ORF522 – Linear and Nonlinear Optimization

3. Geometry and polyhedra

Bartolomeo Stellato — Fall 2024

Today's agenda

Readings [Chapter 2, LO]

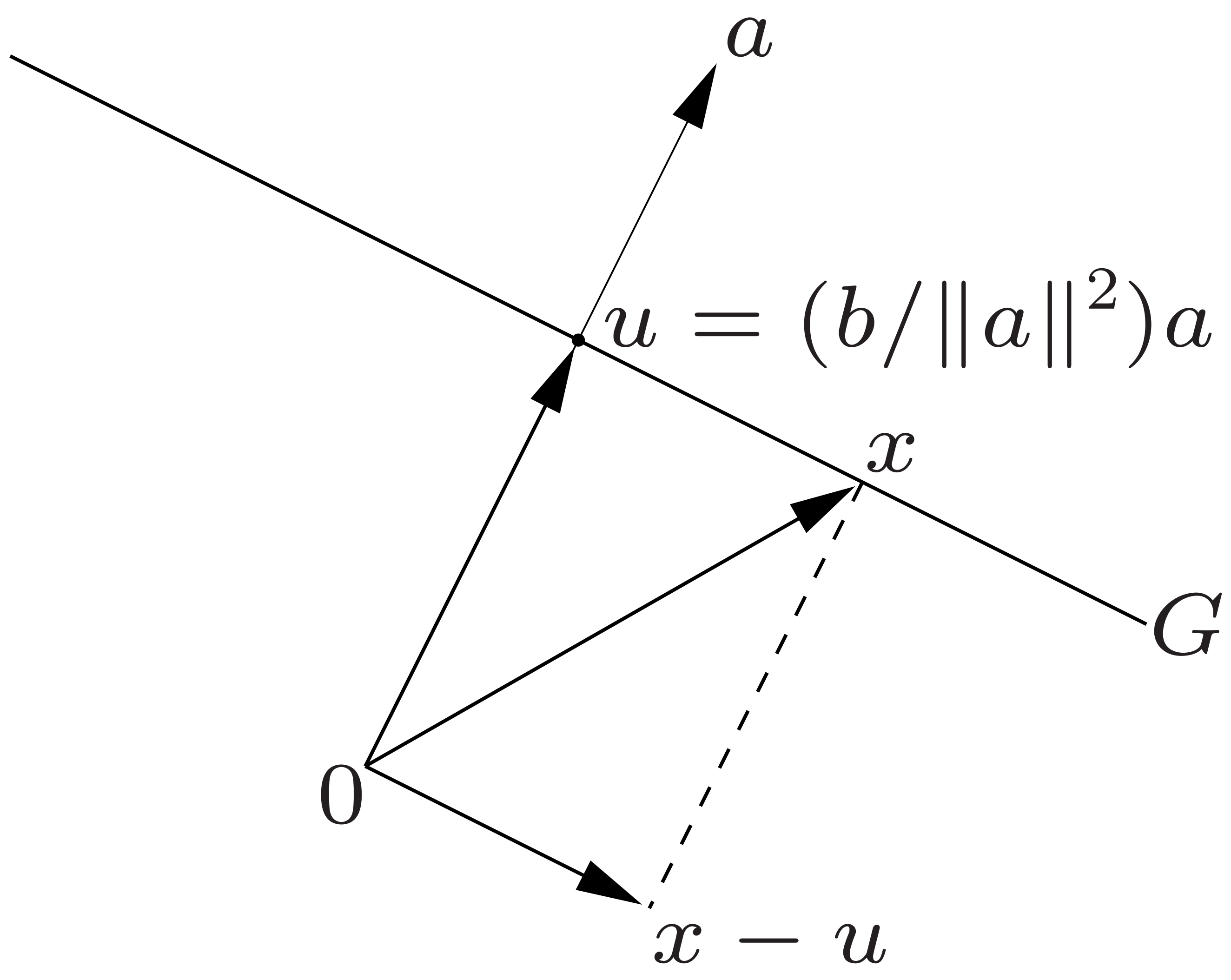
- Polyhedra and linear algebra
- Corners: extreme points, vertices, basic feasible solutions
- Constructing basic solutions
- Existence and optimality of extreme points

Polyhedra and linear algebra

Hyperplanes and half spaces

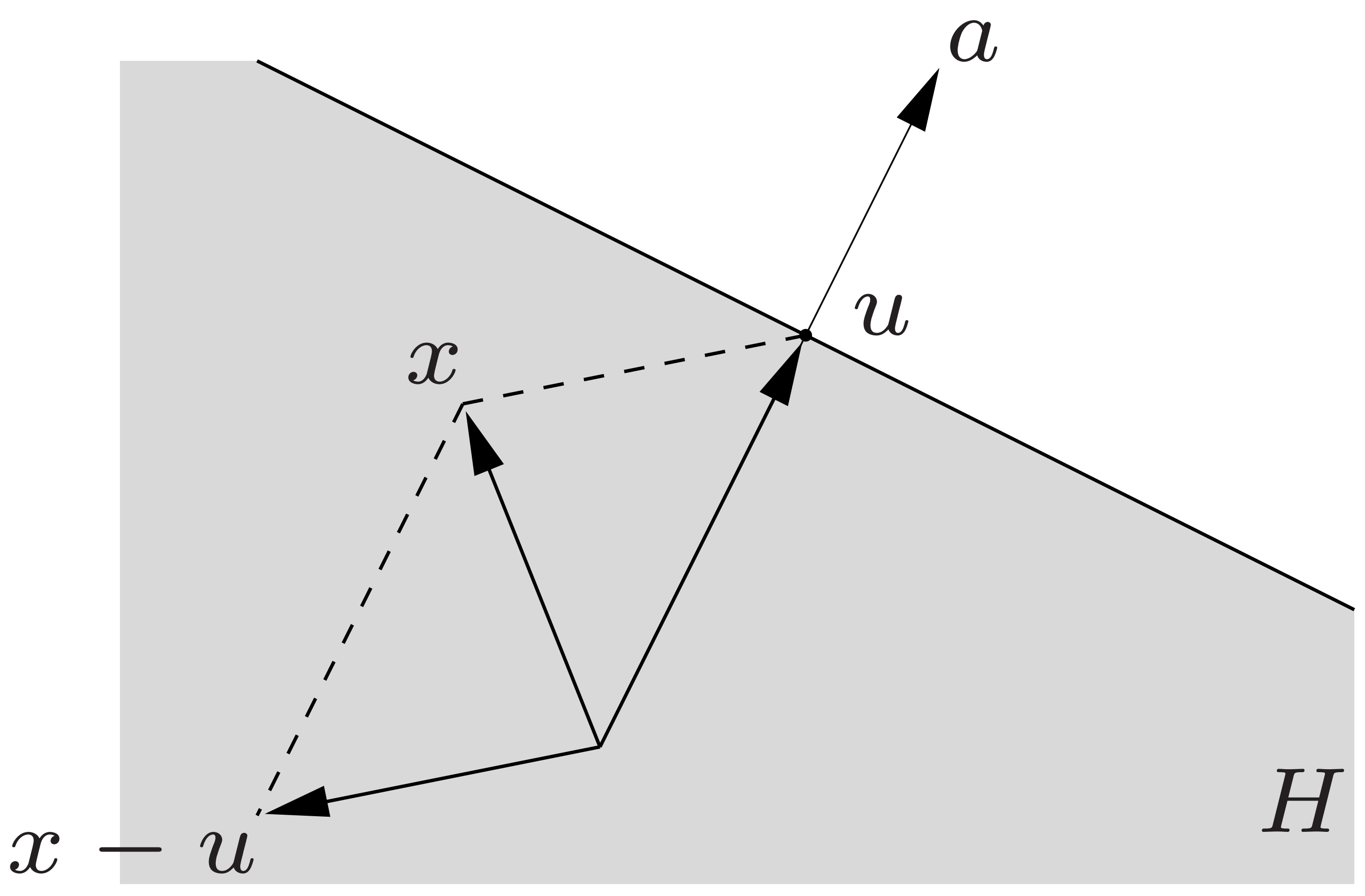
hyperplane

$$G = \{x \mid a^T x = b\}$$



halfspace

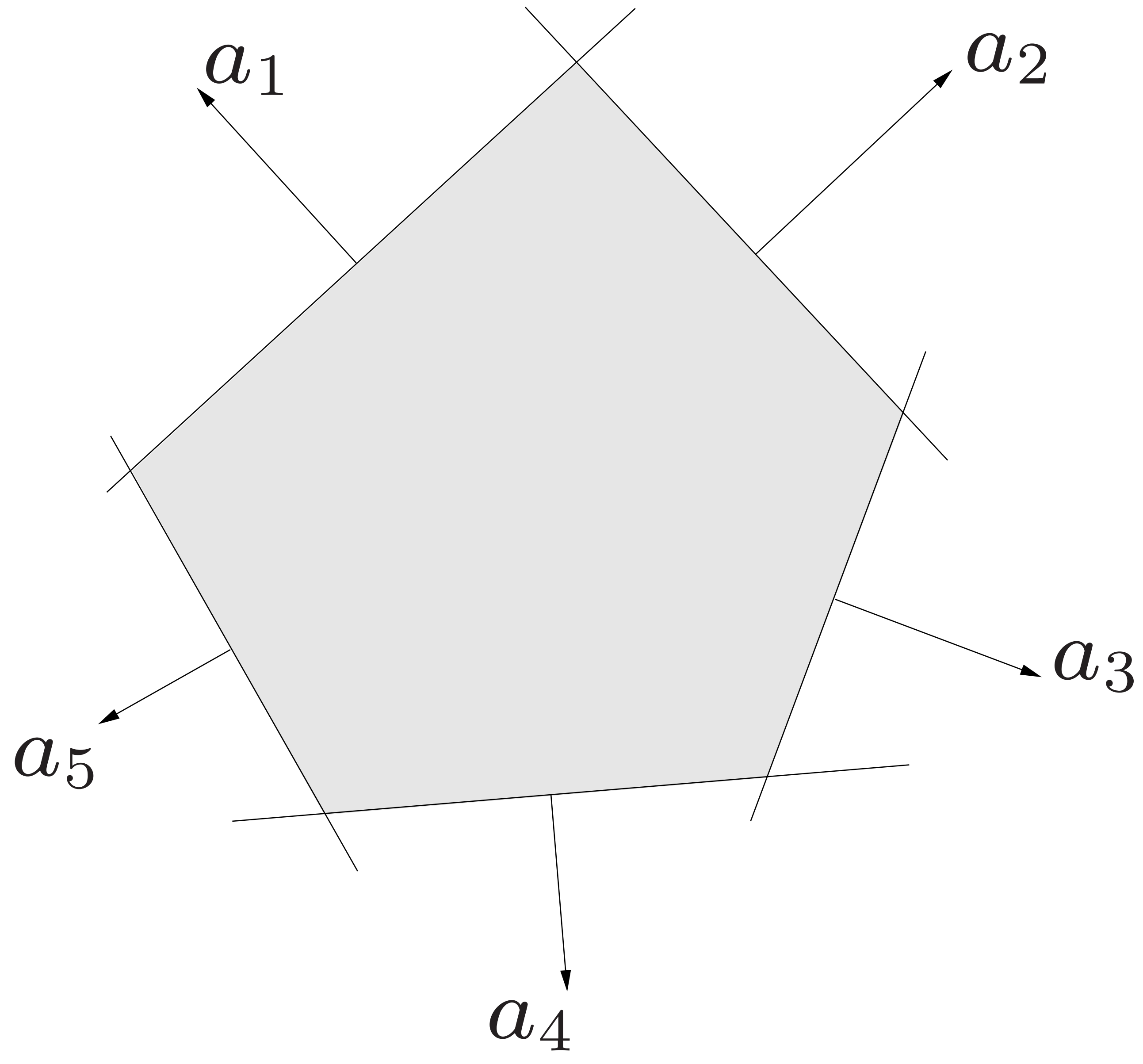
$$H = \{x \mid a^T x \leq b\}$$



- the vector $u = (b/\|a\|^2)a$ satisfies $a^T u = b$
- x is in hyperplane G if $a^T(x - u) = 0$ ($x - u$ is orthogonal to a)
- x is in halfspace H if $a^T(x - u) \leq 0$ (angle $\angle(x - u, a) \geq \pi/2$)

