ORF307 – Optimization

16. Network optimization

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Recap

Primal and dual basic feasible solutions

Primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual problem

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Given a **basis** matrix A_B

Reduced costs

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

Dual feasible: $A^T y + c \geq 0.$ Set $y = -A_B^{-T} c_B$. Dual feasible if $\bar{c} = c + A^T y \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T} c_B = c_B x_B - c_B^T A_B^{-1} b = 0$

(by construction)

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Assumption: $p^\star(0)$ is finite

Properties

- $p^\star(u) > -\infty$ everywhere (from global lower bound)
- $p^\star(u)$ is piecewise-linear on its domain

Optimal value function is piecewise linear

Proof

Dual feasible set

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

$$D = \{y \mid A^T y + c \geq 0\}$$

Assumption: $p^\star(0)$ is finite

If $p^\star(u)$ finite

$$p^\star(u) = \max_{y \in D} -(b+u)^T y = \max_{k=1,\dots,r} -y_k^T u - b^T y_k$$

$y_1, \dots, y_r$ are the extreme points of D

Derivative of the optimal value function

Modified optimal solution

$$x^\star_B(u) = A_B^{-1}(b+u) = x^\star_B + A_B^{-1}u$$
$$y^\star(u) = y^\star$$

Optimal value function

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c_B^T A_B^{-1} u \\ &= p^\star(0) - y^{\star T} u \end{aligned}$$

(affine for small u)

Local derivative

$$\nabla p^\star(u) = -y^\star$$

($y^\star$ are the **shadow prices**)

Today's lecture

Network optimization

- Network flows
- Minimum cost network flow problem
- Network flow solutions
- Examples: maximum flow, shortest path, assignment

Network flows

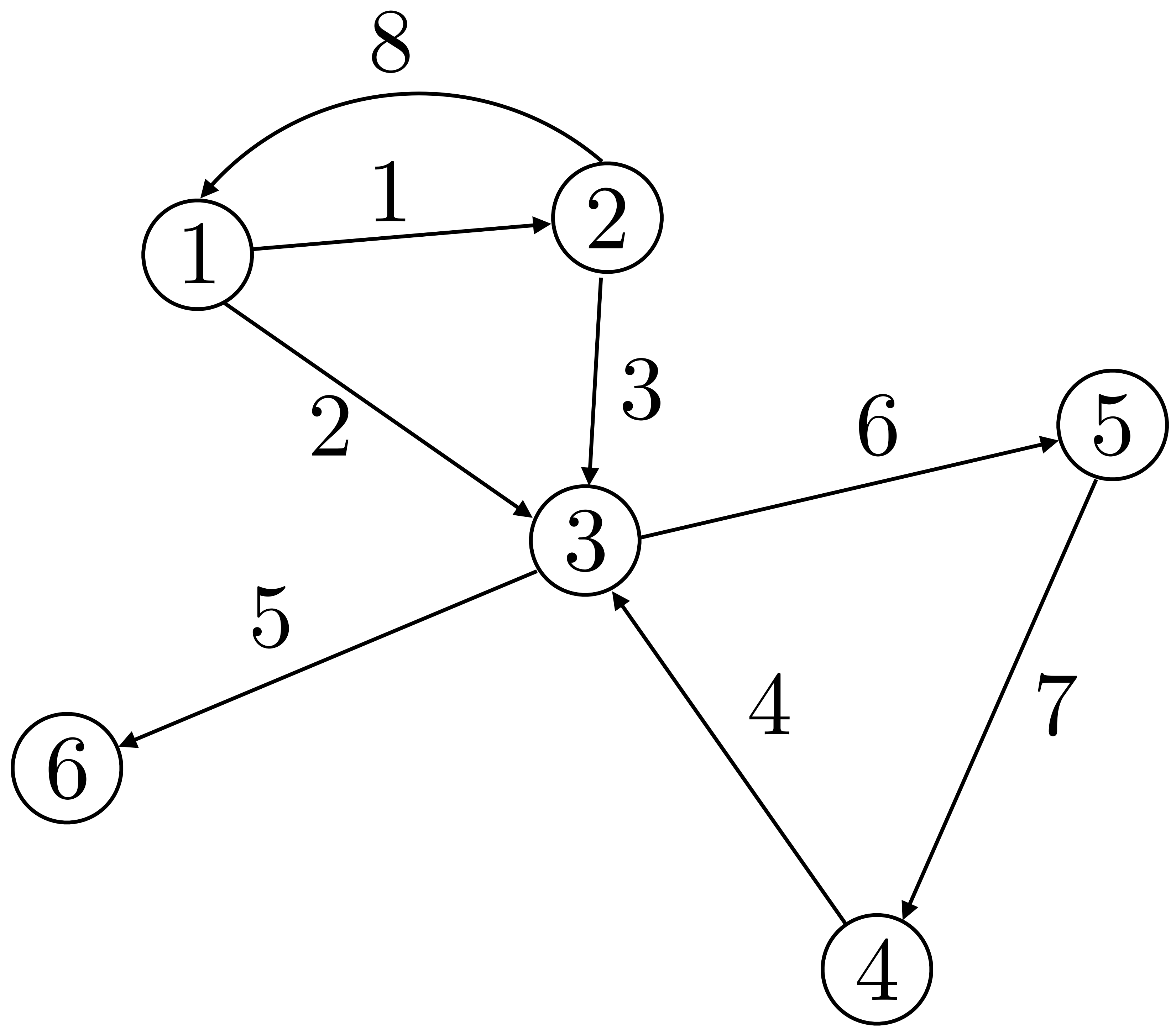
Networks

- Electrical and power networks
- Road networks
- Airline routes
- Printed circuit boards
- Social networks

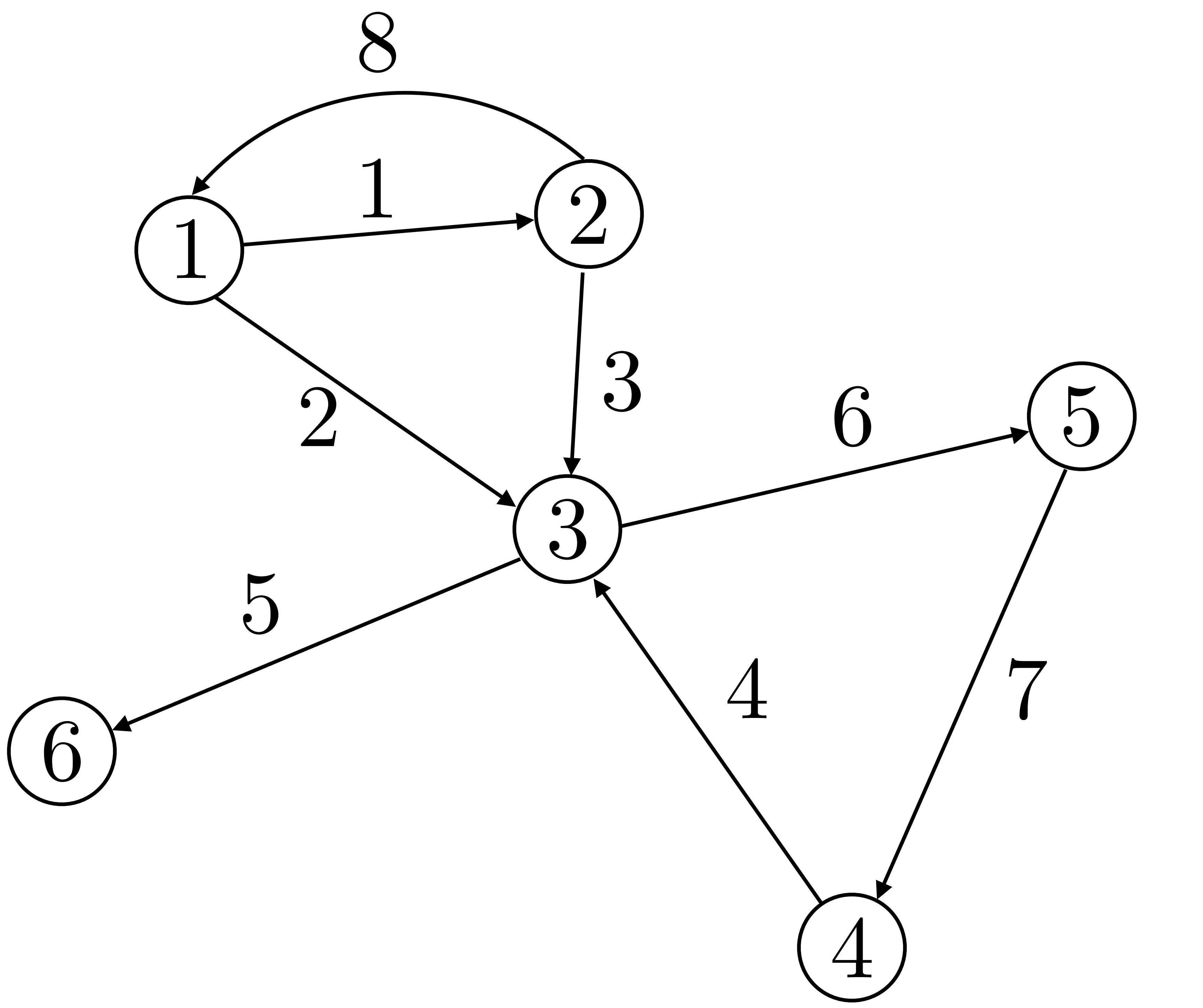
Network modelling

A **network** (or *directed graph*, or *digraph*)
is a set of m nodes and n directed arcs

