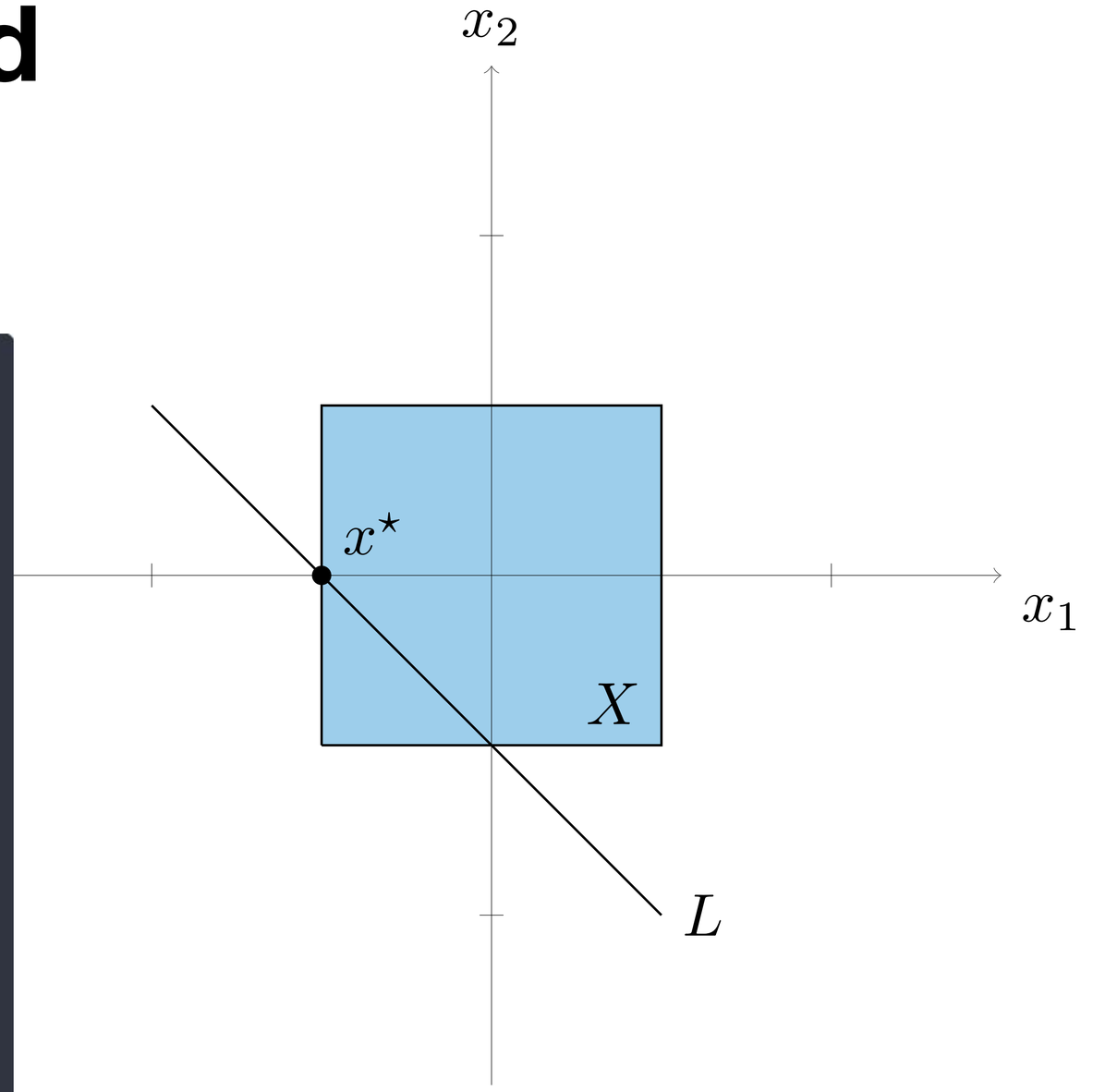
```
import cvxpy as cp

#make decision variable
x = cp.Variable(2)

#define objective
objective = x[0]

#define constraints
constraints = [ cp.norm(x, 'inf') <= 1, #inequalities
                cp.sum(x) == -1]   #equalities

#make problem and solve
prob = cp.Problem(cp.Minimize(objective), constraints)
prob.solve()
```



References

- Bertsimas, Tsitsiklis: Introduction to Linear Optimization
  - Chapter 1: Introduction
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 1: intro to linear programming

Next time

Piecewise linear optimization

- Optimization problems with norms and max functions
- Some applications