Polyhedron
Definition

$$P = \{x \mid a_i^T x \le b_i, \quad i = 1, \ldots, m\} = \{x \mid Ax \le b\}$$

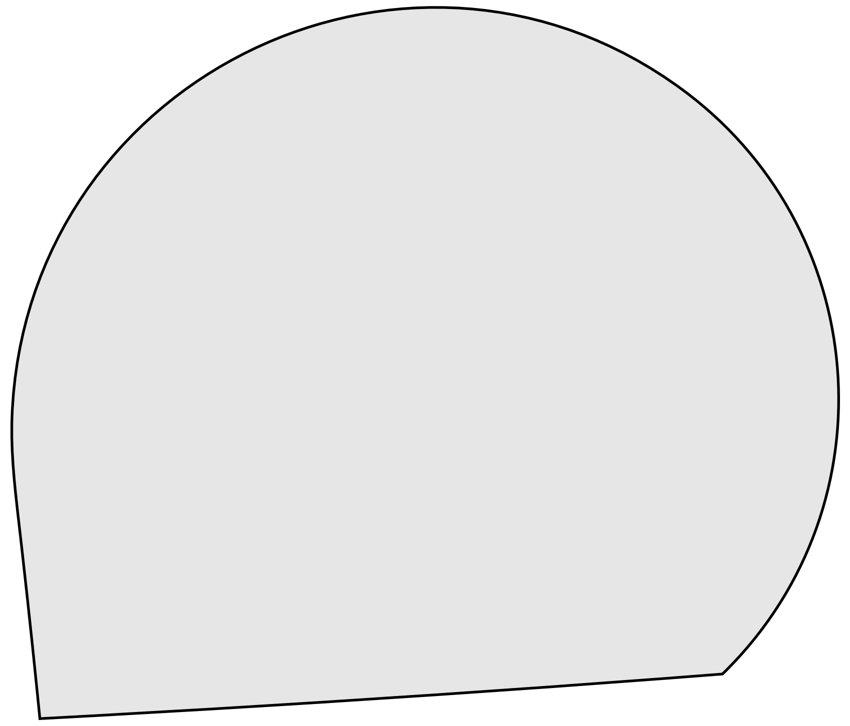


- Intersection of finite number of halfspaces
- Can include equalities

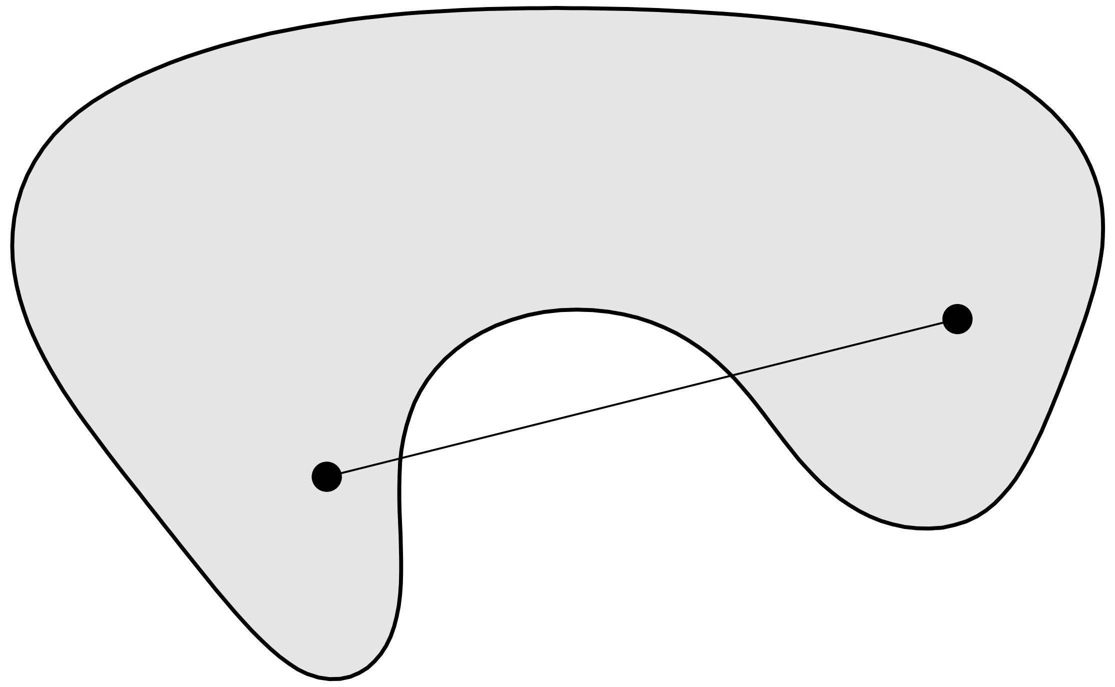
Convex set

Definition

For any $x, y \in C$ and any $\alpha \in [0, 1]$ $\alpha x + (1 - \alpha) y \in C$



Convex



Not convex

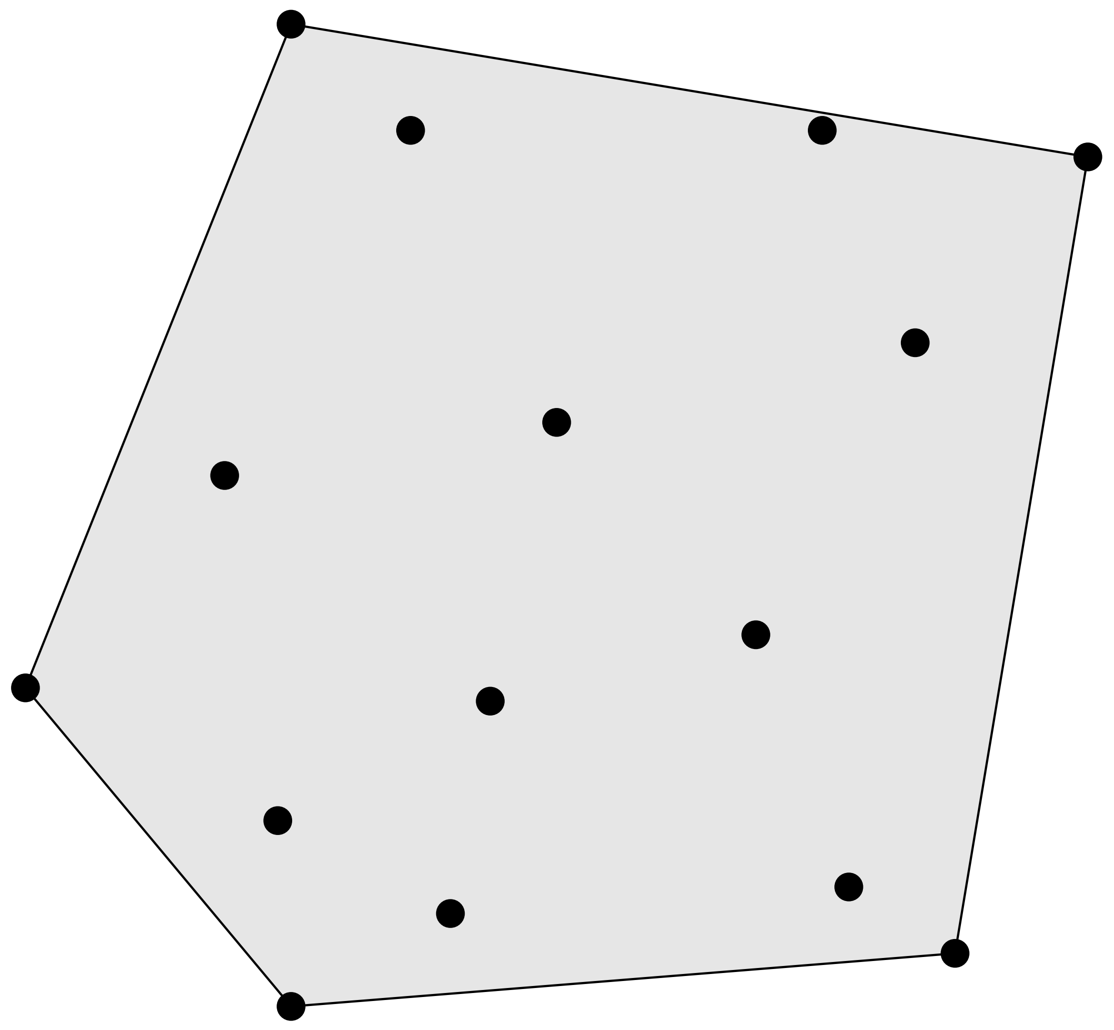
Examples

- $\mathbf{R}^n$
- Hyperplanes
- Halfspaces
- Polyhedra

Convex combinations

Convex combination

$\alpha_1 x_1 + \cdots + \alpha_k x_k$ for any $x_1, \ldots, x_k$ and $\alpha_1, \ldots, \alpha_k$ such that $\alpha_i \geq 0,\ \sum_{i=1}^k \alpha_i = 1$



Convex hull

$$\mathbf{conv}\, C = \left\{ \sum_{i=1}^{k} \alpha_i x_i \mid x_i \in C,\ \alpha \geq 0,\ \mathbf{1}^T \alpha = 1 \right\}$$

Linear independence

a nonempty set of vectors $\{v_1, \ldots, v_k\}$ is **linearly indepdendent** if

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$$

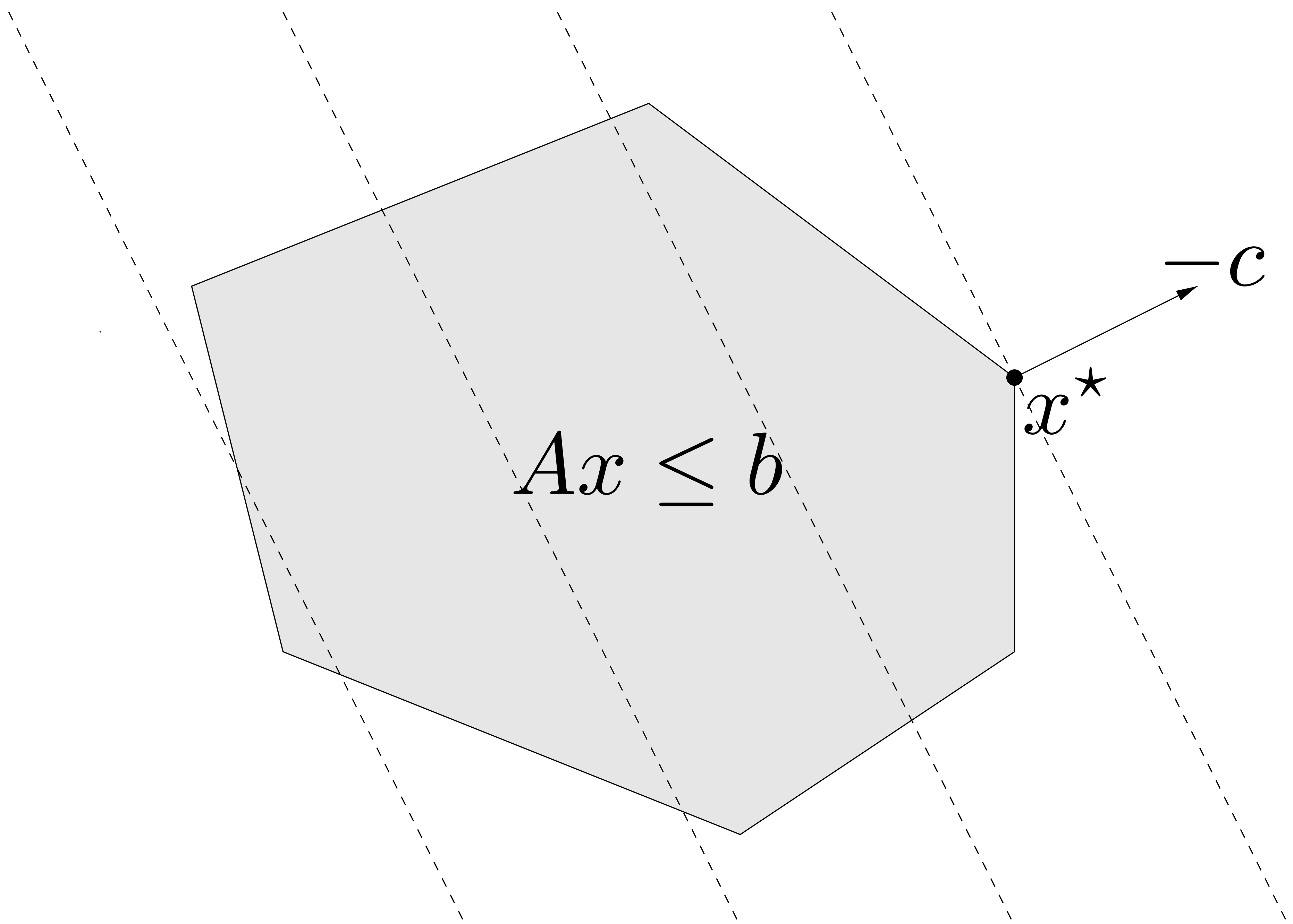
holds only for $\alpha_1 = \cdots = \alpha_k = 0$