- Arcs are ordered pairs of nodes (a, b) (leaves a, enters b)
- **Assumption** there is at most one arc from node a to node b
- There are no loops (arcs from a to a)



Arc-node incidence matrix



$m \times n$ matrix A with entries

$$A_{ij} = \begin{cases} 1 & \text{if arc } j \text{ starts at node } i \\ -1 & \text{if arc } j \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

Note Each column has one -1 and one 1

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

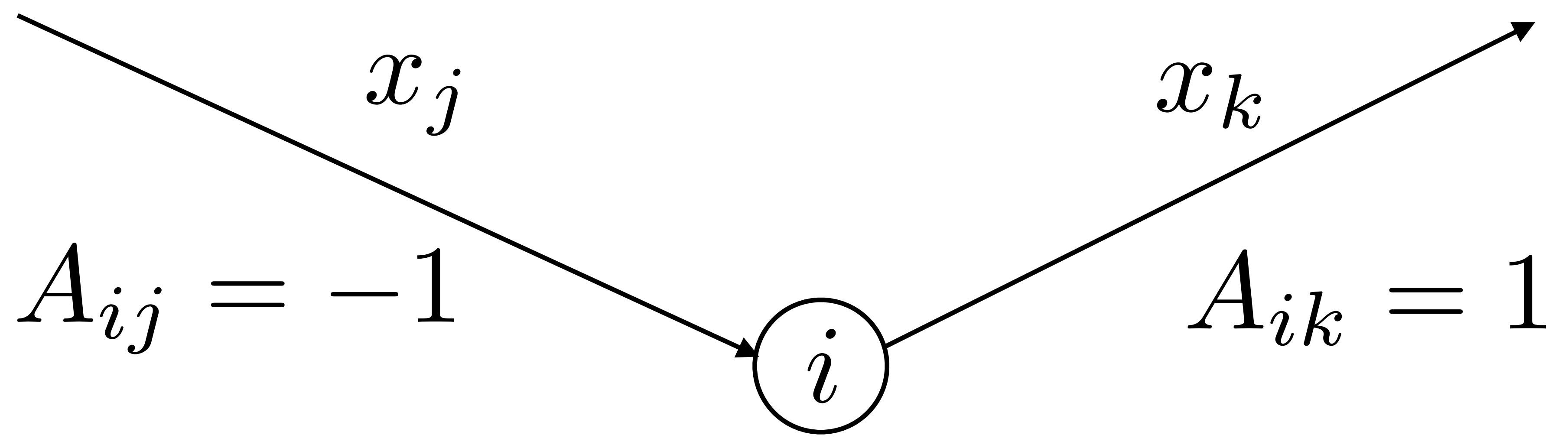
Network flow

flow vector $x \in \mathbf{R}^n$

x_j: flow (of material, traffic, information, electricity, etc) through arc j

total flow leaving node i

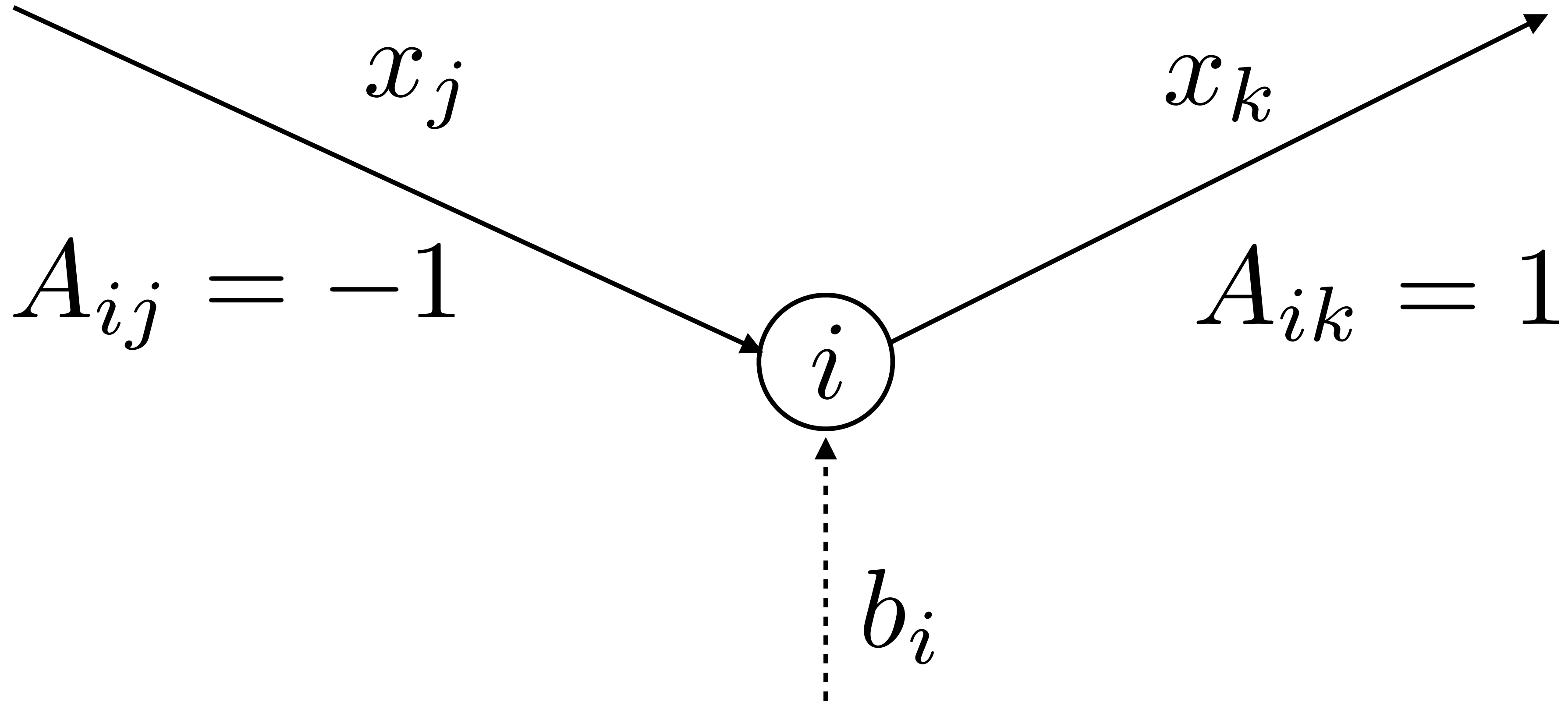
$$\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i$$



External supply

supply vector $b \in \mathbf{R}^m$

- b_i is the external supply at node i (if $b_i < 0$, it represents demand)
- We must have $\mathbf{1}^T b = 0$ (total supply = total demand)



Balance equations

$$\sum_{j=1}^{n} A_{ij} x_j = \underbrace{(Ax)_i}_{\text{Total leaving flow}} = \underbrace{b_i}_{\text{Supply}}, \quad \text{for all } i \quad \longrightarrow \quad Ax = b$$

