ORF522 – Linear and Nonlinear Optimization

7. Linear optimization duality I

Bartolomeo Stellato — Fall 2024

Today's agenda
[Chapter 4, LO][Chapter 5, LP]

- Obtaining lower bounds
- The dual problem
- Weak and strong duality
- Complementary slackness

Obtaining lower bounds

Obtaining lower bounds

A simple example

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + 3x_2 \geq 2 \end{array}$$

What is a **lower bound** on the optimal cost?

A lower bound is 2 because $x_1 + 3x_2 \geq 2$

Obtaining lower bounds

Another example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \end{aligned}$$

What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$\begin{aligned} & 1 \cdot (x_1 + x_2 \geq 2) \\ + & 2 \cdot (x_2 \geq 1) \\ = & x_1 + 3x_2 \geq 4 \end{aligned}$$

A lower bound is 4

Obtaining lower bounds

A more interesting example

minimize $x_1 + 3x_2$

subject to $x_1 + x_2 \geq 2$

$x_2 \geq 1$

$x_1 - x_2 \geq 3$

How can we obtain a lower bound?

Add constraints

$$
\begin{aligned}
& y_1 \cdot (x_1 + x_2 \geq 2) \\
+ & y_2 \cdot (x_2 \geq 1) \\
+ & y_3 \cdot (x_1 - x_2 \geq 3) \\
= & x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3
\end{aligned}
$$

Bound

Match cost coefficients

$$y_1 + y_3 = 1$$

$$y_1 + y_2 - y_3 = 3$$

$$y_1, y_2, y_3 \geq 0$$

Many options

$y = (1, 2, 0) \Rightarrow$ Bound 4

$y = (0, 4, 1) \Rightarrow$ Bound 7

How can we get the **best one**?

Obtaining lower bounds

A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$
\begin{array}{ll}
\text{maximize} & 2y_1 + y_2 + 3y_3 \\
\text{subject to} & y_1 + y_3 = 1 \\
& y_1 + y_2 - y_3 = 3 \\
& y_1, y_2, y_3 \geq 0
\end{array}
$$

This linear optimization problem is called the **dual problem**

The dual problem

Lagrangian dual function

Consider the Linear Program

$$\begin{aligned} p^\star = \quad & \text{minimize} && c^T x \\ & \text{subject to} && Ax \le b \\ & && Cx = d \end{aligned}$$

$\longrightarrow$

Lagrangian

$$L(x, y, z) = c^T x + y^T (Ax - b) + z^T (Cx - d)$$

Lagrange dual function

$$g(y, z) = \inf_x L(x, y, z)$$

Lower bound property

For any $y \ge 0$ and z, $\quad g(y, z) \le p^\star$

Proof. Let $\tilde{x}$ be a feasible point. Then,

$$g(y, z) = \inf_x L(x, y, z) \le c^T \tilde{x} + \underbrace{y^T}_{\ge 0} (\underbrace{A\tilde{x} - b}_{\le 0}) + z^T (\underbrace{C\tilde{x} - d}_{= 0}) \le c^T \tilde{x}$$

$$\implies \quad g(y, z) \le p^\star$$

$\blacksquare$

Dual problem

Lagrangian

$$L(x,y,z) = c^T x + y^T(Ax - b) + z^T(Cx - d)$$

Lower bound property

For any $y \geq 0$ and z, $\quad g(y,z) \leq p^\star$

Lagrange dual function

$$g(y,z) = \inf_x L(x,y,z)$$

When $g(y,z) = -\infty$ the bound is *vacuous*.

when is it non-vacuous?