Properties

- The coefficients α_k in a linear combination $x = \alpha_1 v_1 + \cdots + \alpha_k v_k$ are unique
- None of the vectors v_i is a linear combination of the other vectors

Geometrical interpretation of linear optimization

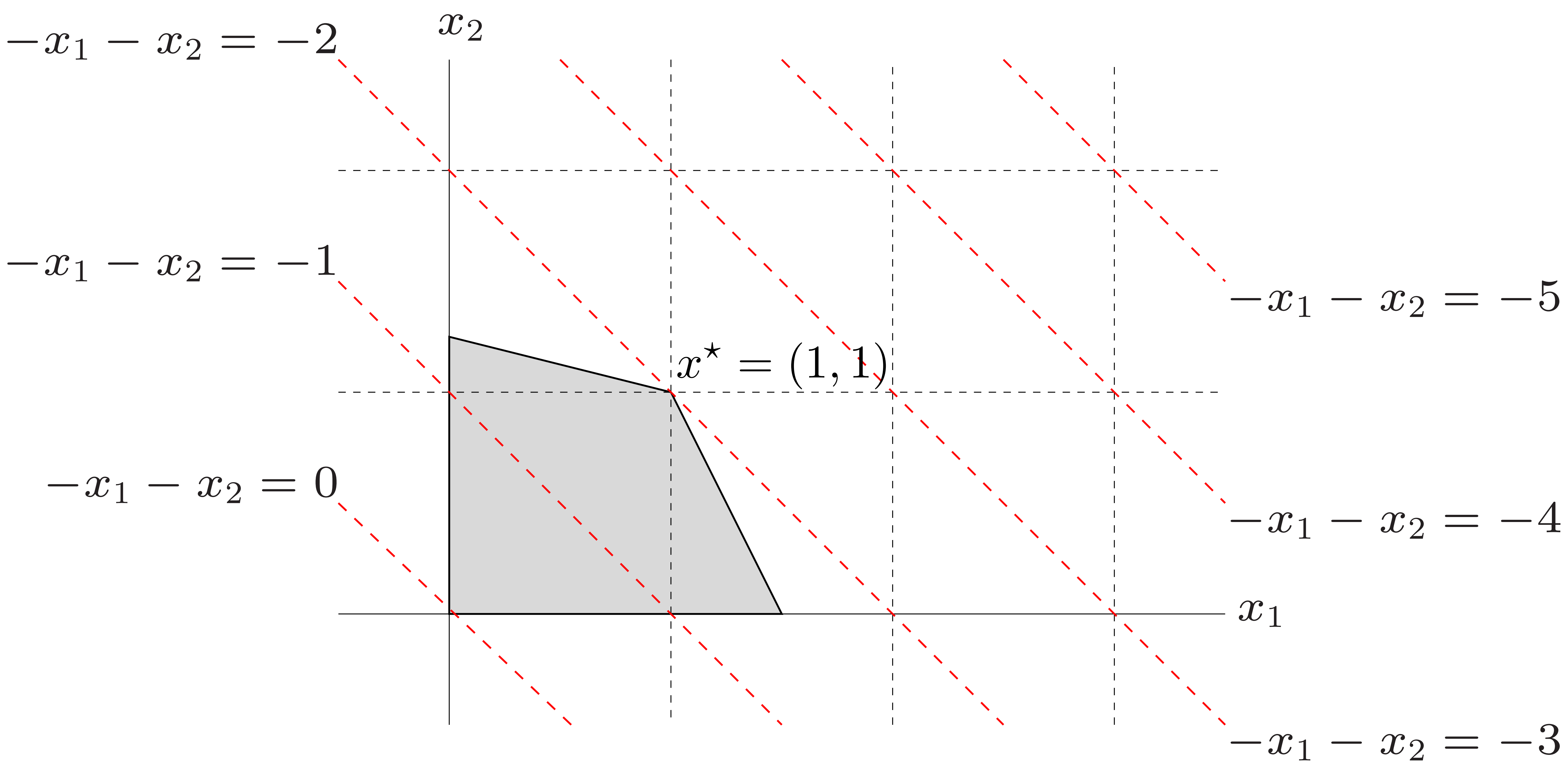
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$



Dashed lines (hyperplanes) are level sets $c^T x = \alpha$ for different α

Example of linear optimization

$$\begin{array}{ll} \text{minimize} & -x_1 - x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 3 \\ & x_1 + 4x_2 \leq 5 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$



Optimal solutions tend to be at a “**corner**” of the feasible set

How do we formalize it?

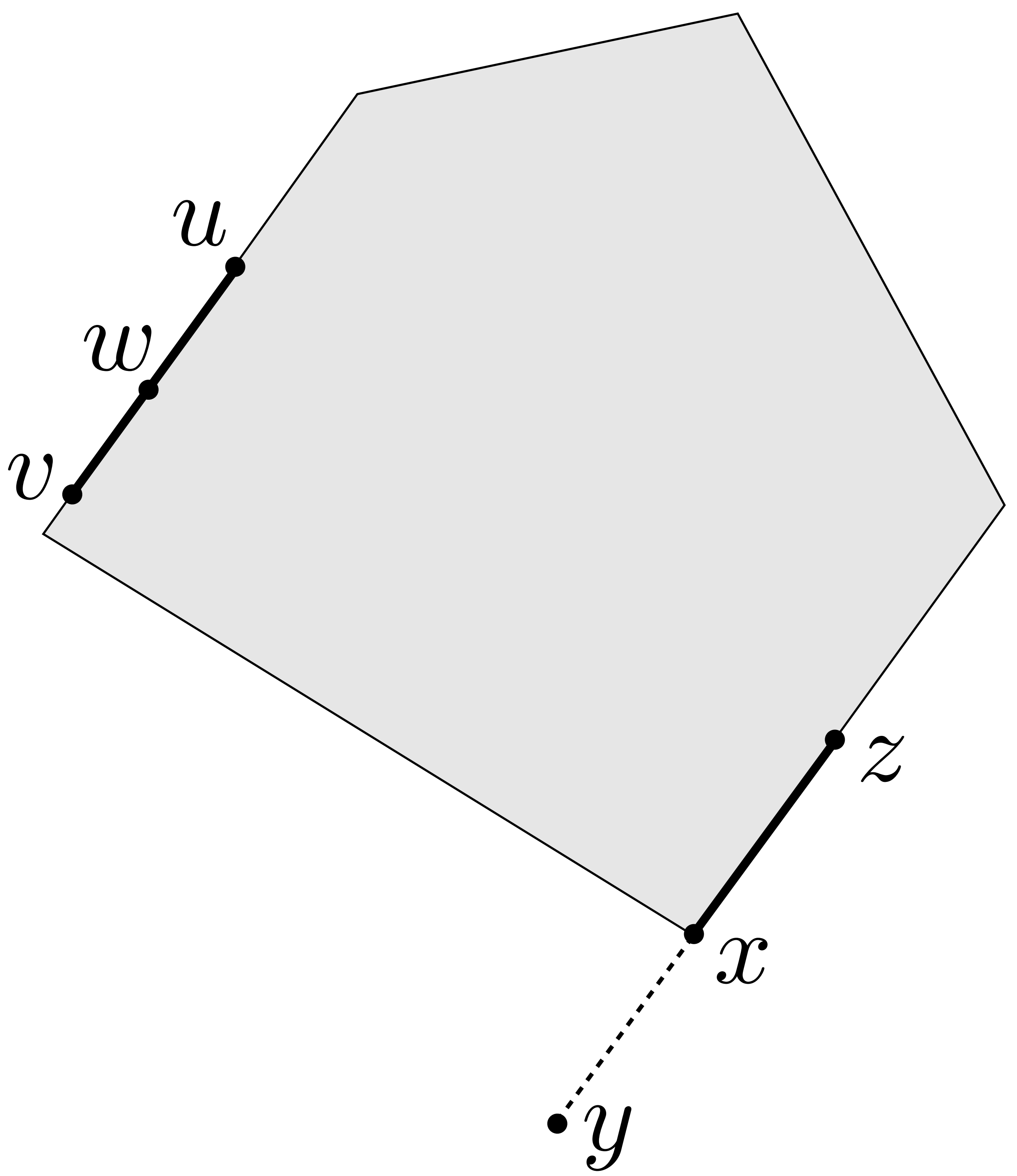
Corners of linear optimization

Extreme points

Definition

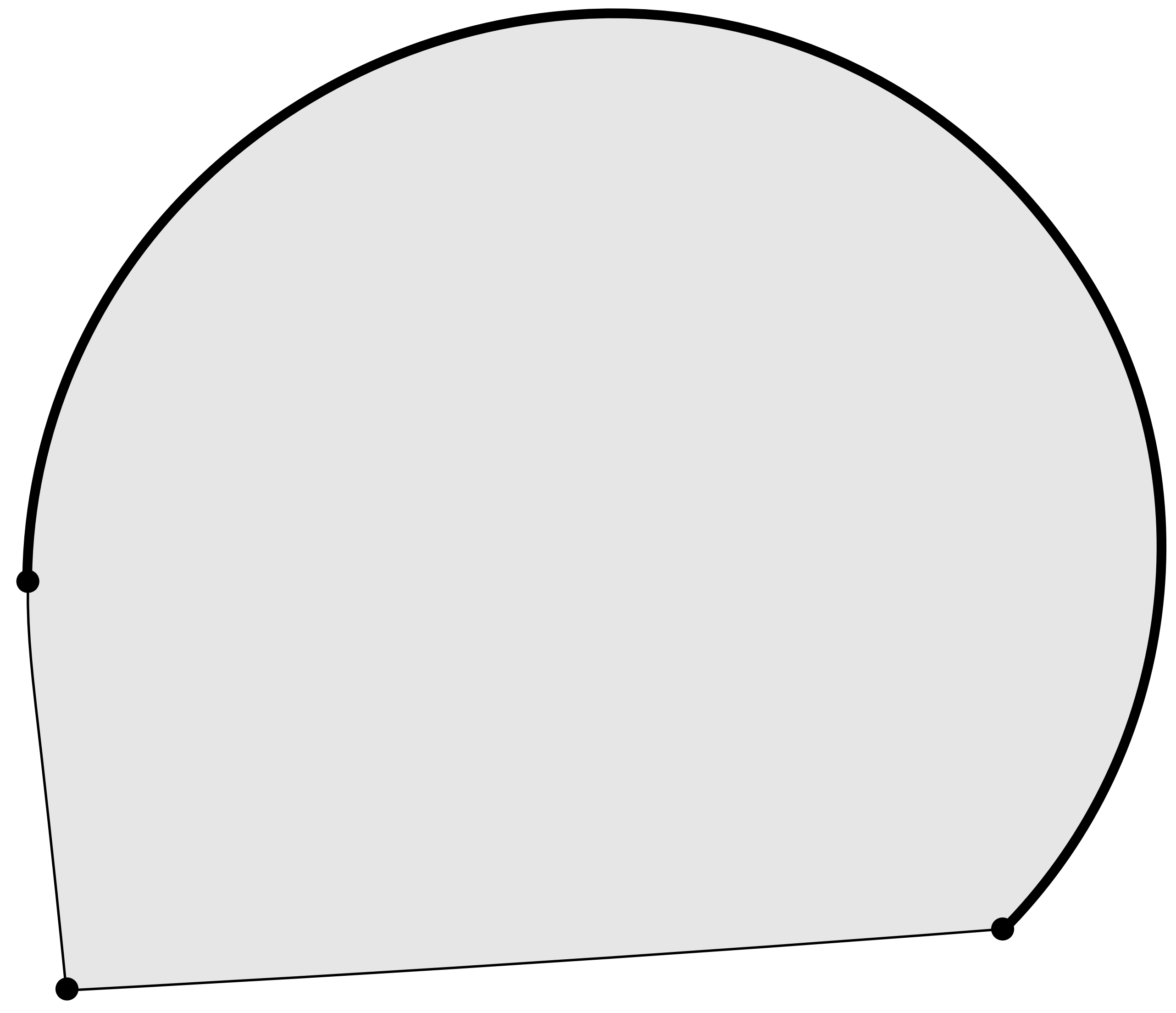
$x \in P$ is said to be an **extreme point** of P if

