ORF522 – Linear and Nonlinear Optimization

8. Linear optimization duality II

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Today's agenda

Readings: [Chapter 4, LO][Chapter 11, LP]

- Two-person zero-sum games
- Farkas lemma
- Adding new variables
- Sensitivity analysis

Two-person zero-sum games

Rock paper scissors

Rules
At count to three declare one of: Rock, Paper, or Scissors

Winners
Identical selection is a draw, otherwise:
- Rock beats ("dulls") scissors
- Scissors beats ("cuts") paper
- Paper beats ("covers") rock

Extremely popular: world RPS society, USA RPS league, etc.

Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \dots, m\}$ (one of m actions)
- Player 2 (P2) chooses a number $j \in \{1, \dots, n\}$ (one of n actions)

Two players make their choice independently

Rule

Player 1 pays A_{ij} to player 2

$A \in \mathbf{R}^{m \times n}$ is the **payoff matrix**

Rock, Paper, Scissors

$$A = \begin{array}{c} \\ \mathrm{R} \\ \mathrm{P} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{P} & \mathrm{S} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

Mixed (randomized) strategies

Deterministic strategies can be **systematically defeated**

Randomized strategies

- P1 chooses randomly according to distribution x:

  $x_i =$ probability that P1 selects action i

- P2 chooses randomly according to distribution y:

  $y_i =$ probability that P2 selects action j

Expected payoff (from P1 P2), if they use mixed-strategies x and y,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

Mixed strategies and probability simplex

Probability simplex in $\mathbf{R}^k$

$$P_k = \{p \in \mathbf{R}^k \mid p \geq 0, \quad \mathbf{1}^T p = 1\}$$

Mixed strategy

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $\longrightarrow$ $x \in P_m, \quad y \in P_n$

Optimal mixed strategies

P1: optimal strategy $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & \max_{y \in P_n} x^T A y \\ \text{subject to} & x \in P_m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \max_{j=1,\dots,n} (A^T x)_j \\ \text{subject to} & x \in P_m \end{array}$$

Inner problem over deterministic strategies (**vertices**)

P2: optimal strategy $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & \min_{x \in P_m} x^T A y \\ \text{subject to} & y \in P_n \end{array} \longrightarrow \begin{array}{ll} \text{maximize} & \min_{i=1,\dots,m} (Ay)_i \\ \text{subject to} & y \in P_n \end{array}$$

Optimal strategies $x^\star$ and $y^\star$ can be computed using **linear optimization**

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

The optimal $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \geq w\mathbf{1} \\ & \mathbf{1}^T y = 1 \\ & y \geq 0 \end{array}$$

The two LPs are **duals** and by **strong duality** the equality follows. ■

Nash equilibrium

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Consequence

The pair of mixed strategies $(x^\star, y^\star)$ attains the **Nash equilibrium** of the two-person matrix game, i.e.,

$$x^T A y^\star \geq x^{\star T} A y^\star \geq x^{\star T} A y, \quad \forall x \in P_m, \, \forall y \in P_n$$

Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

Optimal mixed strategies

$$x^\star = (0.37, 0.33, 0.3), \qquad y^\star = (0.4, 0, 0.13, 0.47)$$

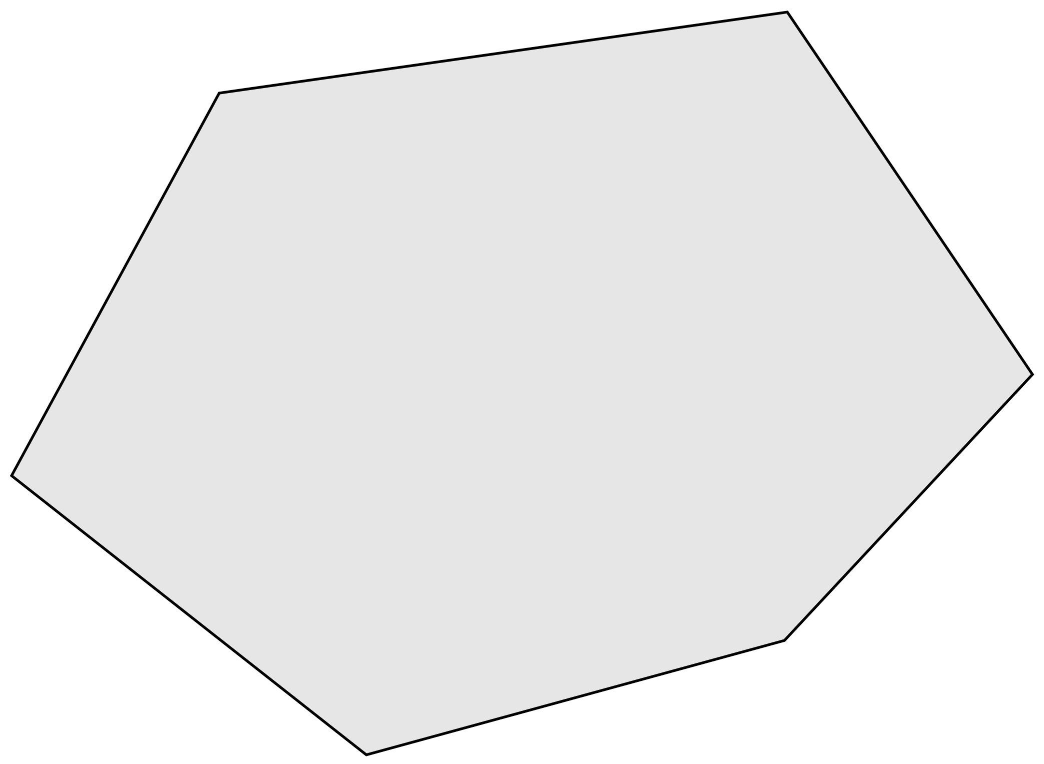
Expected payoff

$$x^{\star T} A y^\star = 0.2$$

Farkas lemma

Feasibility of polyhedra

$$P = \{x \mid Ax = b, \quad x \geq 0\}$$



How to show that P is **feasible**?

Easy: we just need to provide an $x \in P$, i.e., a **certificate**

How to show that P is **infeasible**?

Farkas lemma

Theorem

Given A and b, exactly one of the following statements is true:

1. There exists an x with $Ax = b$, $x \geq 0$
2. There exists a y with $A^T y \geq 0$, $b^T y < 0$

Farkas lemma

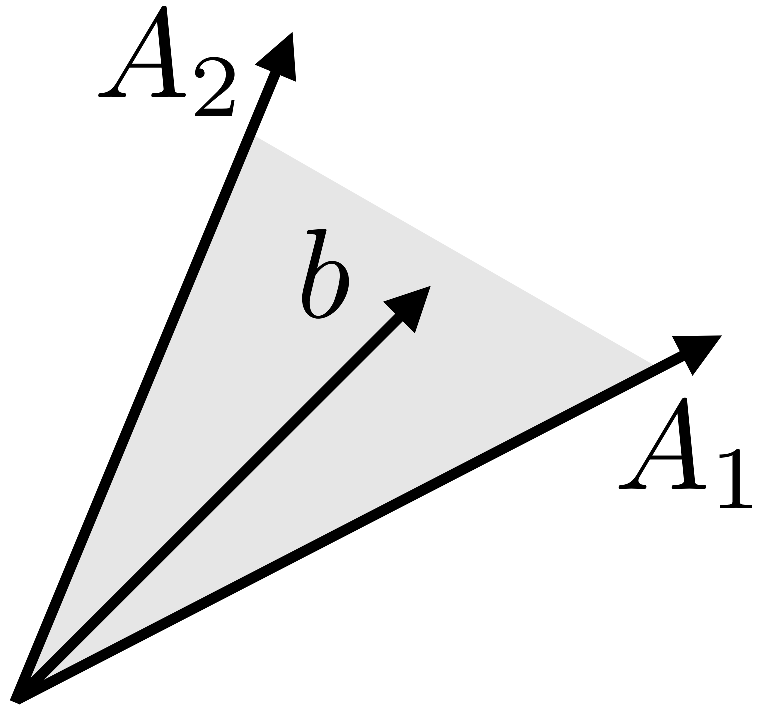
Geometric interpretation

1. First alternative

There exists an x with $Ax = b$, $x \geq 0$

$$b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0, \ i = 1, \ldots, n$$

b is in the cone generated by the columns of A

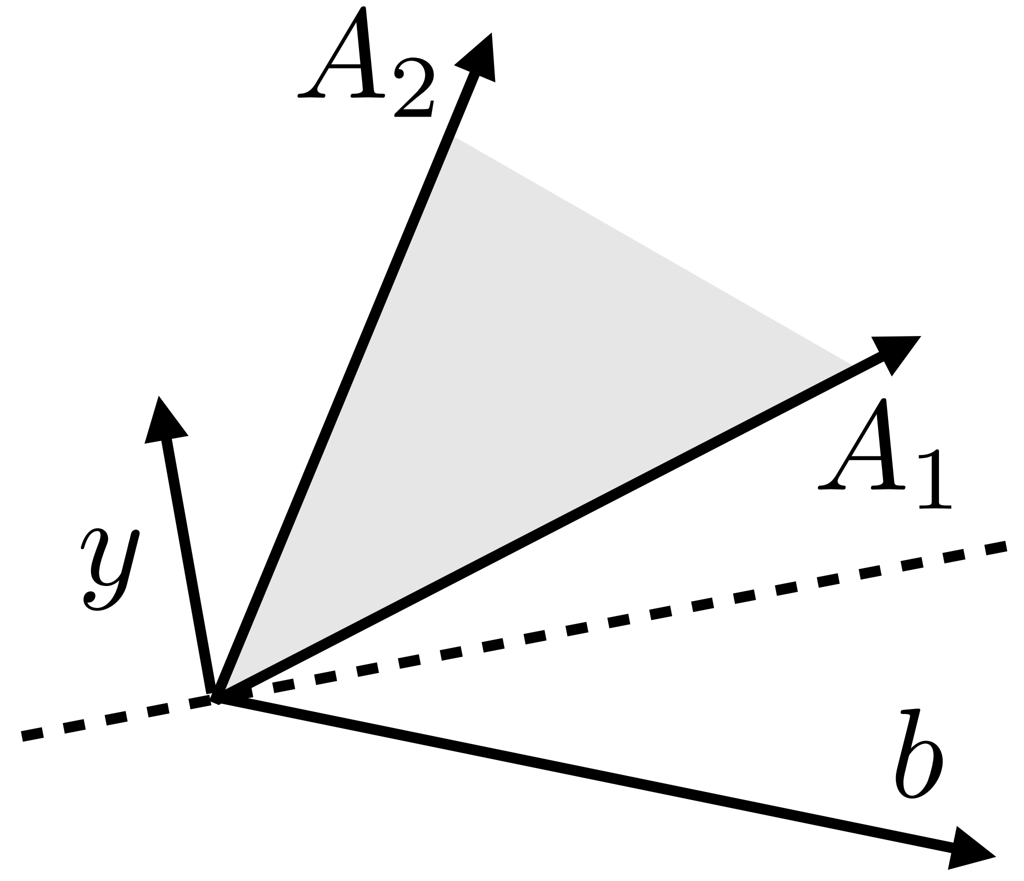


2. Second alternative

There exists a y with $A^T y \geq 0$, $b^T y < 0$

$$y^T A_i \geq 0, \quad i = 1, \ldots, m, \qquad y^T b < 0$$

The hyperplane $y^T z = 0$ separates b from $A_1, \ldots, A_n$



Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both true (easy)

$x \geq 0$, $Ax = b$ and $y^T A \geq 0$ ⟶ $y^T b = y^T Ax \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0

subject to $Ax = b$

$x \geq 0$

Dual

maximize $-b^T y$

subject to $A^T y \geq 0$

↑

$y = 0$ always feasible

Strong duality holds

$$d^\star \neq -\infty, \quad p^\star = d^\star$$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y \geq 0 \end{array}$$

Alternative 1: primal feasible $p^\star = d^\star = 0$

$b^T y \geq 0$ for all y such that $A^T y \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0

subject to $Ax = b$

$x \geq 0$

Dual

maximize $-b^T y$

subject to $A^T y \geq 0$

Alternative 2: primal infeasible $p^\star = d^\star = +\infty$

There exists y such that $A^T y \geq 0$ and $b^T y < 0$

y is an **infeasibility certificate**

Farkas lemma

Many variations

There exists x with $Ax = b,\ x \geq 0$

OR

There exists y with $A^T y \geq 0,\ b^T y < 0$

There exists x with $Ax \leq b,\ x \geq 0$

OR

There exists y with $A^T y \geq 0,\ b^T y < 0,\ y \geq 0$

There exists x with $Ax \leq b$

OR

There exists y with $A^T y = 0,\ b^T y < 0,\ y \geq 0$

Adding new variables

Adding new variables

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

$\longrightarrow$

minimize $c^T x + c_{n+1} x_{n+1}$
subject to $Ax + A_{n+1} x_{n+1} = b$
$x, x_{n+1} \geq 0$

Solution $x^\star, y^\star$

Solution $(x^\star, 0), y^\star$ **optimal** for the new problem?