$\nabla_x L(x,y,z)$

$$g(y,z) = \inf_x (c + A^T y + C^T z)^T x - b^T y - d^T z$$

$$= \begin{cases} -b^T y - d^T z & c + A^T y + C^T z = 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem

find the best lower bound

$$\begin{array}{ll} \text{maximize} & -b^T y - d^T z \\ \text{subject to} & c + A^T y + C^T z = 0 \\ & y \geq 0 \end{array}$$

Primal and dual problems

Primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$

Dual problem

$$\begin{array}{ll} \text{maximize} & -b^T y - d^T z \\ \text{subject to} & c + A^T y + C^T z = 0 \\ & y \geq 0 \end{array}$$

Primal variable $x \in \mathbf{R}^n$

Dual variables $y \in \mathbf{R}^m$, $z \in \mathbf{R}^p$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

Example from before

minimize $x_1 + 3x_2$
subject to $x_1 + x_2 \geq 2$
$x_2 \geq 1$
$x_1 - x_2 \geq 3$

$\longrightarrow$

Inequality form LP

minimize $c^T x$
subject to $Ax \leq b$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$b = (-2, -1, -3)$$

Dual

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

$\longrightarrow$

maximize $2y_1 + y_2 + 3y_3$
subject to $-y_1 - y_3 = -1$
$-y_1 - y_2 + y_3 = -3$
$y_1, y_2, y_3 \geq 0$

To memorize

Ways to get the dual

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention

- Consider constraints as $g(x) \leq 0$ or $g(x) = 0$
- Each dual variable is associated to a primal constraint
- z free for primal equalities and $y \geq 0$ for primal inequalities

Dual of an LP in standard form

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
$$

Dual of the dual

Theorem

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

Exercise

Derive dual and dualize again

Primal		**Dual**	
minimize	$c^T x$	maximize	$-b^T y - d^T z$
subject to	$Ax \leq b$	subject to	$A^T y + C^T z + c = 0$
	$Cx = d$		$y \geq 0$

Theorem

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

Weak and strong duality

Optimal objective values

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

⟶ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$

■

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Strong duality

Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow$$

- optimal basis B
- optimal solution $x^\star$ with $A_B x_B^\star = b$
- reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$

Define $y^\star$ such that $y^\star = -A_B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

$$-b^T y^\star = -b^T(-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

By weak duality theorem corollary, $y^\star$ is an optimal solution of the dual.
Therefore, $d^\star = p^\star$. ■

Exception to strong duality

Primal

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

Optimal value is $p^\star = +\infty$

Dual

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & 0 \cdot y + 1 = 0 \\ & y \geq 0 \end{array}$$

Optimal value is $d^\star = -\infty$

Both **primal** and **dual infeasible**

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1,1)$ and $(3,3)$ proven by **weak duality**
- $(3,1)$ and $(2,2)$ proven by **strong duality**

Example

Production problem

maximize $x_1 + 2x_2$ ← Profits

subject to
$$x_1 \leq 100$$
$$2x_2 \leq 200$$
$$x_1 + x_2 \leq 150$$
← Resources
$$x_1, x_2 \geq 0$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Dualize

1. Transform in inequality form

minimize $c^T x$

subject to $Ax \leq b$

2. Derive dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Production problem

The dual

$$\begin{aligned} \text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs $1\times$ resource 1 and 3): $y_1 + y_3 \geq 1$
  - Product 2 (price 2, needs $2\times$ resource 2 and $1\times$ resource 3): $2y_2 + y_3 \geq 2$

Complementary slackness

Optimality conditions

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax \leq b$
- y is **dual feasible:** $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Can we **relate** x and y (not only the objective)?

Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Theorem

Primal,dual feasible x, y are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \dots, m$$

i.e., at optimum, $b - Ax$ and y have a **complementary sparsity** pattern:

$$y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$$
$$a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$$

Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Proof

The duality gap at primal feasible x and dual feasible y can be written as

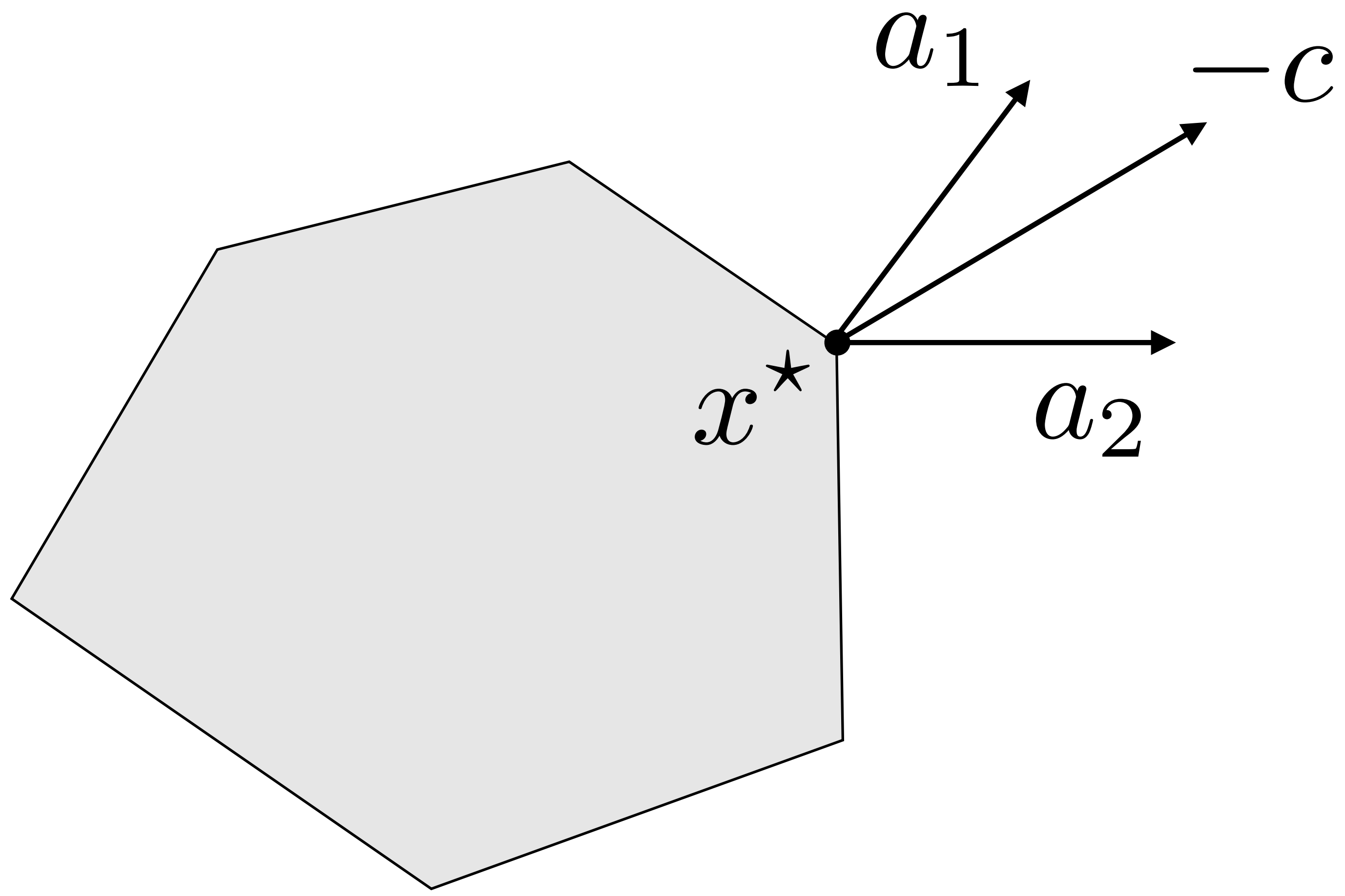
$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0 ■

For **feasible** x and y **complementary slackness** = **zero duality gap**

Geometric interpretation

Example in $\mathbf{R}^2$



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Optimal dual solution y satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$

Geometric interpretation: $-c$ lies in the cone generated by a_1 and a_2

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at x $\longrightarrow$ $y = (0, y_2, 0, y_4)$

$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

and

$$y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Complementary slackness is useful to recover $y^\star$ from $x^\star$

Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Geometrically link** primal and dual solutions with complementary slackness

Next lecture

More on duality:

- Game theoretic interpretation
- Alternative systems
- Adding new variables
- Sensitivity analysis