$\nexists y, z \in P \ (y \neq x, z \neq x)$ and $\alpha \in (0,1)$ such that $x = \alpha y + (1-\alpha) z$



Extreme points

Convex sets



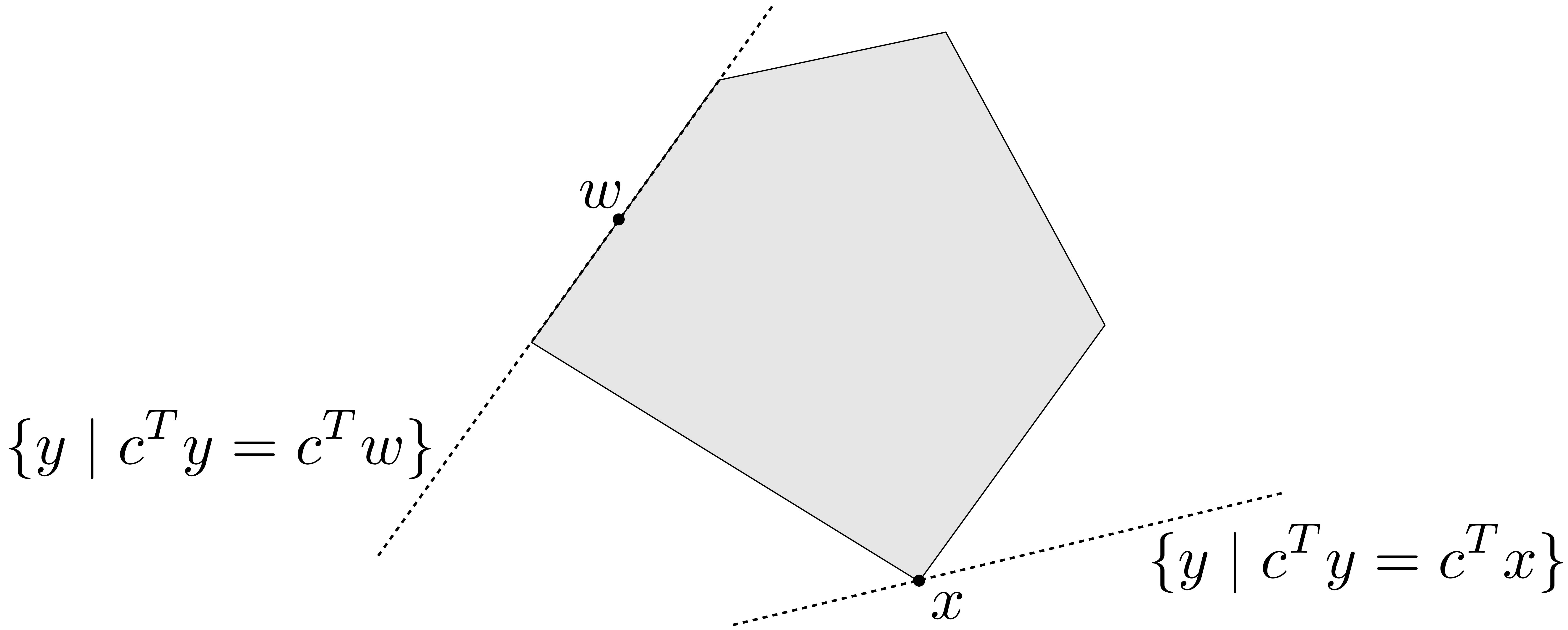
- Convex sets can have an infinite number of extreme points
- Polyhedra are convex sets with a finite number of extreme points

Vertices

Definition

$x \in P$ is a **vertex** if $\exists c$ such that x is the unique optimum of

$$\begin{array}{ll} \text{minimize} & c^T y \\ \text{subject to} & y \in P \end{array}$$



Basic feasible solution

$$P = \{x \mid a_i^T x \le b_i, \quad i = 1, \dots, m\}$$

Active constraints at $\bar{x}$

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \dots, m\} \mid a_i^T \bar{x} = b_i\}$$

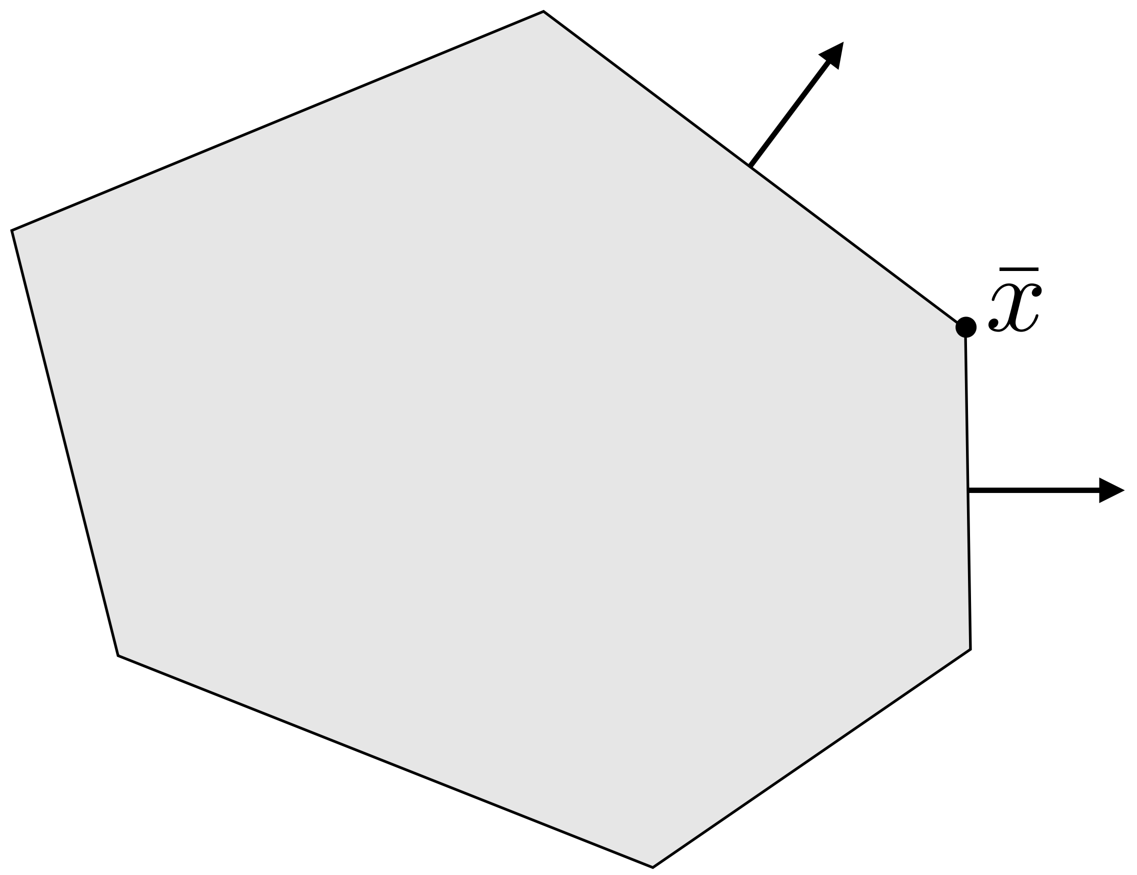
Index of all the constraints satisfied as **equality**

Basic solution $\bar{x}$

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

Basic feasible solution $\bar{x}$

- $\bar{x} \in P$
- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

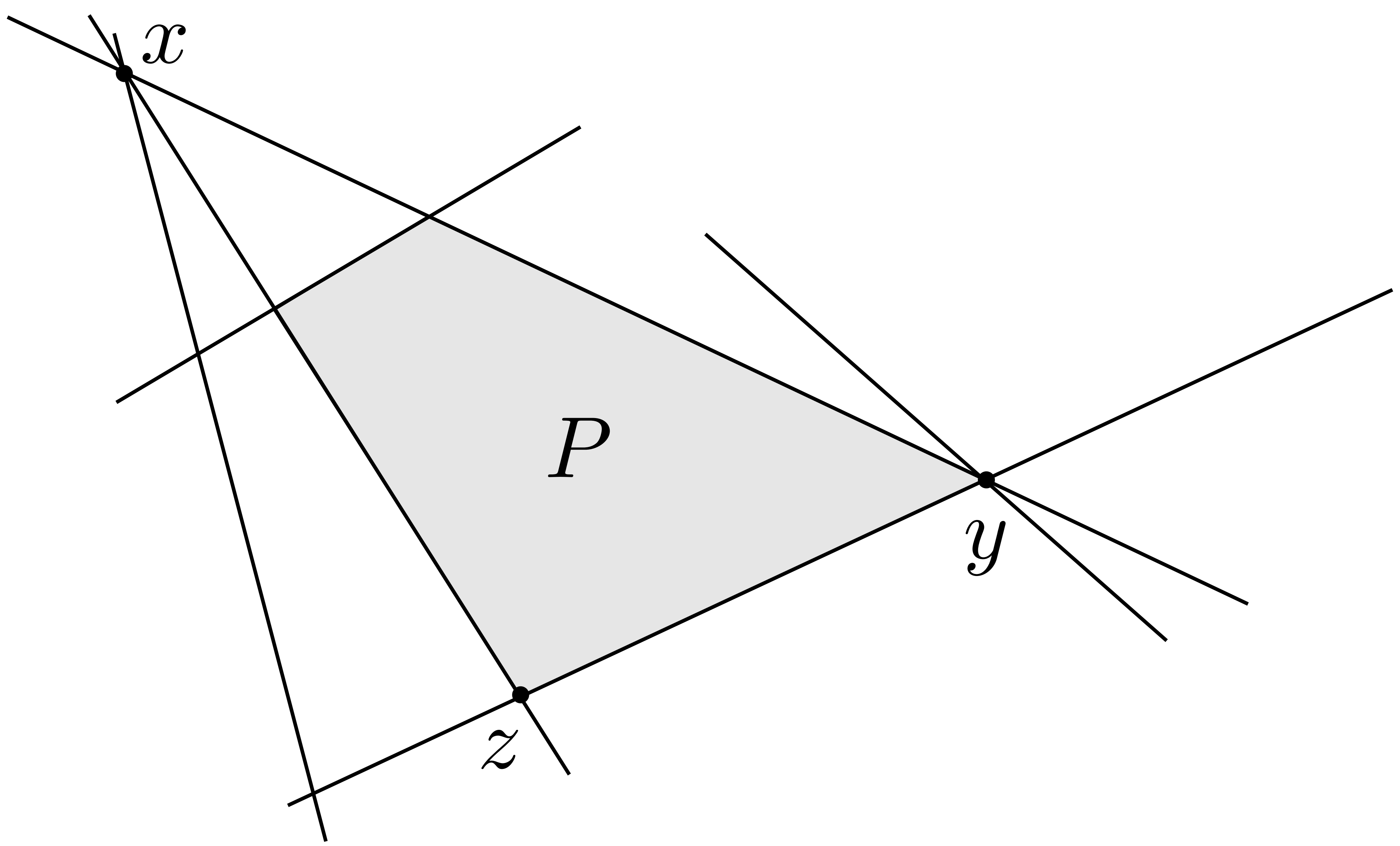


Degenerate basic feasible solutions

A solution $\bar{x}$ is degenerate if $|\mathcal{I}(\bar{x})| > n$

True or False?