Minimum cost network flow problem

Minimum cost network flow problem

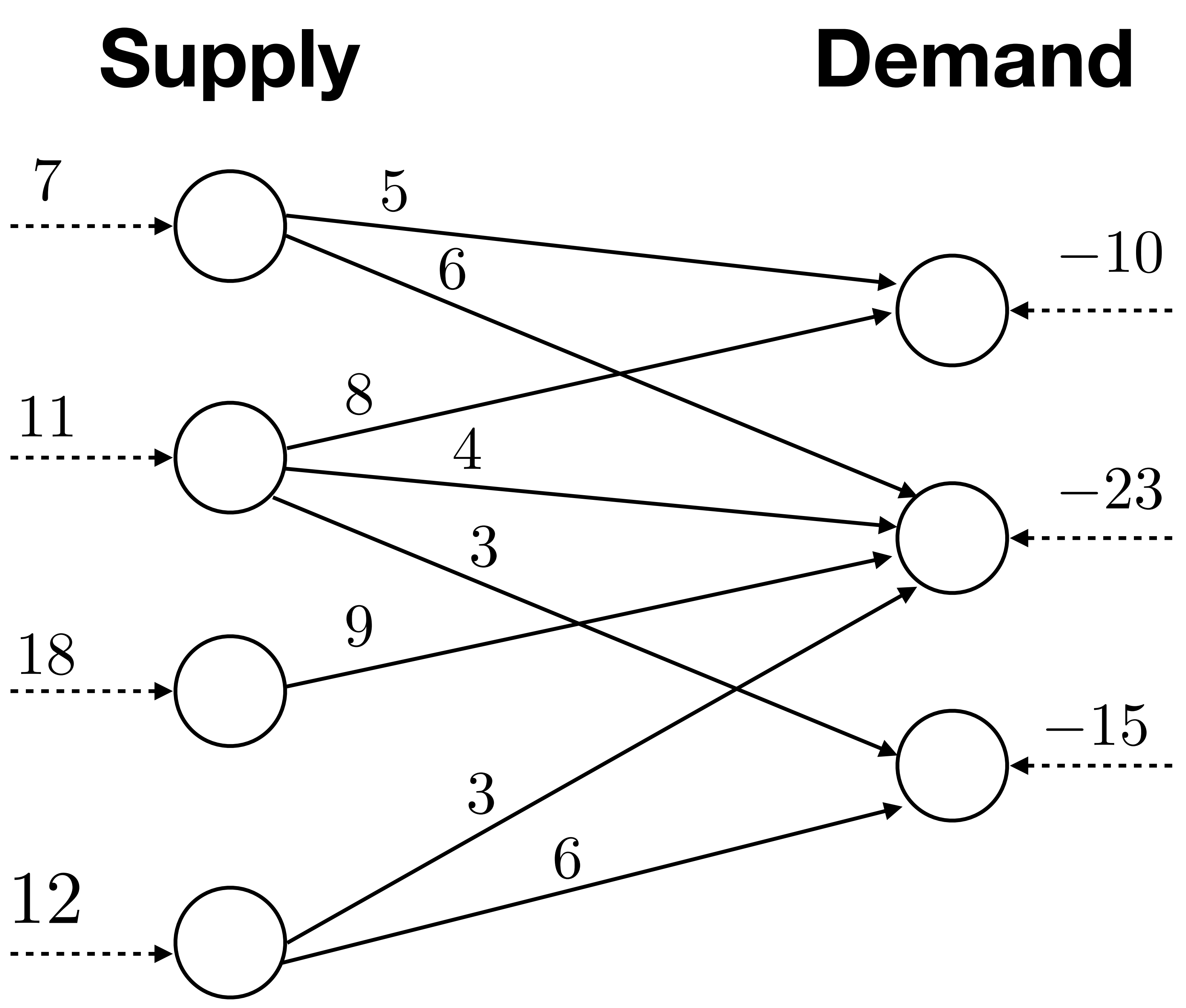
$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& 0 \leq x \leq u
\end{array}
$$

- c_i is unit cost of flow through arc i
- Flow x_i must be nonnegative
- u_i is the maximum flow capacity of arc i
- Many network optimization problems are just special cases

Example

Transportation

Goal ship $x \in \mathbf{R}^n$ to satisfy demand



(arc costs shown)
All capacities 20

$$c = (5, 6, 8, 4, 3, 9, 3, 6)$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$$b = (7, 11, 18, 12, -10, -23, -15)$$

$$u = 20\,\mathbf{1}$$

Minimum cost network flow

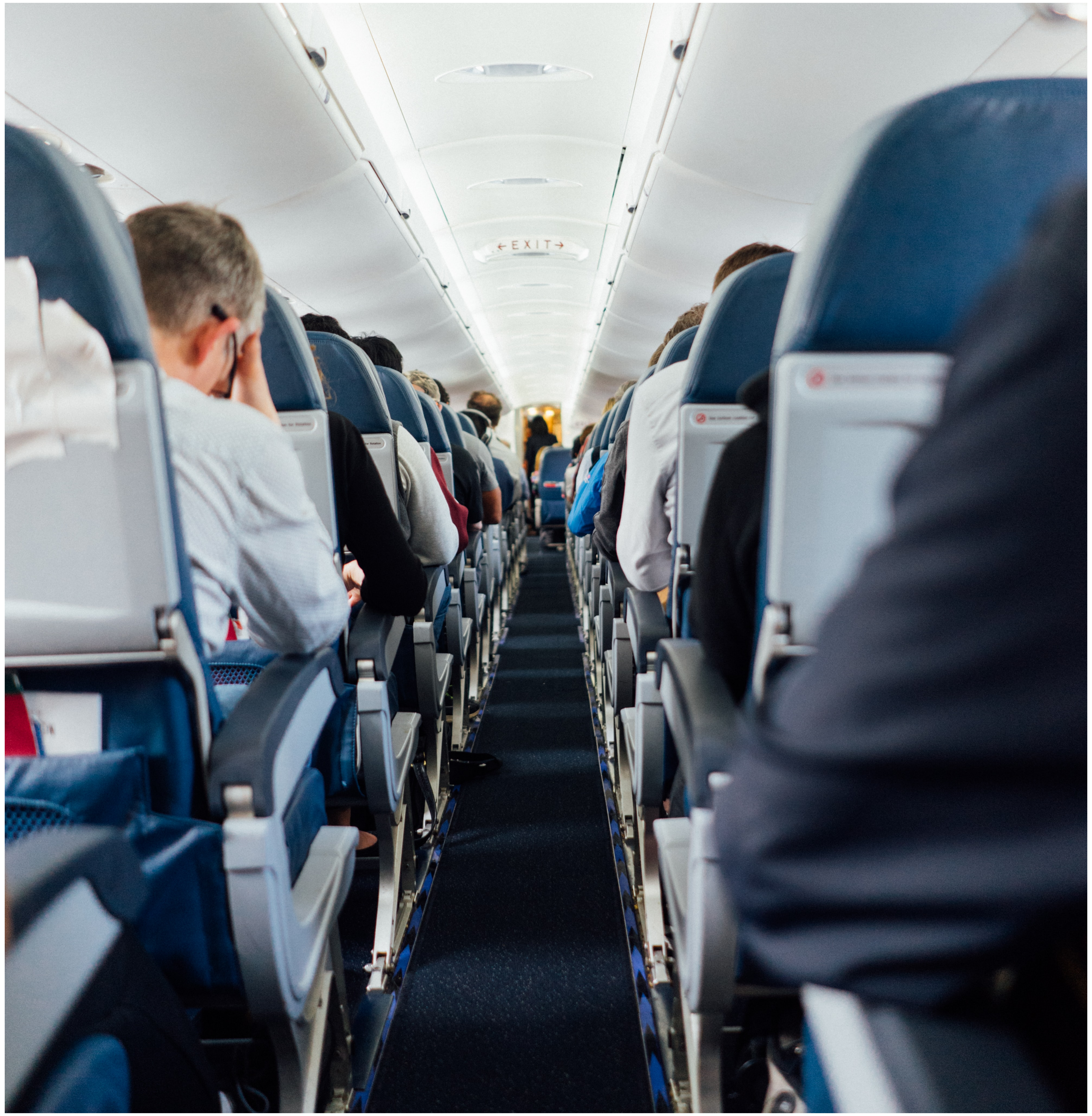
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$