Adding new variables

Optimality conditions

minimize $\quad c^T x + c_{n+1} x_{n+1}$

subject to $\quad Ax + A_{n+1} x_{n+1} = b$ ⟶ Solution $(x^\star, 0)$ is still **primal feasible**

$\quad x, x_{n+1} \geq 0$

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Yes

$(x^\star, 0)$ still **optimal** for new problem

Otherwise

Primal simplex

Adding new variables

Example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
	$4x_1 + 2x_2 + 1.5x_3 \leq 20$	production
	$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$	quality control
	$x \geq 0$	

minimize	$c^T x$
subject to	$Ax = b$
	$x \geq 0$

$$c = (-60, -30, -20, 0, 0, 0)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Adding new variables

Example: add new product?

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

$$\begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Previous solution

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Still optimal

$$A_{n+1}^T y^\star + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

Shall we add a new product?

Sensitivity analysis

Information from primal-dual solution

Goal: extract information from $x^\star, y^\star$ about their sensitivity with respect to changes in problem data

Modified LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b + u \\
& x \geq 0
\end{array}
$$

Optimal cost $p^\star(u)$

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Global lower bound

Given $y^\star$ a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T y^\star \\ &= p^\star(0) - u^T y^\star \end{aligned}$$

(from weak duality and dual feasibility)