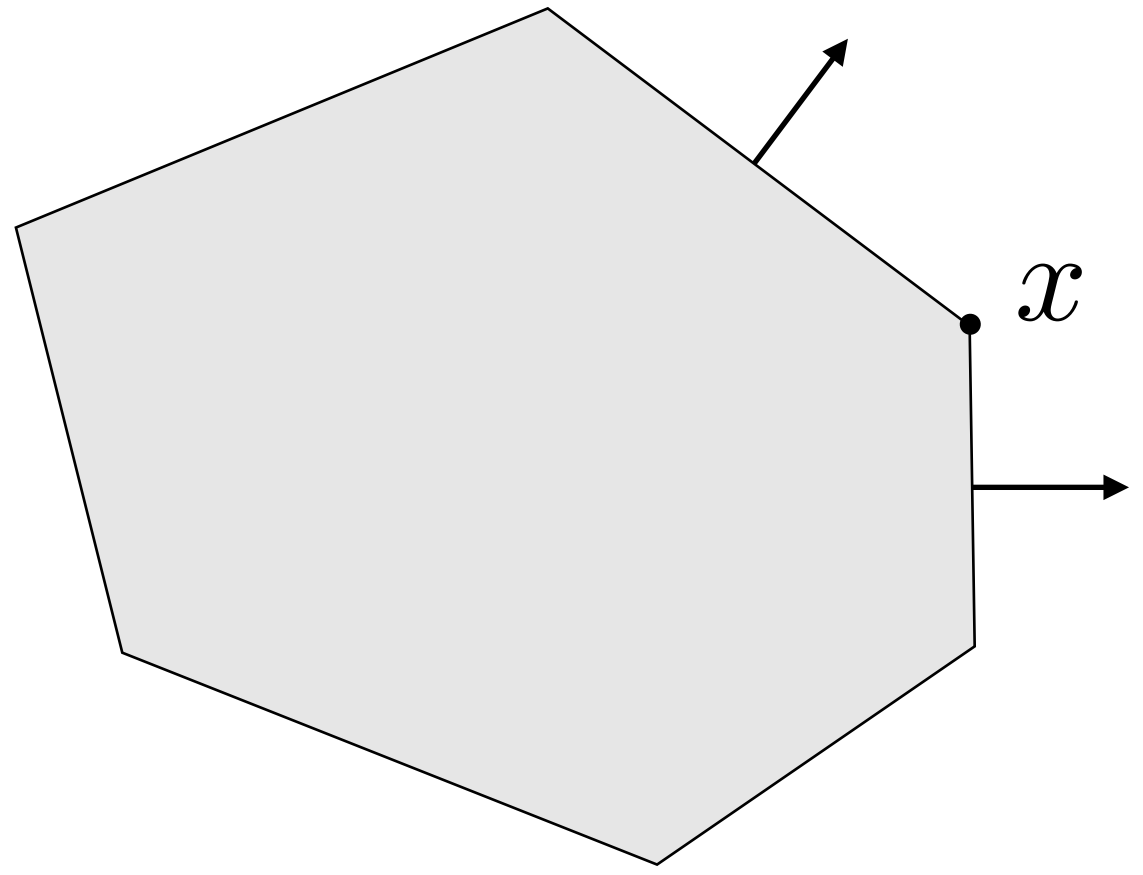
	Basic	Feasible	Degenerate
x			
y			
z			



Equivalence

Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



Let $x \in P$

x is a **vertex** $\iff$ x is an **extreme point** $\iff$ x is a **basic feasible solution**

Equivalent theorem proof

Vertex $\implies$ Extreme point

If x is a vertex, $\exists c$ such that $c^T x < c^T y, \quad \forall y \in P, y \neq x$

Let's assume x is not an extreme point:

$\exists y, z \neq x$ such that $x = \lambda y + (1 - \lambda) z$ with $0 < \lambda < 1$

Since x is a vertex, $c^T x < c^T y$ and $c^T x < c^T z$

Therefore, $c^T x = \lambda c^T y + (1 - \lambda) c^T z > \lambda c^T x + (1 - \lambda) c^T x = c^T x$

$\implies$ **contradiction** $\blacksquare$

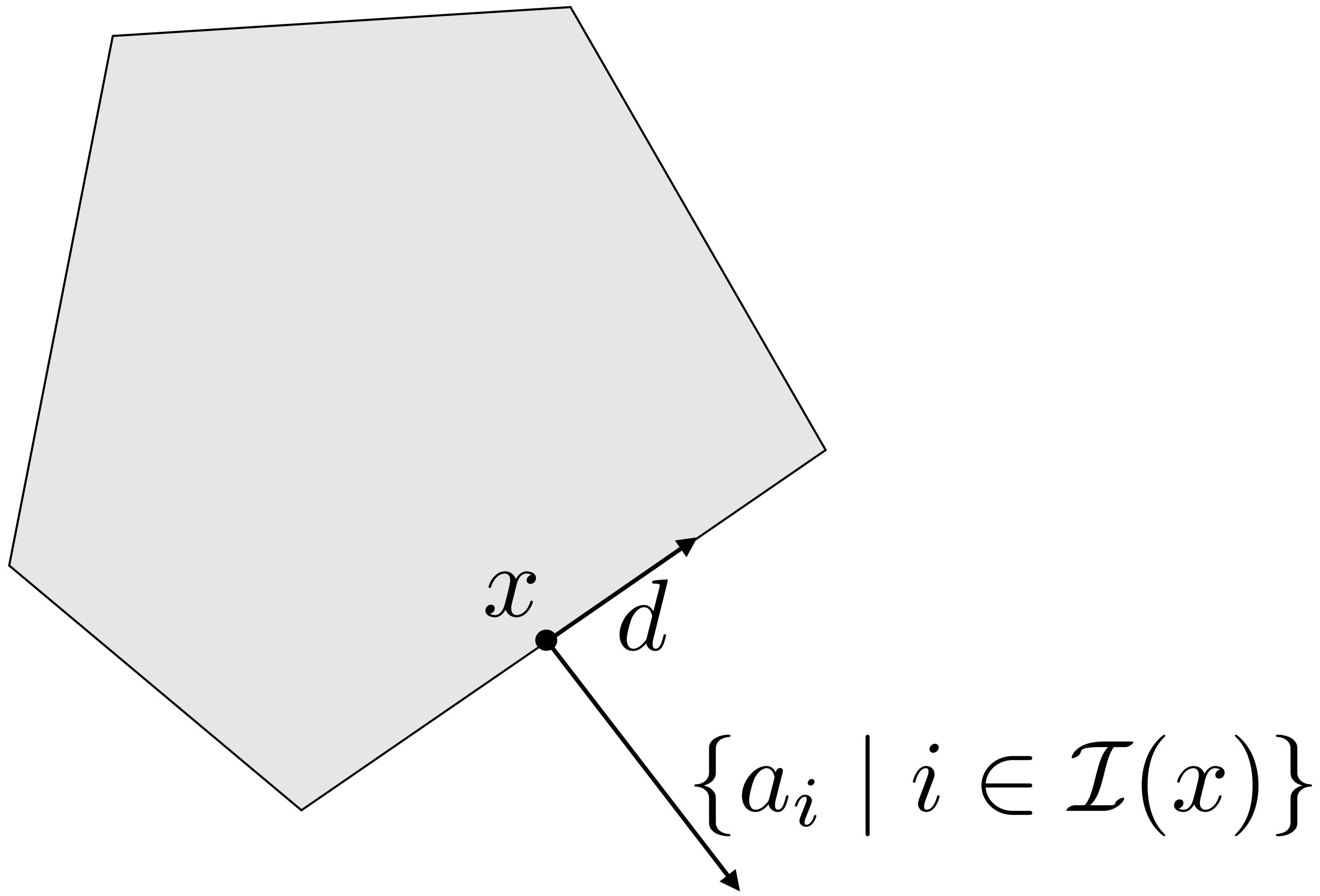
Equivalent theorem proof

Extreme point $\implies$ **Basic feasible solution** (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span $\mathbf{R}^n$

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$



Equivalent theorem proof

Extreme point $\implies$ **Basic feasible solution** (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span $\mathbf{R}^n$

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$

Let $\epsilon > 0$ and define $y = x + \epsilon d$ and $z = x - \epsilon d$

For $i \in \mathcal{I}(x)$ we have $a_i^T y = b_i$ and $a_i^T z = b_i$

For $i \notin \mathcal{I}(x)$ we have $a_i^T x < b_i \quad \Rightarrow \quad a_i^T(x + \epsilon d) < b_i$ and $a_i^T(x - \epsilon d) < b_i$

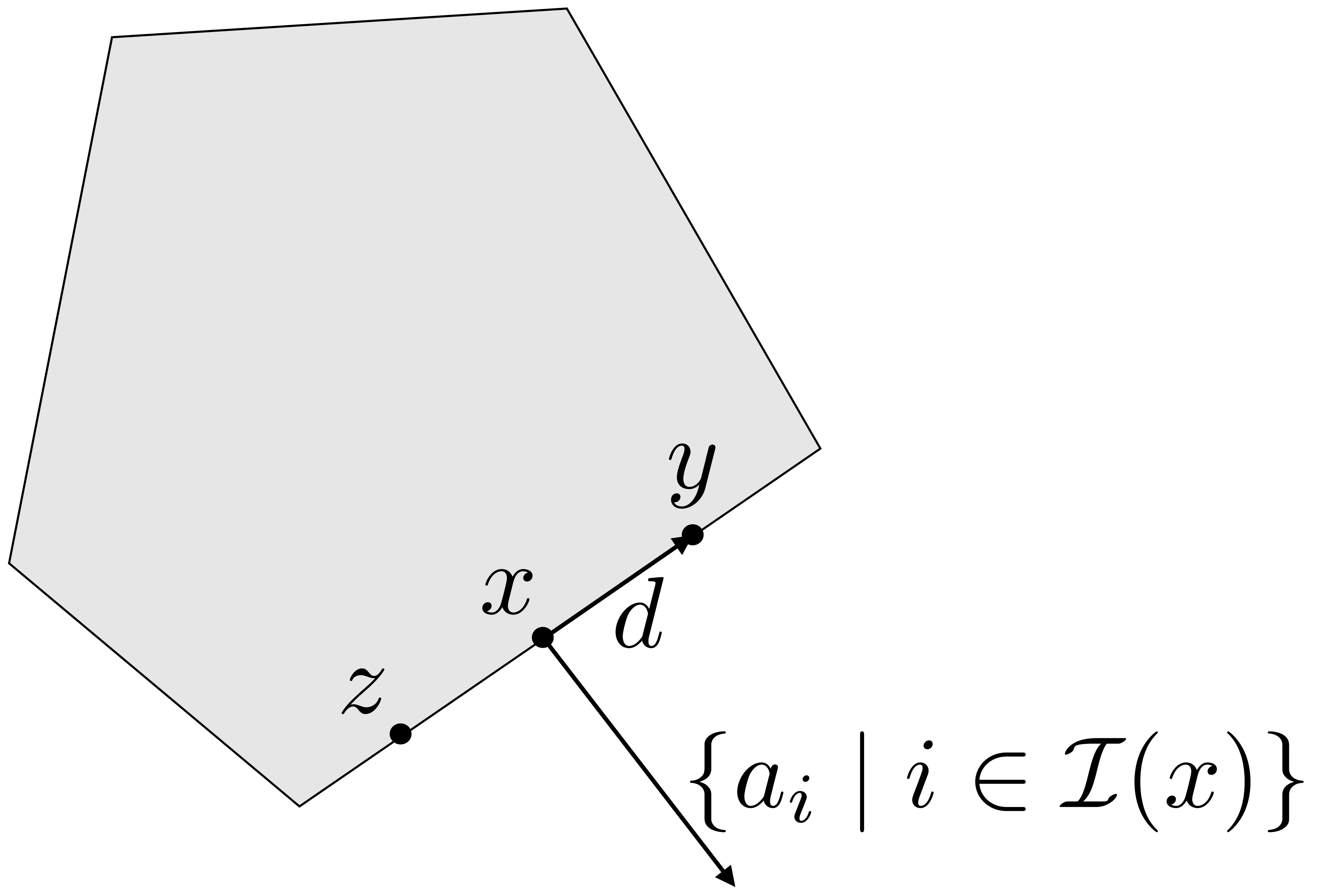
Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda)z$ with $\lambda = 0.5$.