$$x^\star = (7, 0, 3, 0, 8, 18, 5, 7)$$

Example

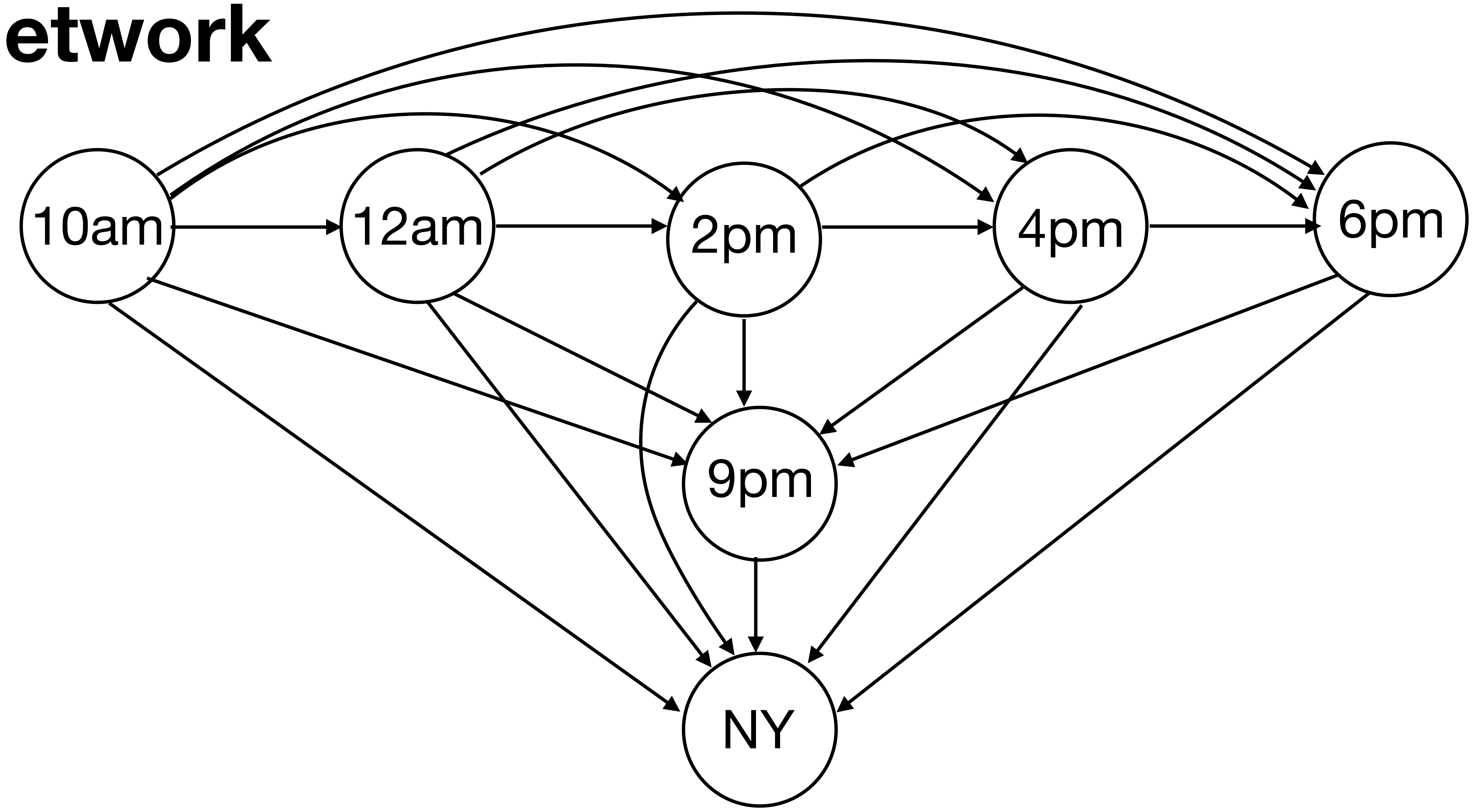
Airline passenger routing

- United Airlines has 5 flights per day from BOS to NY (10am, 12pm, 2pm, 4pm, 6pm)
- Flight capacities (100, 100, 100, 150, 150)
- Costs: $50/hour of delay
- Last option: 9pm flight with other company (additional cost $75)
- Today’s reservations (110, 118, 103, 161, 140)



Airline passenger routing

Network



Network flow formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \le x \le u \end{array}$$

Decisions

x_j: passengers flowing on arc j

Costs

c_j: cost of moving passenger on arc j

- Between flights: $\$50$/hour
- To 9pm flight: $\$75$ additional
- To NY: $\$0$ (as scheduled)

Supplies

b_i reserved passengers for flight i

- 9pm flight: $b_i = 0$
- NY supply: - total reserved passeng.

Capacities

u_j maximum passengers over arc j

- Between flights: $u_j = \infty$
- To NY: $u_i =$ flight capacity

Network flow solutions

Remove arc capacities

Goal: create equivalent network without arc capacities

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \le x \le u \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} = \tilde{b} \\ & \tilde{x} \ge 0 \end{array}$$

Standard form LP with arc-node incidence matrix

Remove arc capacities

Idea: slack variables

$$x_j \le u_j \quad \Rightarrow \quad x_j + s_j = u_j,\ s_j \ge 0$$

Network structure lost
no longer one -1
and one 1 per column

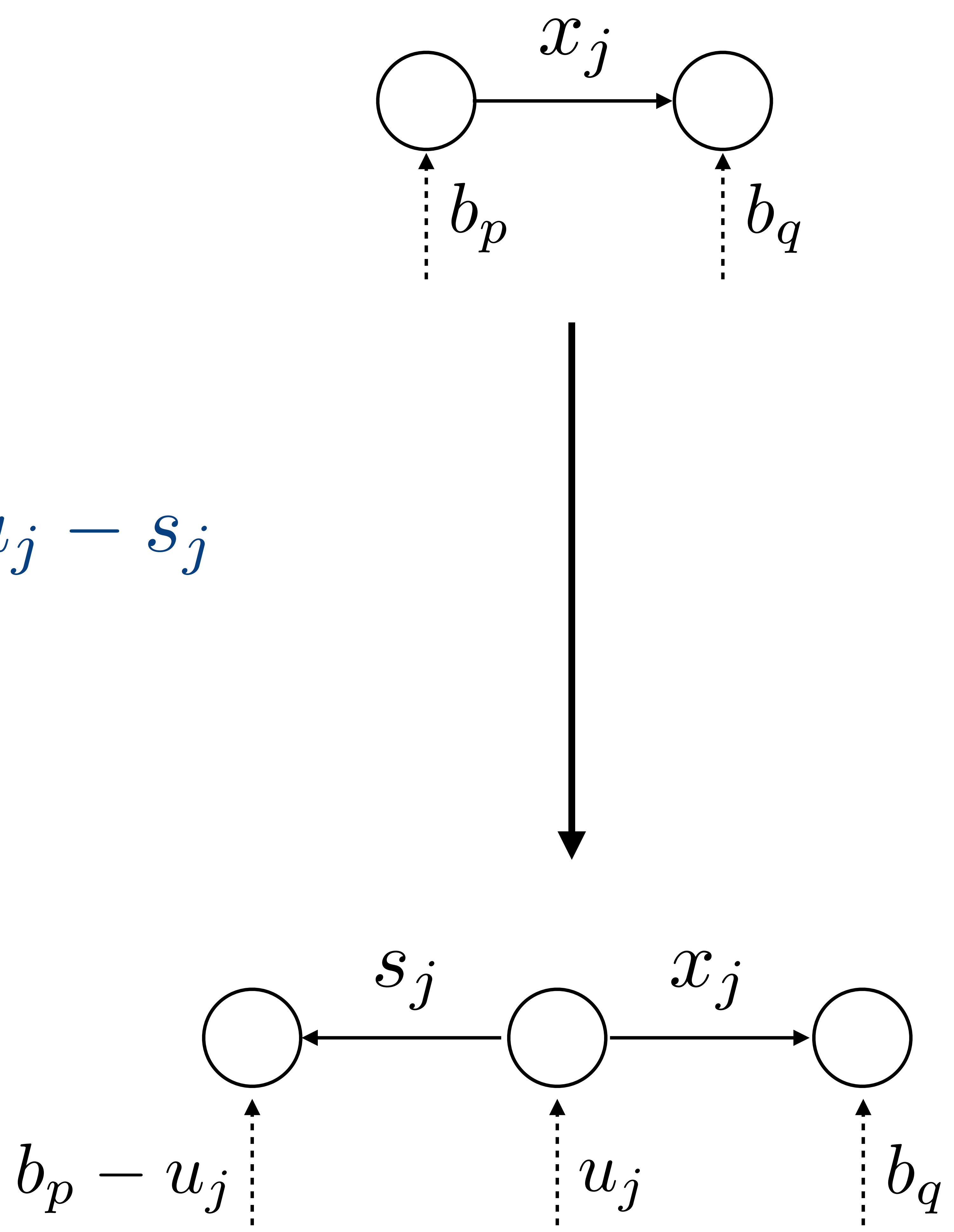
$$\cdots + x_j \ldots = b_p$$
$$\cdots - x_j \ldots = b_q$$
$$x_j + s_j = u_j$$

$$x_j = u_j - s_j$$

Network structure recovered
(new node and new arc)

$$\cdots - s_j = b_p - u_j$$
$$\cdots - x_j \ldots = b_q$$
$$x_j + s_j = u_j$$

Nodes/arcs interpretation



Equivalent uncapacitated network flow

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- A still an arc-node incidence matrix
- Can we say something about the extreme points?