It holds for any u

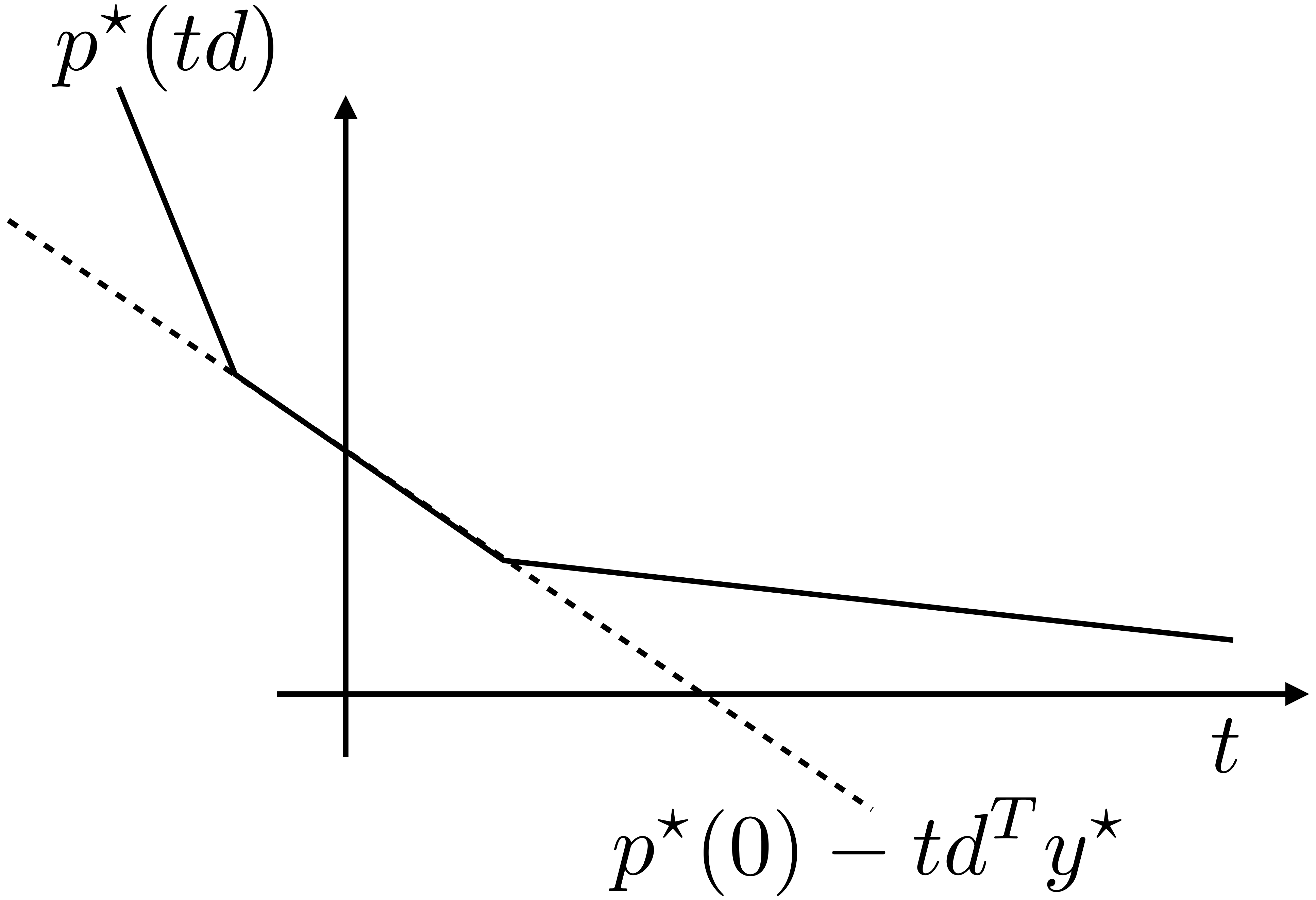
Global sensitivity

Example

Take $u = td$ with $d \in \mathbf{R}^m$ fixed

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + td \\ & x \geq 0 \end{array}$$

$p^\star(td)$ is the optimal value as a function of t



Sensitivity information (assuming $d^T y^\star \geq 0$)

- $t < 0$ the optimal value increases
- $t > 0$ the optimal value decreases (not so much if t is small)

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Assumption: $p^\star(0)$ is finite

Properties

- $p^\star(u) > -\infty$ everywhere (from global lower bound)
- $p^\star(u)$ is piecewise-linear on its domain

Optimal value function is piecewise linear

Proof

Dual feasible set

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

$$D = \{y \mid A^T y + c \geq 0\}$$

Assumption: $p^\star(0)$ is finite

If $p^\star(u)$ finite

$$p^\star(u) = \max_{y \in D} -(b+u)^T y = \max_{k=1,\ldots,r} -y_k^T u - b^T y_k$$

$y_1, \ldots, y_r$ are the extreme points of D

Local sensitivity

u in neighborhood of the origin

Original LP

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

→

Optimal solution

Primal $x_i = 0, \quad i \notin B$

$x_B^\star = A_B^{-1} b$

Dual $y^\star = -A_B^{-T} c_B$

Modified LP

minimize $c^T x$

subject to $Ax = b + u$

$x \geq 0$

Modified dual

maximize $-(b+u)^T y$

subject to $A^T y + c \geq 0$

Optimal basis does not change

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1} u$$

$$y^\star(u) = y^\star$$

Derivative of the optimal value function

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1}u$$
$$y^\star(u) = y^\star$$

Optimal value function

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c_B^T A_B^{-1} u \\ &= p^\star(0) - y^{\star T} u \end{aligned}$$

(affine for small u)

Local derivative

$$\frac{\partial p^\star(u)}{\partial u} = -y^\star$$

($y^\star$ are the **shadow prices**)

Sensitivity example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
	$4x_1 + 2x_2 + 1.5x_3 \leq 20$	production
	$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$	quality control
	$x \geq 0$	

$x^\star = (2, 0, 8, 24, 0, 0)$, $y^\star = (0, 10, 10)$, $c^T x^\star = -280$, basis $\{1, 3, 4\}$

What does $y_3^\star = 10$ mean?

Let's increase the quality control budget by 1, i.e., $u = (0, 0, 1)$

$$p^\star(10) = p^\star(0) - y^{\star T} u = -280 - 10 = -290$$

Linear optimization duality

Today, we learned to:

- **Interpret** linear optimization duality using game theory
- **Prove** Farkas lemma using duality
- **Understand** how the solution changes if we add new variables to the problem
- **Analyze sensitivity** of the cost with respect to changes in the data

Next lecture

- Nonlinear optimization