$\implies x$ is **not an extreme point** ■

Equivalent theorem proof

Extreme point $\implies$ Basic feasible solution (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**



Hence, $y, z \in P$ and $x = \lambda y + (1-\lambda)z$ with $\lambda = 0.5$.

$\implies x$ is **not an extreme point** ∎

Equivalent theorem proof

Basic feasible solution $\implies$ Vertex

Left as exercise

Hint

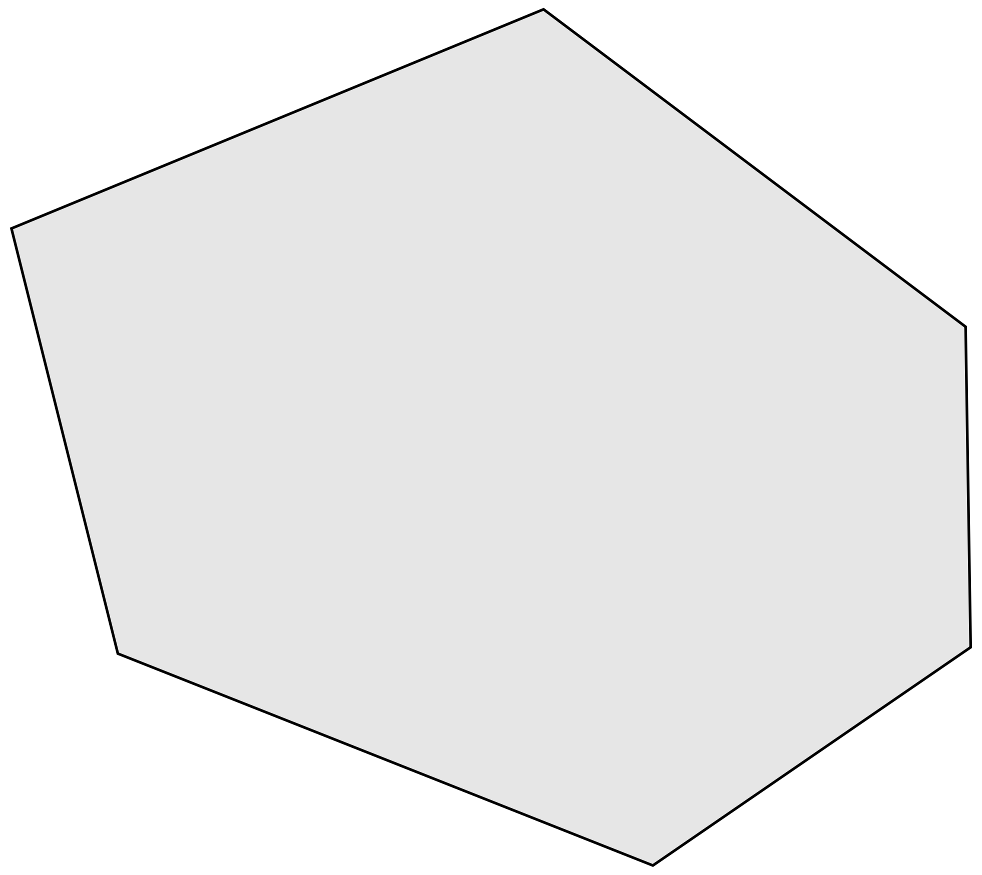
Define $c = -\sum_{i \in \mathcal{I}(x)} a_i$

How about nonlinear optimization?