Total unimodularity

A matrix is **totally unimodular** if all its minors are $-1, 0$ or 1
(minor is the determinant of a square submatrix of A)

example: a node-arc incidence matrix of a directed graph

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

properties

- the entries of A_{ij} (i.e., its minors of order 1) are $-1, 0$, or 1
- The inverse of any nonsingular square submatrix of A has entries $+1, -1$, or 0

Integrality theorem

Given a polyhedron $P = \{x \in \mathbf{R}^n \mid Ax = b, \quad x \geq 0\}$

where

- A is totally unimodular
- b is an integer vector

$\longrightarrow$ all the extreme points of P are integer vectors.

Proof

- All extreme points are basic feasible solutions with $x_B = A_B^{-1}b$ and $x_i = 0, \ i \neq B$
- A_B^{-1} has integer components because of total unimodularity of A
- b has also integer components
- Therefore, also x is integral

$\blacksquare$

Implications for network and combinatorial optimization

Minimum cost network flow

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& 0 \le x \le u
\end{array}
$$

If b and u are integral
solutions $x^\star$ are integral

Integer linear programs

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& 0 \le x \le u \\
& x \in \mathbf{Z}^n
\end{array}
$$

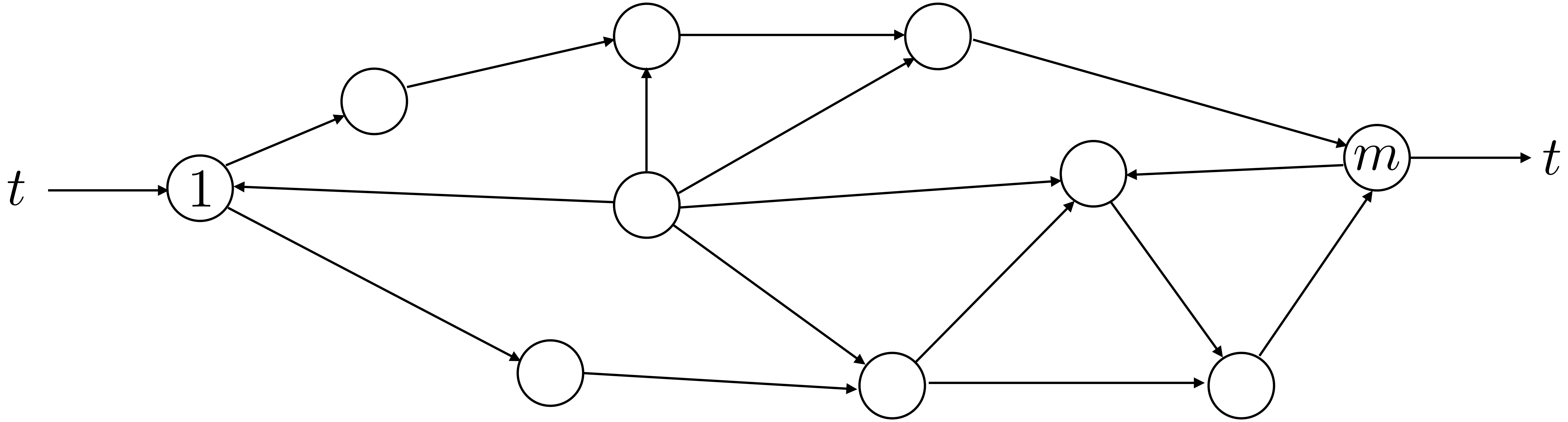
Very difficult in general
(more on this in a few weeks)

If A totally unimodular
and b, u integral, we can
relax integrality and solve
a fast LP instead

Examples

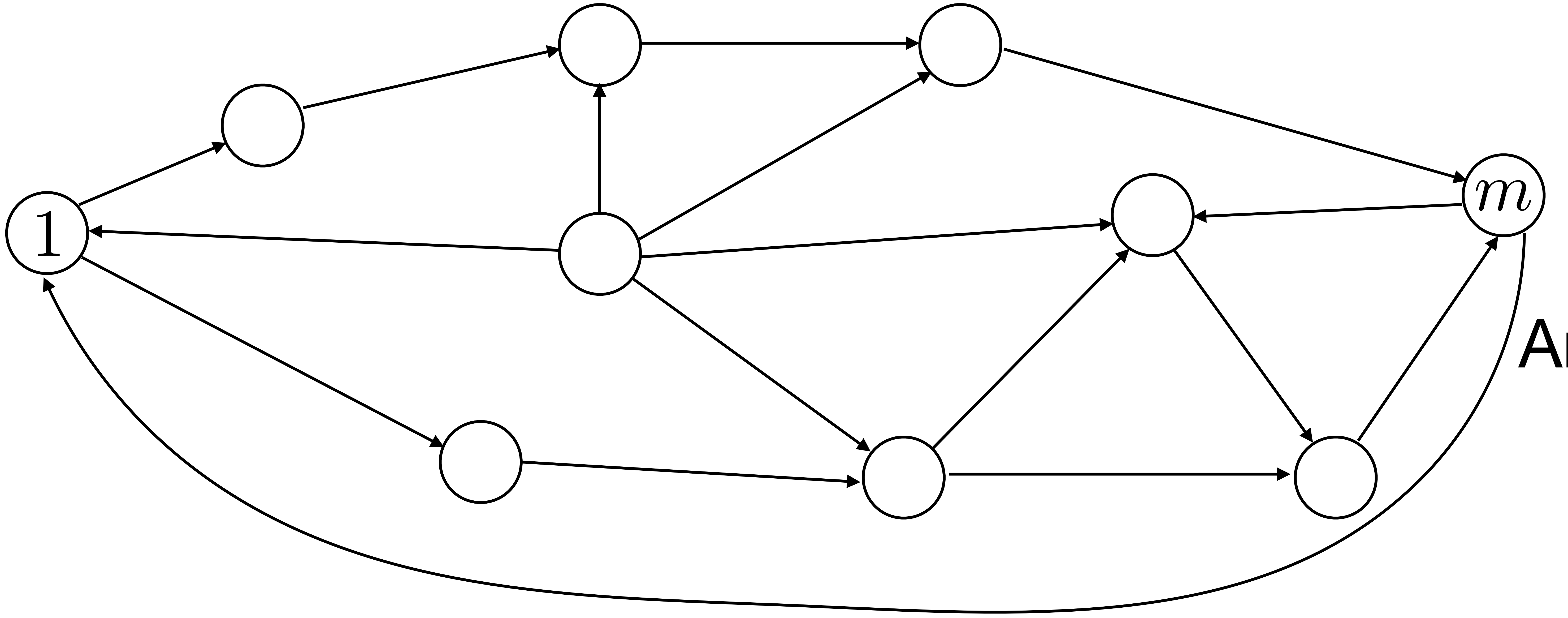
Maximum flow problem

Goal maximize flow from node 1 (source)
to node m (sink) through the network



$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & Ax = te \\ & 0 \leq x \leq u \end{array} \qquad e = (1, 0, \ldots, 0, -1)$$