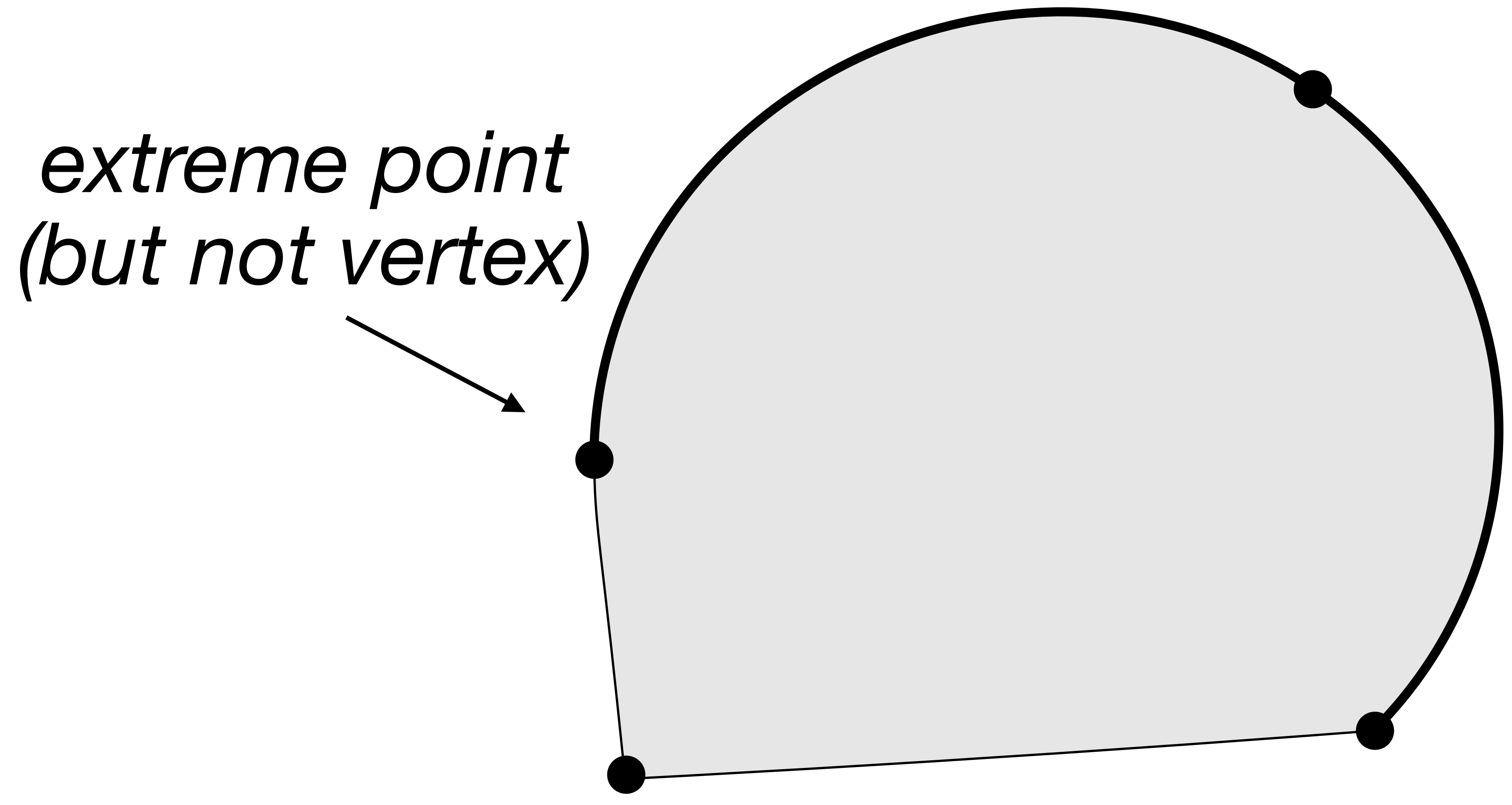
Polyhedral sets

- Extreme points
- Vertices
- Basic feasible solutions

→ *all equivalent*



Nonpolyhedral sets → *equivalence fails!*



Constructing basic solutions

Standard form polyhedra

Definition

Standard form LP

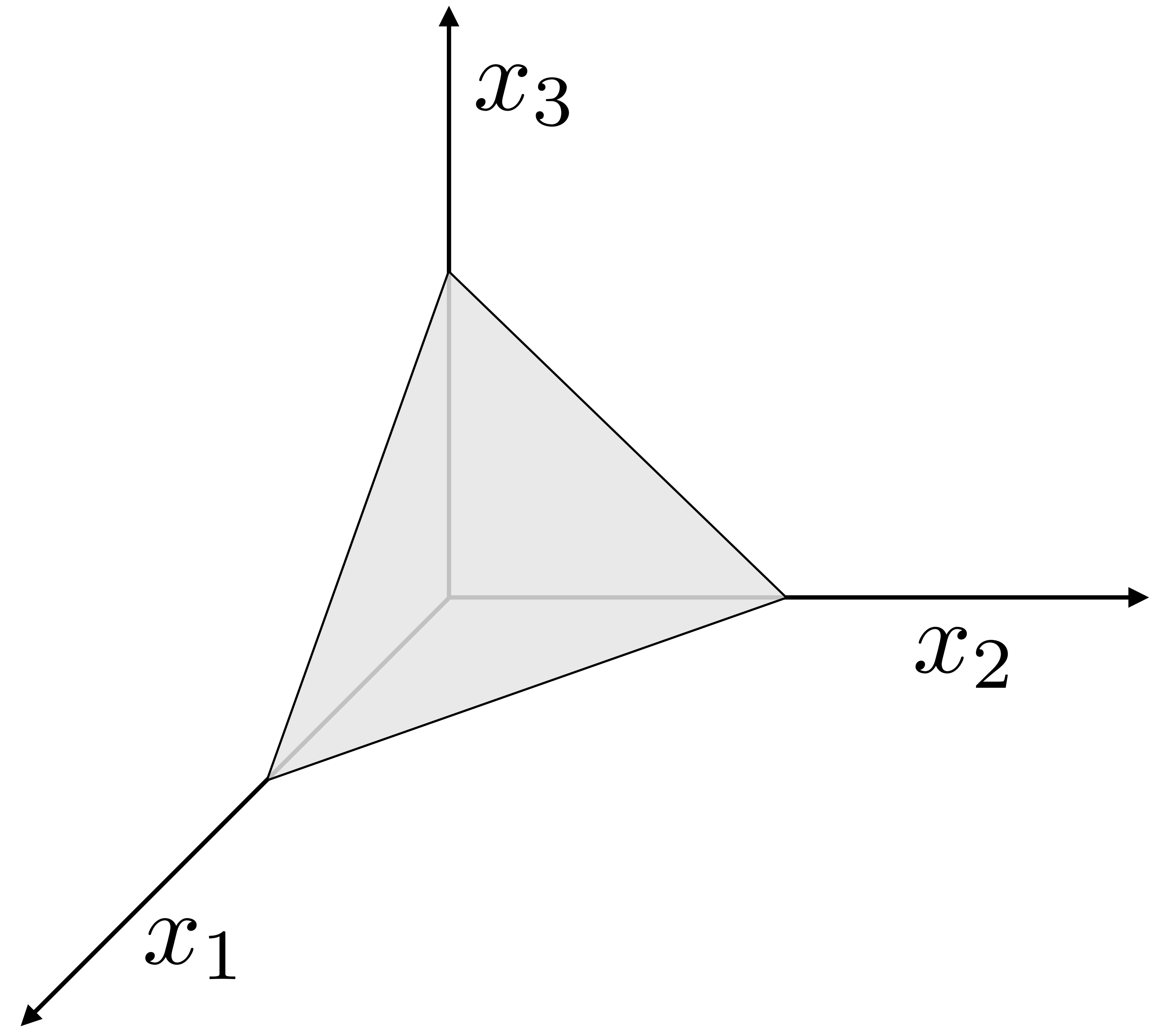
minimize $\quad c^T x$

subject to $\quad Ax = b$

$\qquad x \geq 0$

Standard form polyhedron

$$P = \{x \mid Ax = b,\ x \geq 0\}$$



Assumption

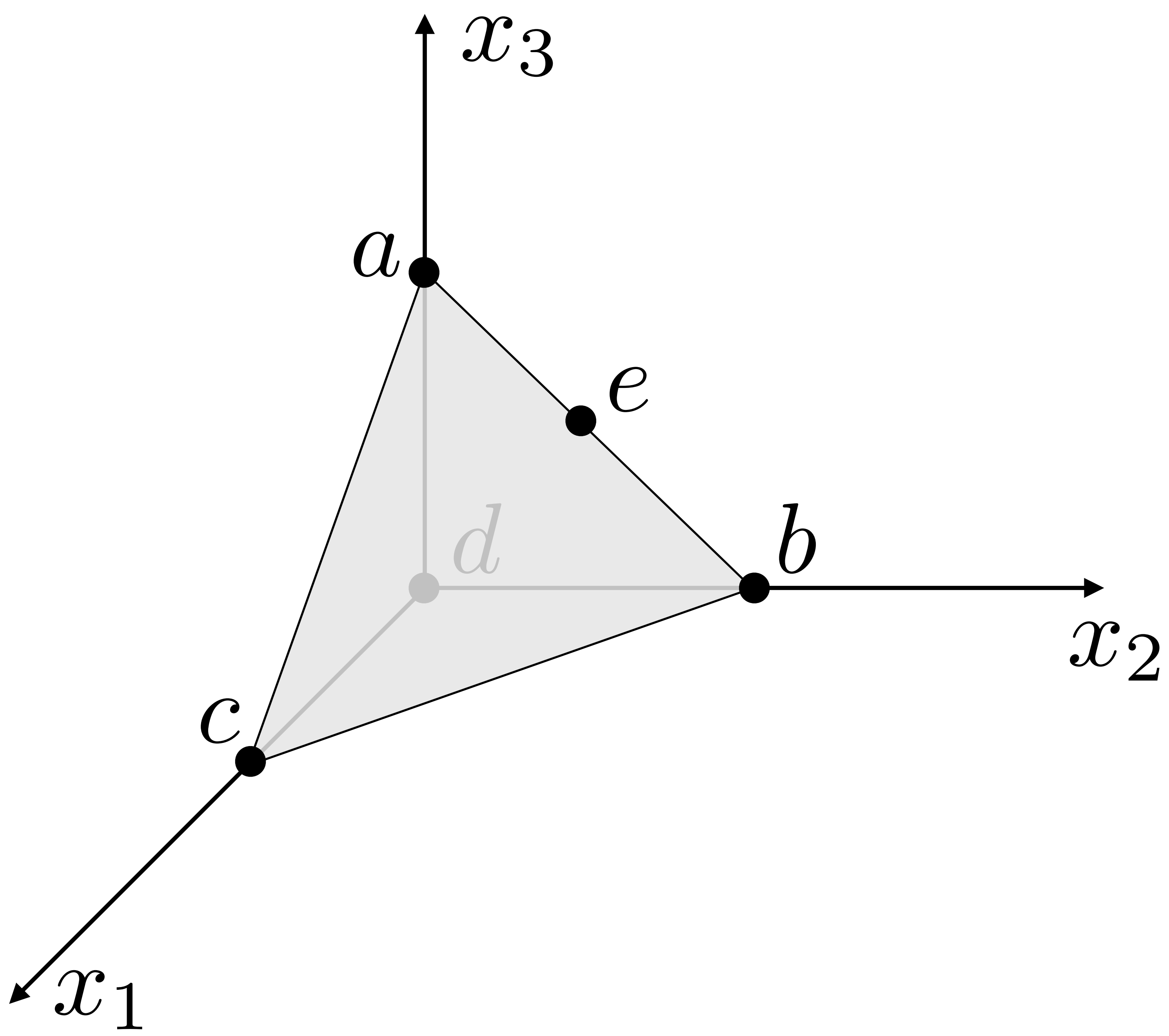
$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P lives in $(n - m)$-dimensional subspace

Example of basic feasible solutions on standard form polyhedra

$$P = \{x \mid x_1 + x_2 + x_3 = 1, \quad x \geq 0\}$$



- a, b, c: basic-feasible solutions
- d: equality constraint not active
- e: only 2 active constraints

Basic solutions

Standard form polyhedra

$P = \{x \mid Ax = b,\ x \geq 0\}$ with $A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Theorem

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
  - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
  - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

Intuition: from geometry to standard form

minimize $c^T x$
subject to $Ax \leq b$ ⟶

minimize $c^T(x^+ - x^-)$
subject to $\begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b$
$(x^+, x^-, s) \geq 0$ ⟶

minimize $\tilde{c}^T \tilde{x}$
subject to $\tilde{A}\tilde{x} = b$
$\tilde{x} \geq 0$

Variables: $\tilde{n} = 2n + m$
(Equality) constraints: $\tilde{m} = m \implies$ **active**

For a **basic solution** ⟶ We need $\tilde{n} - \tilde{m} = 2n$ active inequalities $\Rightarrow \tilde{x}_i = 0$ (non basic)

Which corresponds to m inequalities inactive $\Rightarrow \tilde{x}_i > 0$ (basic)

Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

Basis matrix

Basis columns

Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow$$

Solve $A_B x_B = b$

If $x_B \geq 0$, then x is a **basic feasible solution**

Finding a basic solution

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 2 & -1 & -3 & 0 & 0 \\ 0 & 2 & 8 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

$A_{B(1)}$ $A_{B(2)}$ $A_{B(3)}$

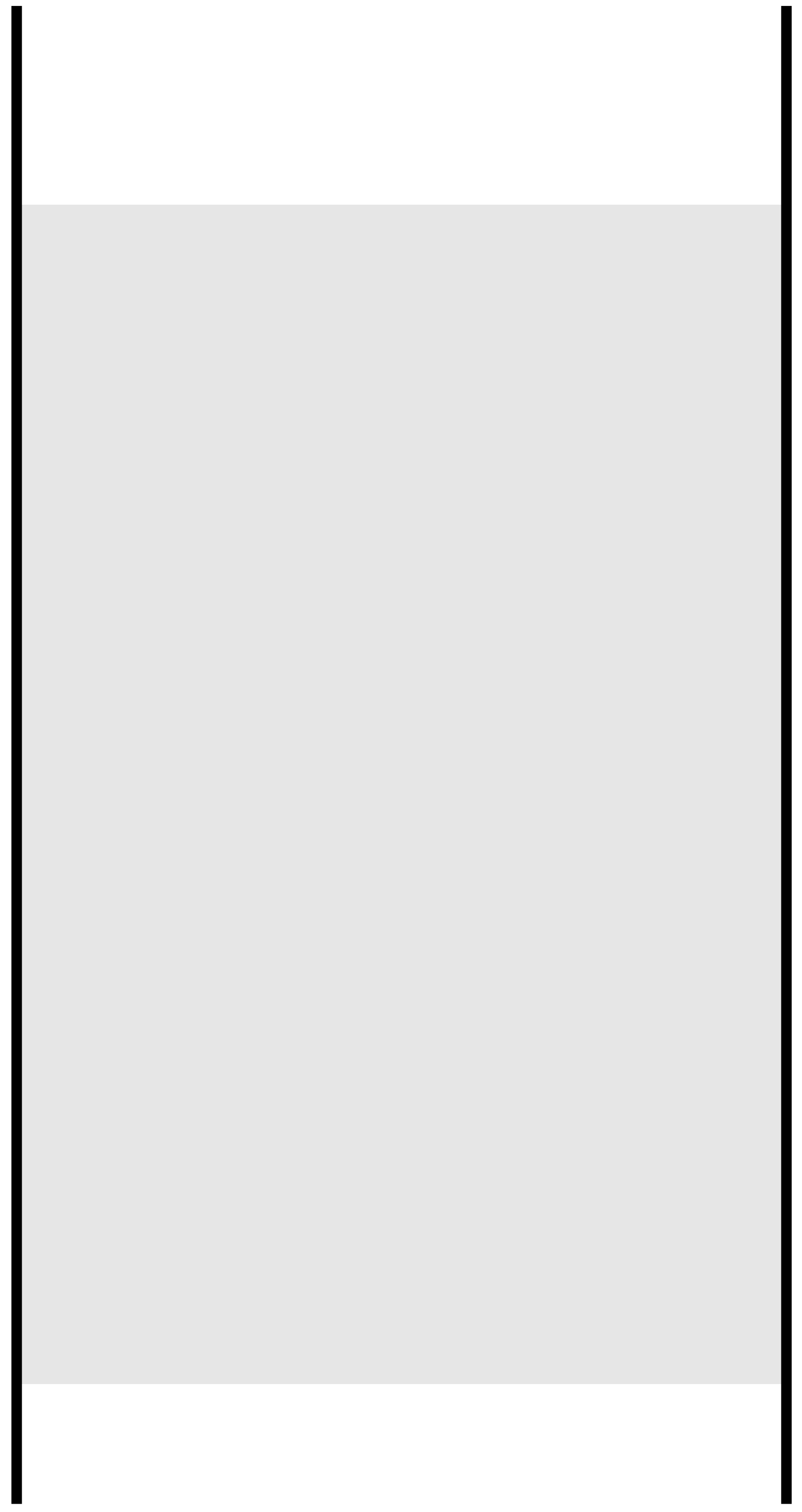
$$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \geq 0$$