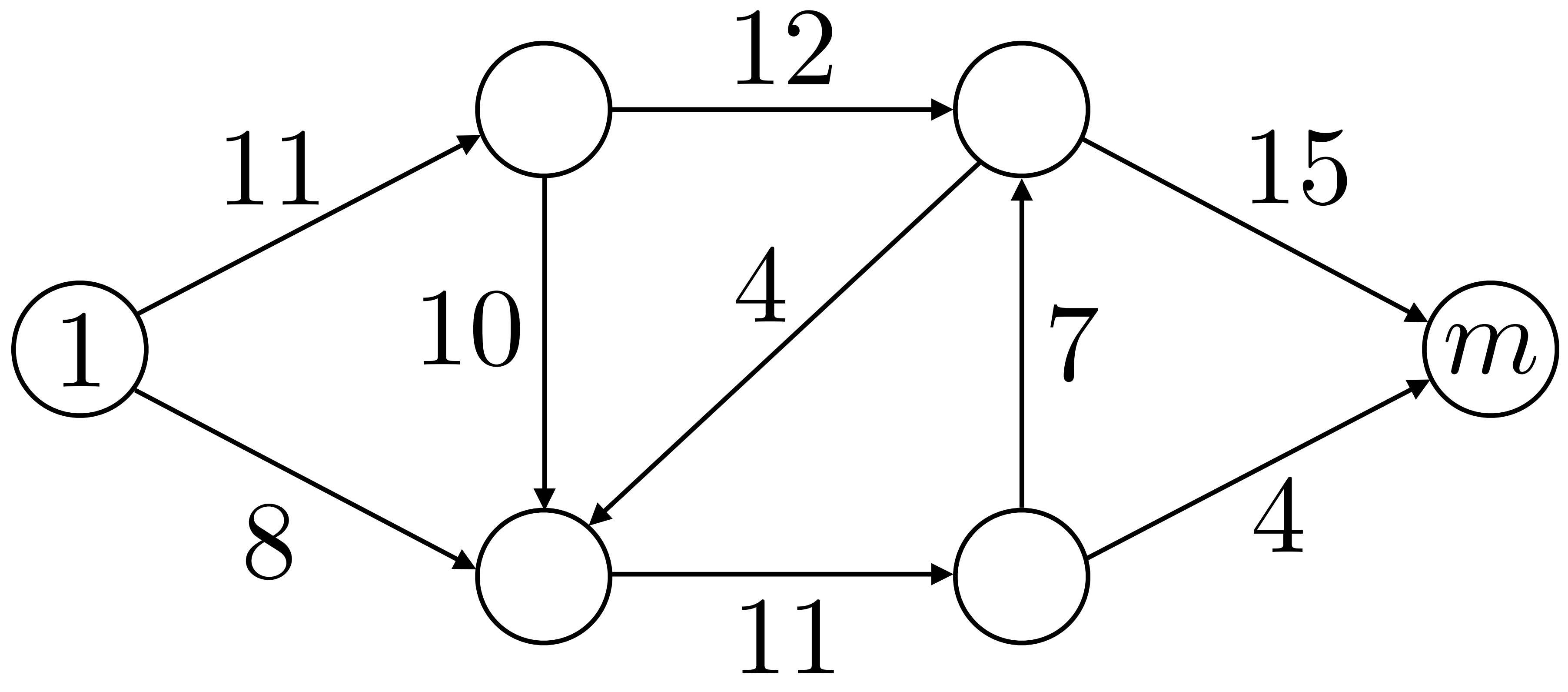
Maximum flow as minimum cost flow



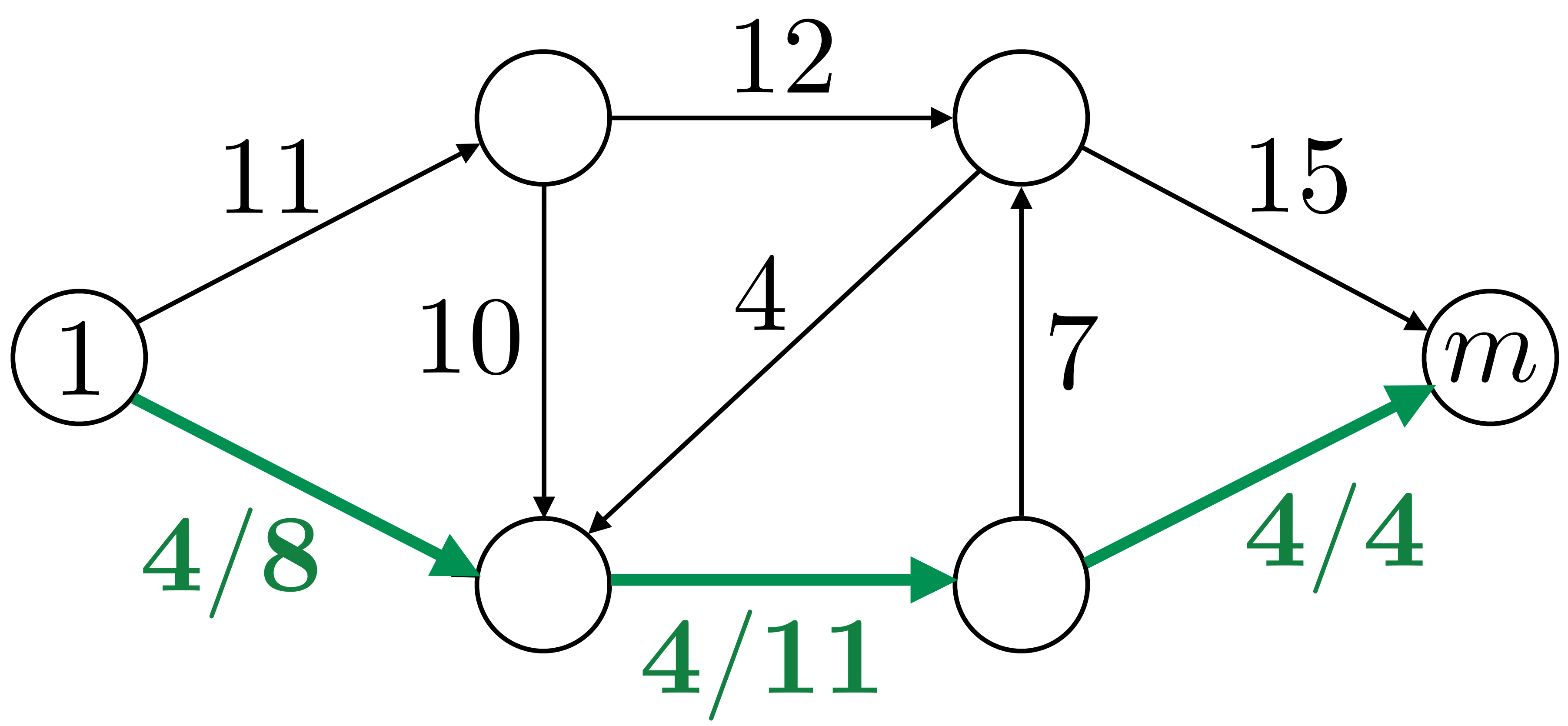
$$
\begin{aligned}
\text{minimize} \quad & -t \\
\text{subject to} \quad & \begin{bmatrix} A & -e \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 0 \\
& 0 \le \begin{bmatrix} x \\ t \end{bmatrix} \le \begin{bmatrix} u \\ \infty \end{bmatrix}
\end{aligned}
$$

Maximum flow example

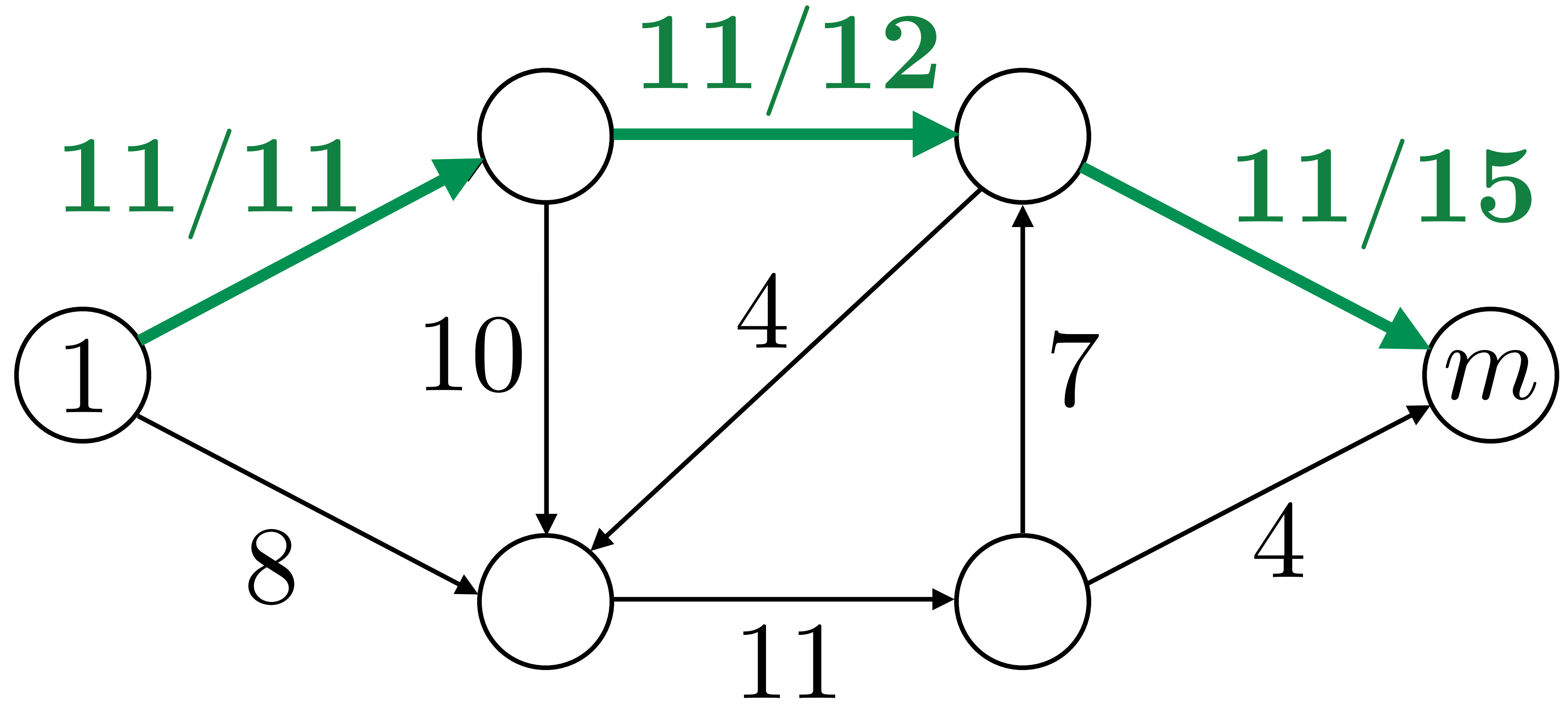
(arc capacities shown)



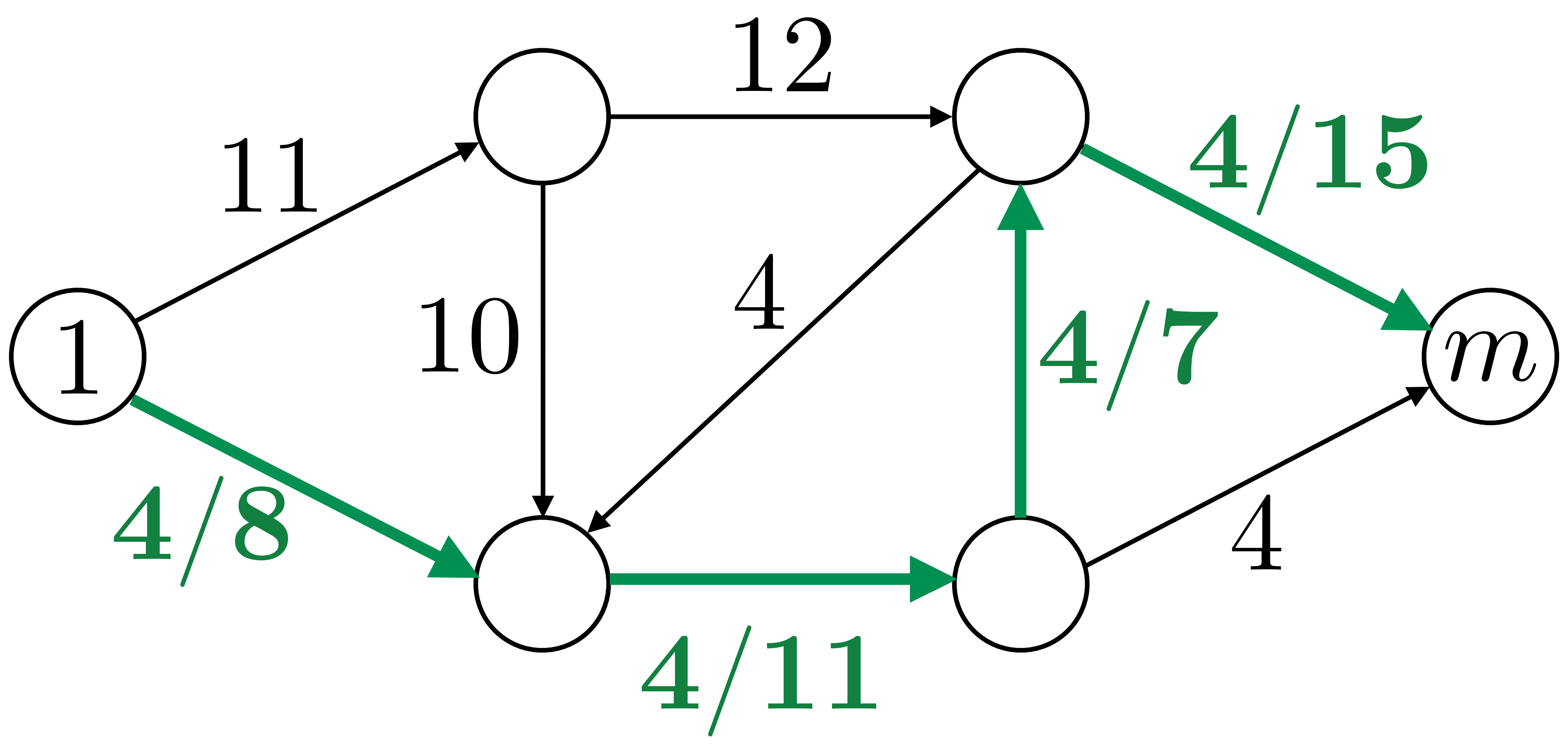
First flow



Second flow



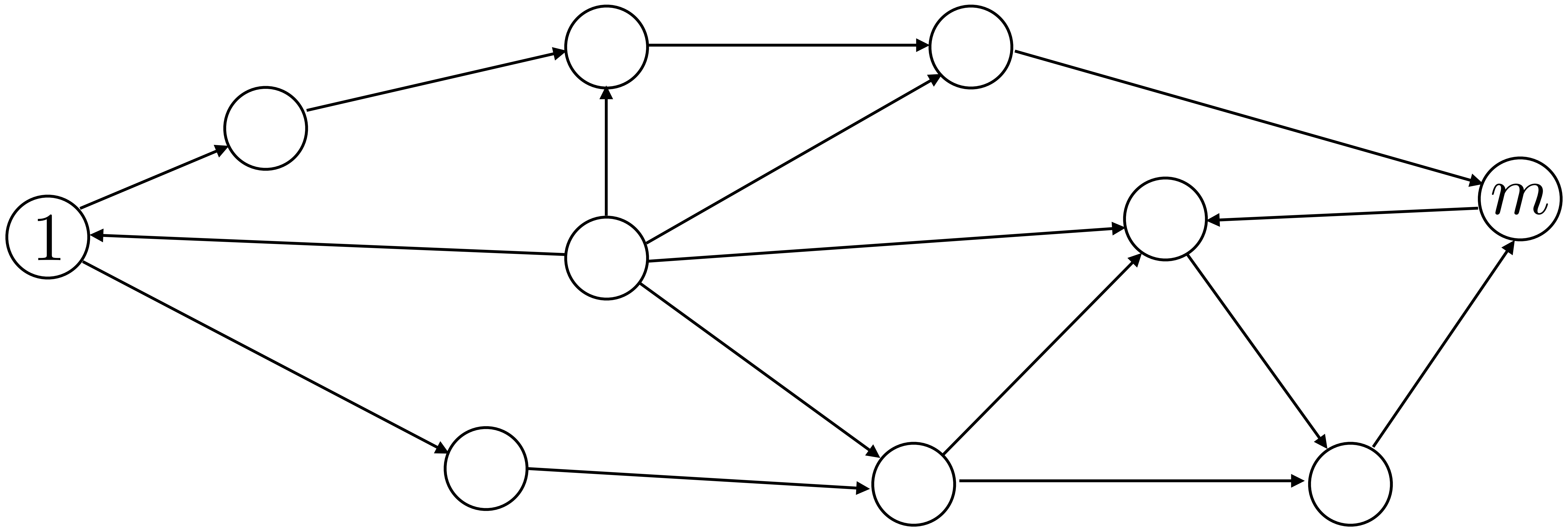
Third flow



Total flow: 19

Shortest path problem

Goal Find the shortest path between nodes 1 and m



paths can be represented as vectors $x \in \{0,1\}^n$

Formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = e \\ & x \in \{0,1\}^n \end{array}$$

- c_j is the “length” of arc j
- $e = (1, 0, \ldots, 0, -1)$
- Variables are binary (include or not arc in path)

Shortest path as minimum cost flow

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = e \\ & x \in \{0,1\}^n \end{array}$$