Solve $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$

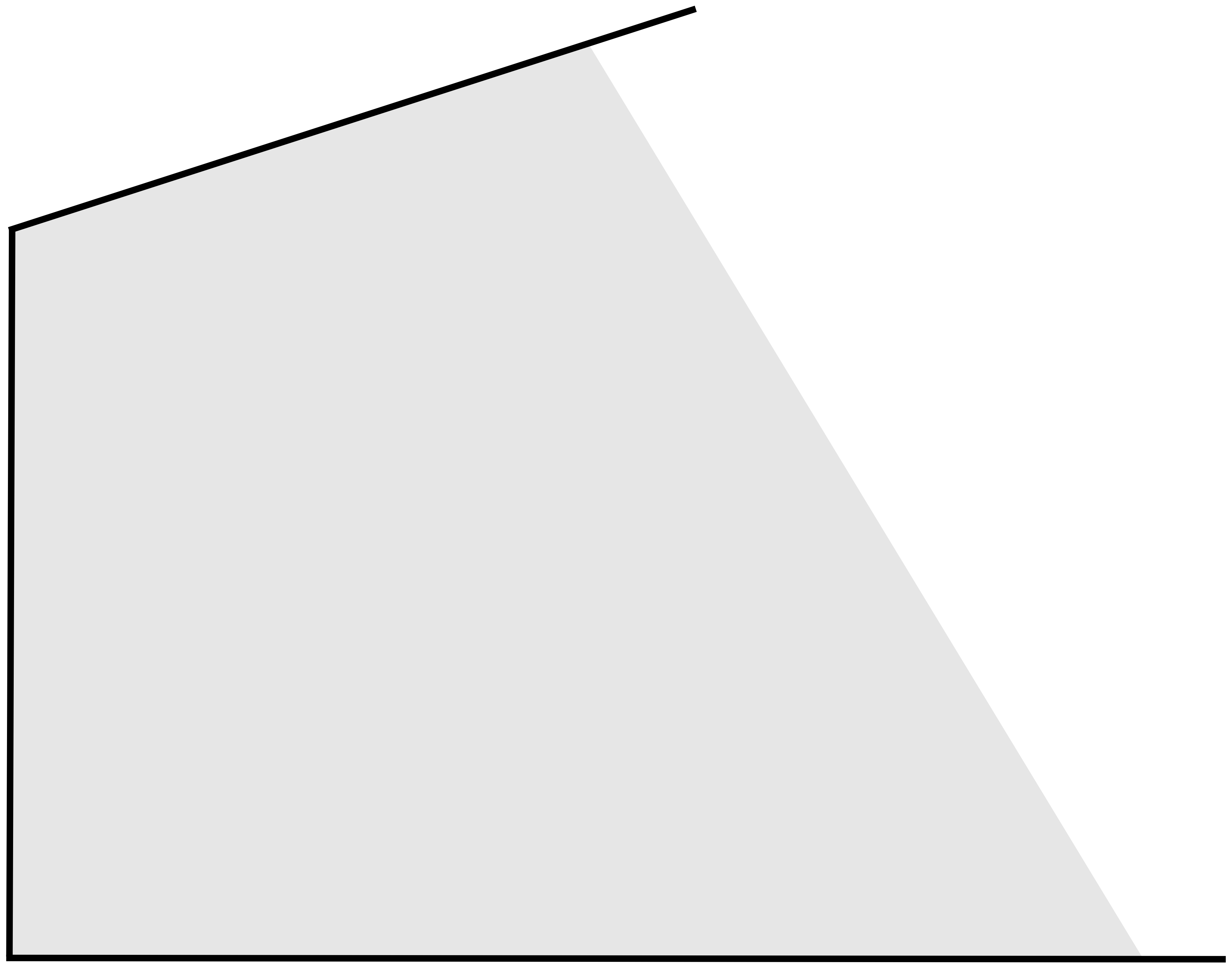
Existence and optimality of extreme points

Existence of extreme points

Example



No extreme points



Extreme points

Why?

Existence of extreme points

Characterization

A polyhedron P **contains a line** if

$\exists x \in P$ and a nonzero vector d such that $x + \lambda d \in P, \forall \lambda \in \mathbf{R}$.

Theorem

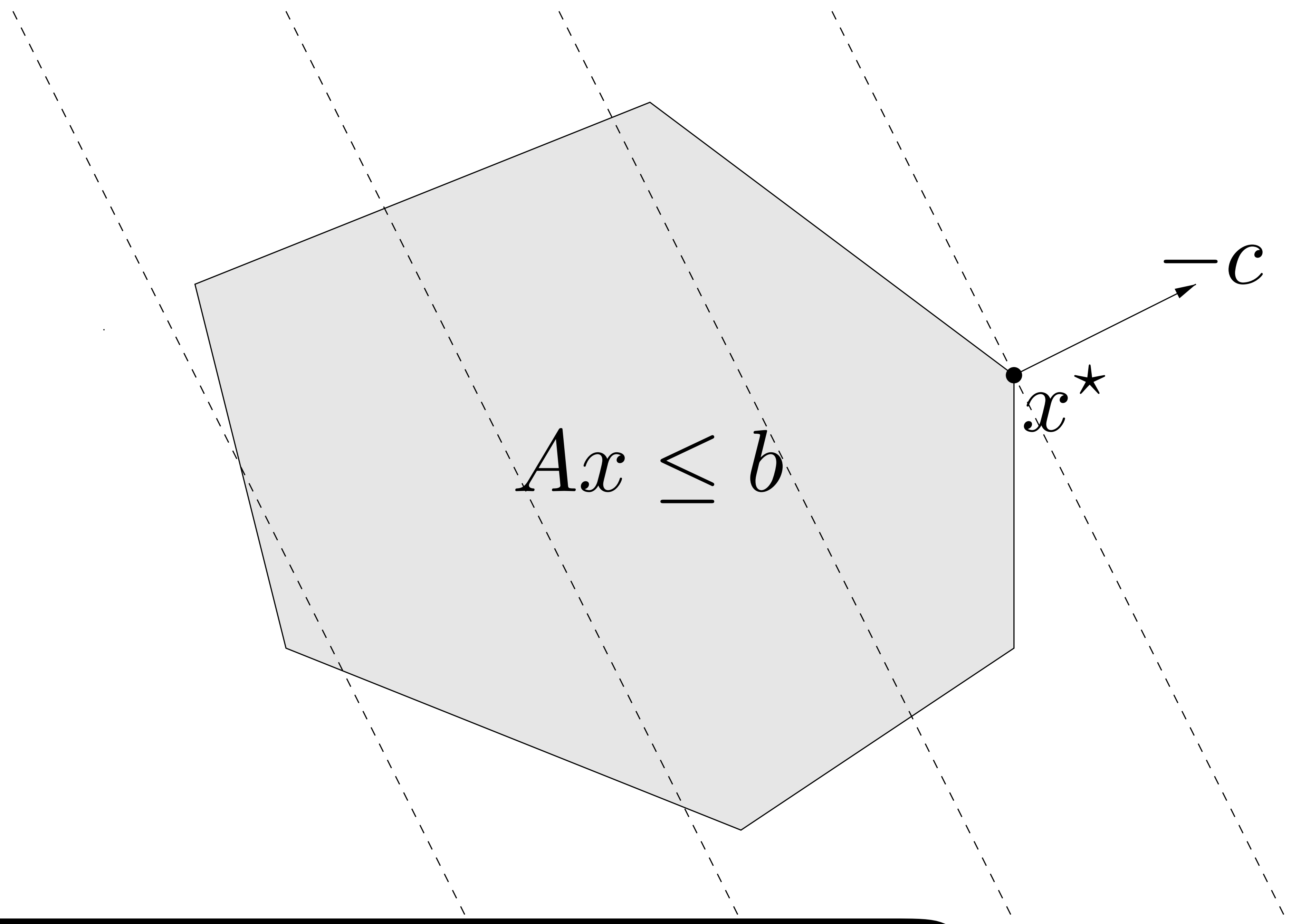
Given a polyhedron $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\}$, the following are **equivalent**

- P does not contain a line
- P has at least one extreme point
- n of the a_i vectors are linearly independent

Corollary

Every nonempty **bounded polyhedron** has **at least one basic feasible solution**

Optimality of extreme points



$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Theorem

If
- P has at least one extreme point
- There exists an optimal solution $x^\star$

Then, there exists an optimal solution which is an **extreme point** of P

We only need to search between **extreme points**

Proof of optimality of extreme points

Theorem

If
- P has at least one extreme point
- There exists an optimal solution $x^\star$

Then, there exists an optimal solution which is an **extreme point** of P

Let v be the optimal value of the problem

Let $X = \{x \mid v = c^T x,\ Ax \leq b\}$ be the set of optimal solutions

We have that $\emptyset \neq X \subseteq P \quad \Longrightarrow \quad X$ contains no line $\quad \Longrightarrow \quad X$ has an extreme point $x^\star$

Claim: $x^\star$ is an extreme point of P

Suppose not. Then $\exists y, w \in P$ with $y, w \neq x^\star$ such that $x^\star = \lambda y + (1-\lambda) w$ and $0 < \lambda < 1$.

Then, we can write the optimal value as $v = c^T x^\star = \lambda c^T y + (1-\lambda) c^T w$.

Because of optimality, we have that $c^T y \geq v$ and $c^T w \geq v$.

Then, the last equality is achieved when $c^T y = v$ and $c^T w = v$, and $y, w \in X$.

$\Longrightarrow \quad x^\star$ is not an extreme point of X (*contradiction*). ■

How to search among basic feasible solutions?

Idea

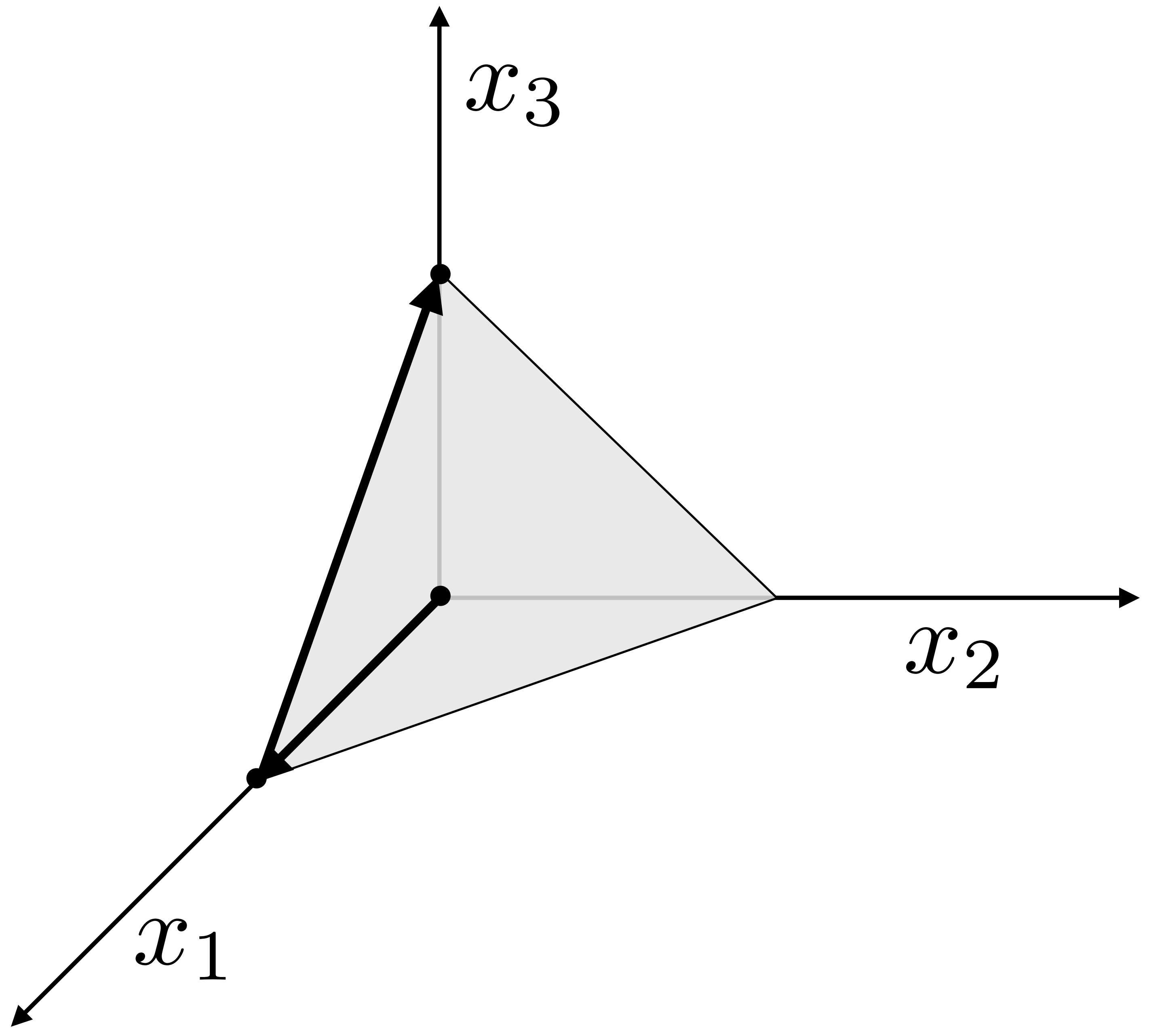
List all the basic feasible solutions, compare objective values and pick the best one.

Intractable!

If $n = 1000$ and $m = 100$, we have 10^{143} combinations!

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Geometry of linear optimization

Today, we learned to:

- **Apply geometric and algebraic properties** of polyhedra to characterize the "corners" of the feasible region.
- **Construct basic feasible solutions** by solving a linear system.
- **Recognize existence and optimality** of extreme points.

Next lecture

The simplex method

- Iterations
- Convergence
- Complexity