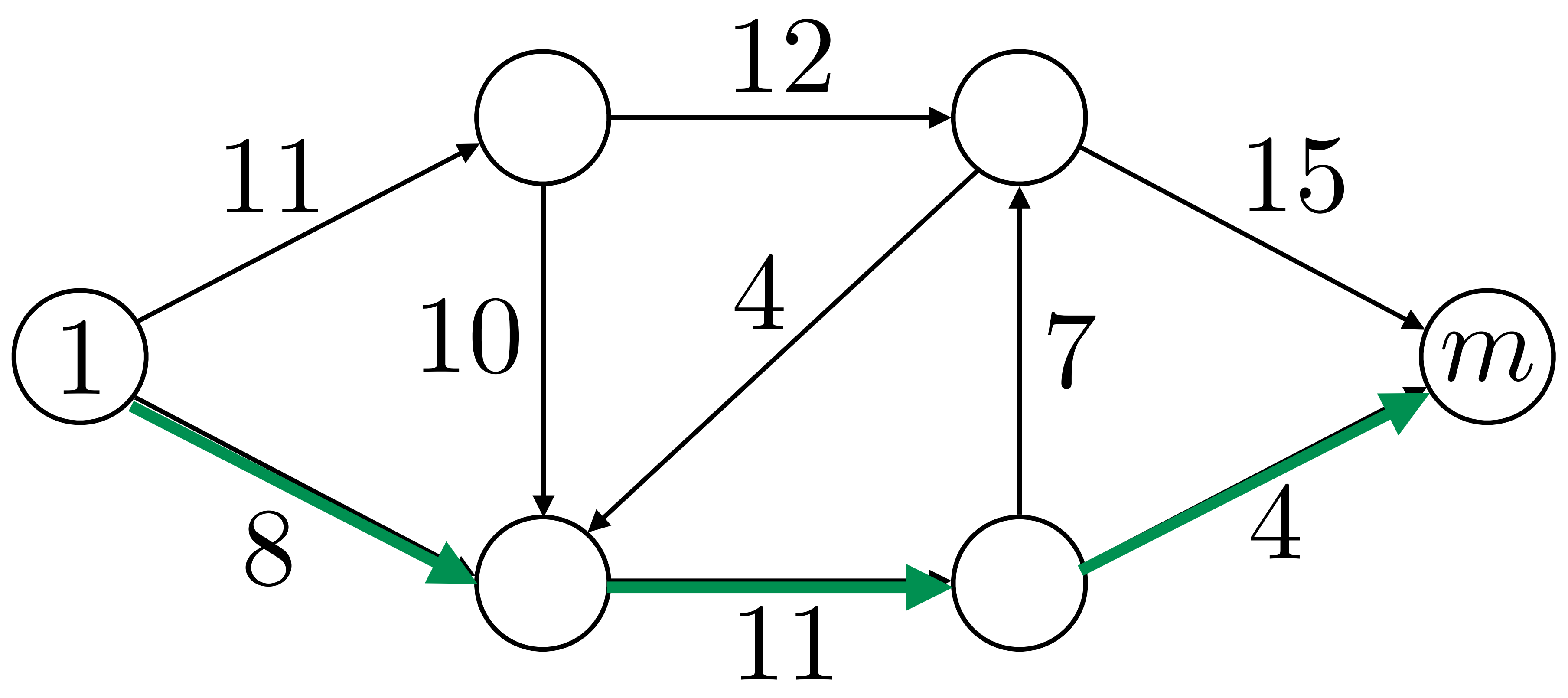
$\longrightarrow$

Relaxation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = e \\ & 0 \leq x \leq \mathbf{1} \end{array}$$

Extreme points
satisfy $x_i \in \{0, 1\}$

Example (arc costs shown)



$$c = (11, 8, 10, 12, 4, 11, 7, 15, 4)$$

$$x^\star = (0, 1, 0, 0, 0, 1, 0, 0, 1)$$

$$c^T x^\star = 24$$

Assignment problem

Goal match N persons to N tasks

- Each person assigned to one task, each task to one person
- C_{ij} Cost of matching person i to task j

LP formulation

minimize $$\sum_{i,j=1}^{N} C_{ij} X_{ij}$$

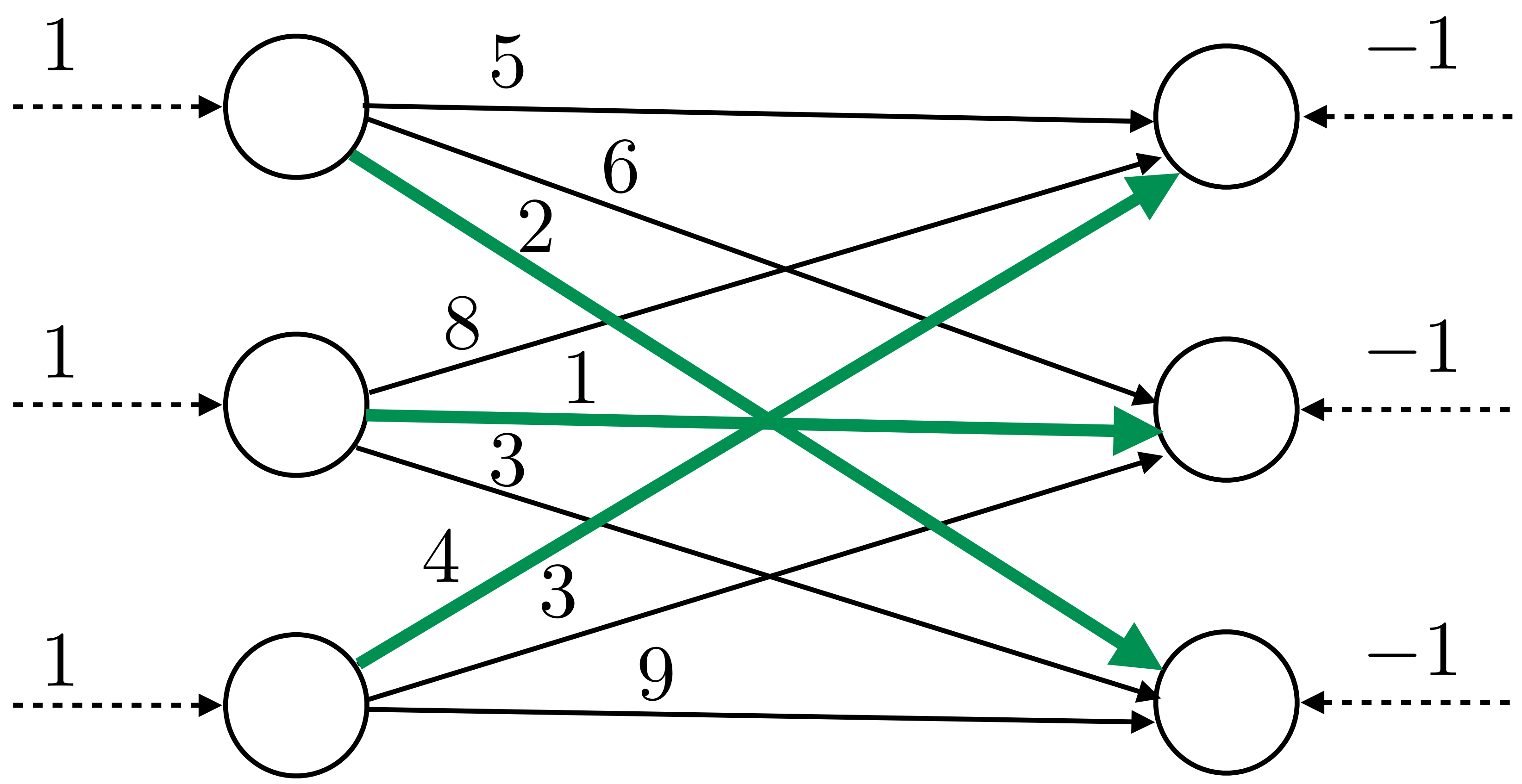
subject to $$\sum_{i=1}^{N} X_{ij} = 1, \quad j = 1, \dots, N$$

$$\sum_{j=1}^{N} X_{ij} = 1, \quad i = 1, \dots, N$$

$$X_{ij} \in \{0, 1\}$$

How do you define the network?

Task assignment as minimum cost network flow



(arc costs shown)

$$c = (5, 6, 2, 8, 1, 3, 4, 3, 9)$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$$b = (1, 1, 1, -1, -1, -1)$$

Minimum cost network flow

minimize $c^T x$

subject to $Ax = b$

$\mathbf{0 \leq x \leq 1}$

Extreme points satisfy $x_i \in \{0, 1\}$ $\longrightarrow$

Optimal solution

$$x^\star = (0, 0, 1, 0, 1, 0, 0, 0, 1)$$
$$c^T x^\star = 7$$

Network optimization

Today, we learned to:

- **Model** flows across networks
- **Formulate** minimum cost network flow problems
- **Analyze** network flow problem solutions (integrality theorem)
- **Formulate** maximum-flow, shortest path, and assignment problems as minimum cost network flows

References

- D. Bertsimas and J. Tsitsiklis: Introduction to Linear Optimization
  - Chapter 7: Network flow problems

- R. Vanderbei: Linear Programming
  - Chapter 14: Network Flow Problems
  - Chapter 15: Applications

Next lecture

- Interior point algorithms