ORF307 — Optimization

14. Duality II

Bartolomeo Stellato — Spring 2025

Recap

Optimal objective values

Primal

minimize $c^T x$
subject to $Ax \leq b$

Dual

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y \qquad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**

Today's agenda

More on duality

- Two-person zero-sum games
- Farkas lemma
- Complementary slackness
- KKT conditions

Two-person games

Rock paper scissors

Rules

At count to three declare one of: Rock, Paper, or Scissors

Winners

Identical selection is a draw, otherwise:

- Rock beats (“dulls”) scissors
- Scissors beats (“cuts”) paper
- Paper beats (“covers”) rock

Extremely popular: world RPS society, USA RPS league, etc.

Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of m actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of n actions)

Two players make their choice independently

Rule

Player 1 pays A_{ij} to player 2

$A \in \mathbf{R}^{m \times n}$ is the **payoff matrix**

Rock, Paper, Scissors

$$A = \begin{array}{c} \\ \mathrm{R} \\ \mathrm{P} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{P} & \mathrm{S} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

Mixed (randomized) strategies

Deterministic strategies can be **systematically defeated**

Randomized strategies

- P1 chooses randomly according to distribution x:

  $x_i =$ probability that P1 selects action i

- P2 chooses randomly according to distribution y:

  $y_j =$ probability that P2 selects action j

Expected payoff (from P1 P2), if they use mixed-strategies x and y,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

Mixed strategies and probability simplex

Probability simplex in $\mathbf{R}^k$
$P_k = \{p \in \mathbf{R}^k \mid p \geq 0, \quad \mathbf{1}^T p = 1\}$

Mixed strategy

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $\longrightarrow$ $x \in P_m, \quad y \in P_n$

Optimal mixed strategies

P1: optimal strategy $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & \max_{y \in P_n} x^T A y \\ \text{subject to} & x \in P_m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \max_{j=1,\dots,n} (A^T x)_j \\ \text{subject to} & x \in P_m \end{array}$$

Inner problem over deterministic strategies (**vertices**)

P2: optimal strategy $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & \min_{x \in P_m} x^T A y \\ \text{subject to} & y \in P_n \end{array} \longrightarrow \begin{array}{ll} \text{maximize} & \min_{i=1,\dots,m} (A y)_i \\ \text{subject to} & y \in P_n \end{array}$$

Optimal strategies $x^\star$ and $y^\star$ can be computed using **linear optimization**

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

The optimal $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \geq w\mathbf{1} \\ & \mathbf{1}^T y = 1 \\ & y \geq 0 \end{array}$$

The two LPs are **duals** and by **strong duality** the equality follows. ■

Nash equilibrium

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Consequence

The pair of mixed strategies $(x^\star, y^\star)$ attains the **Nash equilibrium** of the two-person matrix game, i.e.,

$$x^T A y^\star \geq x^{\star T} A y^\star \geq x^{\star T} A y, \quad \forall x \in P_m, \; \forall y \in P_n$$

Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

Optimal deterministic strategies

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

Optimal mixed strategies

$$x^\star = (0.37, 0.33, 0.3), \qquad y^\star = (0.4, 0, 0.13, 0.47)$$

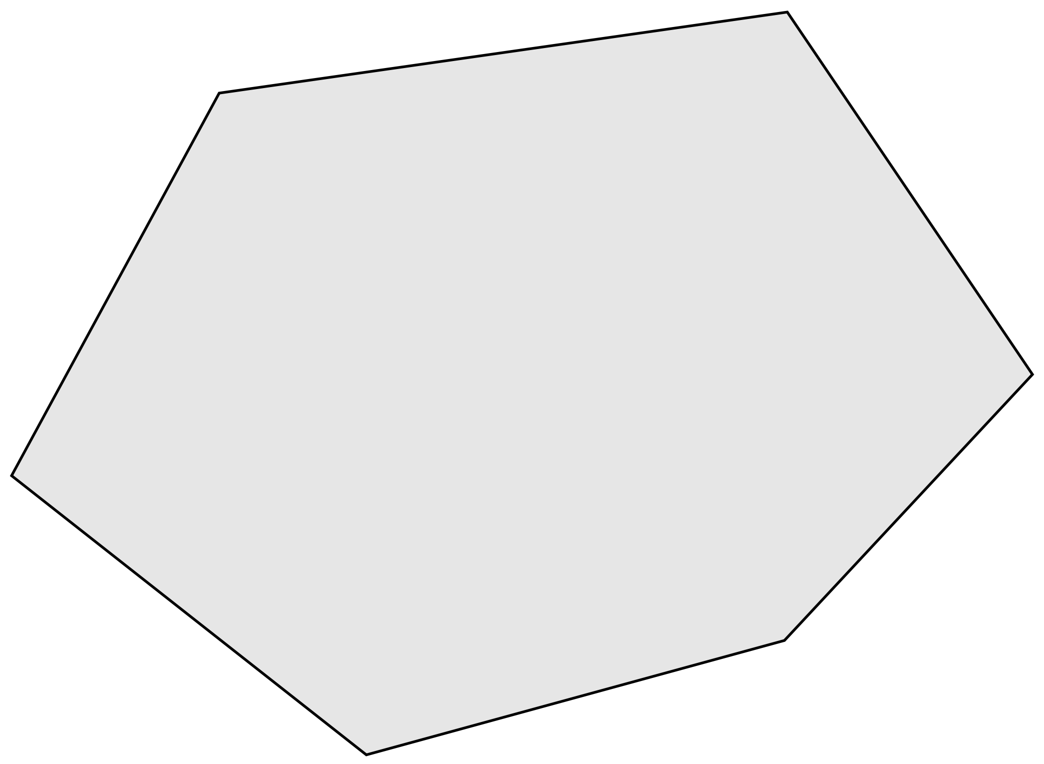
Expected payoff

$$x^{\star T} A y^\star = 0.2$$

Farkas lemma

Feasibility of polyhedra

$$P = \{x \mid Ax = b, \quad x \geq 0\}$$



How to show that P is **feasible**?

Easy: we just need to provide an $x \in P$, i.e., a **certificate**

How to show that P is **infeasible**?

Farkas lemma

Theorem

Given A and b, exactly one of the following statements is true:

1. There exists an x with $Ax = b$, $x \geq 0$
2. There exists a y with $A^T y \geq 0$, $b^T y < 0$

Farkas lemma

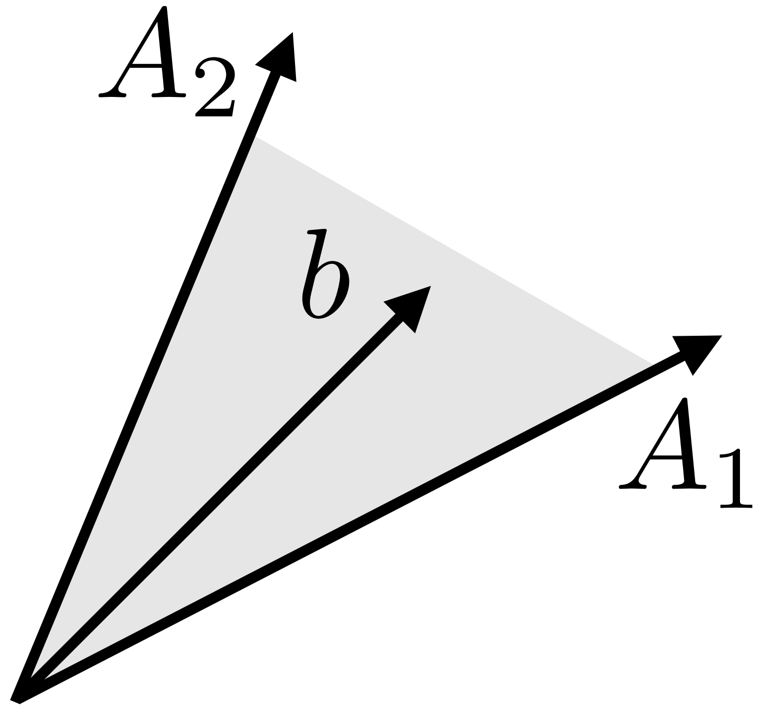
Geometric interpretation

1. First alternative

There exists an x with $Ax = b$, $x \geq 0$

$$b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0, \; i = 1, \dots, n$$

b is in the cone generated by the columns of A

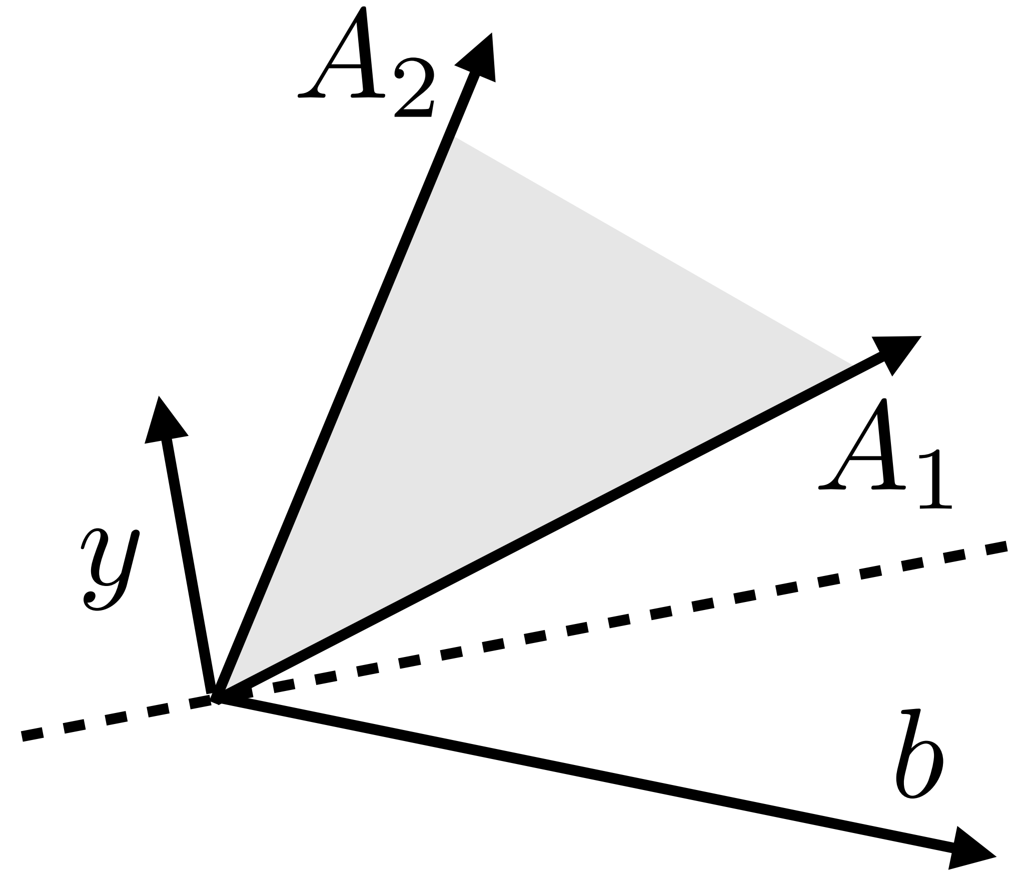


2. Second alternative

There exists a y with $A^T y \geq 0$, $b^T y < 0$

$$y^T A_i \geq 0, \quad i = 1, \dots, m, \qquad y^T b < 0$$

The hyperplane $y^T z = 0$ separates b from $A_1, \dots, A_n$



Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both true (easy)

$x \geq 0$, $Ax = b$ and $y^T A \geq 0$ ⟶ $y^T b = y^T Ax \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0
subject to $Ax = b$
$x \geq 0$

Dual

maximize $-b^T y$
subject to $A^T y \geq 0$

↑
$y = 0$ always feasible

Strong duality holds

$d^\star \neq -\infty, \quad p^\star = d^\star$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y \geq 0 \end{array}$$

Alternative 1: primal feasible $p^\star = d^\star = 0$

$b^T y \geq 0$ for all y such that $A^T y \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0
subject to $Ax = b$
$x \geq 0$

Dual

maximize $-b^T y$
subject to $A^T y \geq 0$

Alternative 2: primal infeasible $p^\star = d^\star = +\infty$

There exists y such that $A^T y \geq 0$ and $b^T y < 0$

y is an **infeasibility certificate**

Farkas lemma

Many variations

There exists x with $Ax = b,\ x \geq 0$

OR

There exists y with $A^T y \geq 0,\ b^T y < 0$

There exists x with $Ax \leq b,\ x \geq 0$

OR

There exists y with $A^T y \geq 0,\ b^T y < 0,\ y \geq 0$

There exists x with $Ax \leq b$

OR

There exists y with $A^T y = 0,\ b^T y < 0,\ y \geq 0$

Complementary slackness

Optimality conditions

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax \leq b$
- y is **dual feasible:** $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Can we **relate** x and y (not only the objective)?

Complementary slackness

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

Theorem

Primal,dual feasible x, y are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and y have a **complementary sparsity** pattern:

$$\begin{aligned} y_i > 0 \quad &\Rightarrow \quad a_i^T x = b_i \\ a_i^T x < b_i \quad &\Rightarrow \quad y_i = 0 \end{aligned}$$

Complementary slackness

Primal

minimize $c^T x$
subject to $Ax \leq b$

Dual

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

Proof

The duality gap at primal feasible x and dual feasible y can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0 ■

For **feasible** x and y **complementary slackness** = **zero duality gap**

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at x ⟶ $y = (0, y_2, 0, y_4)$

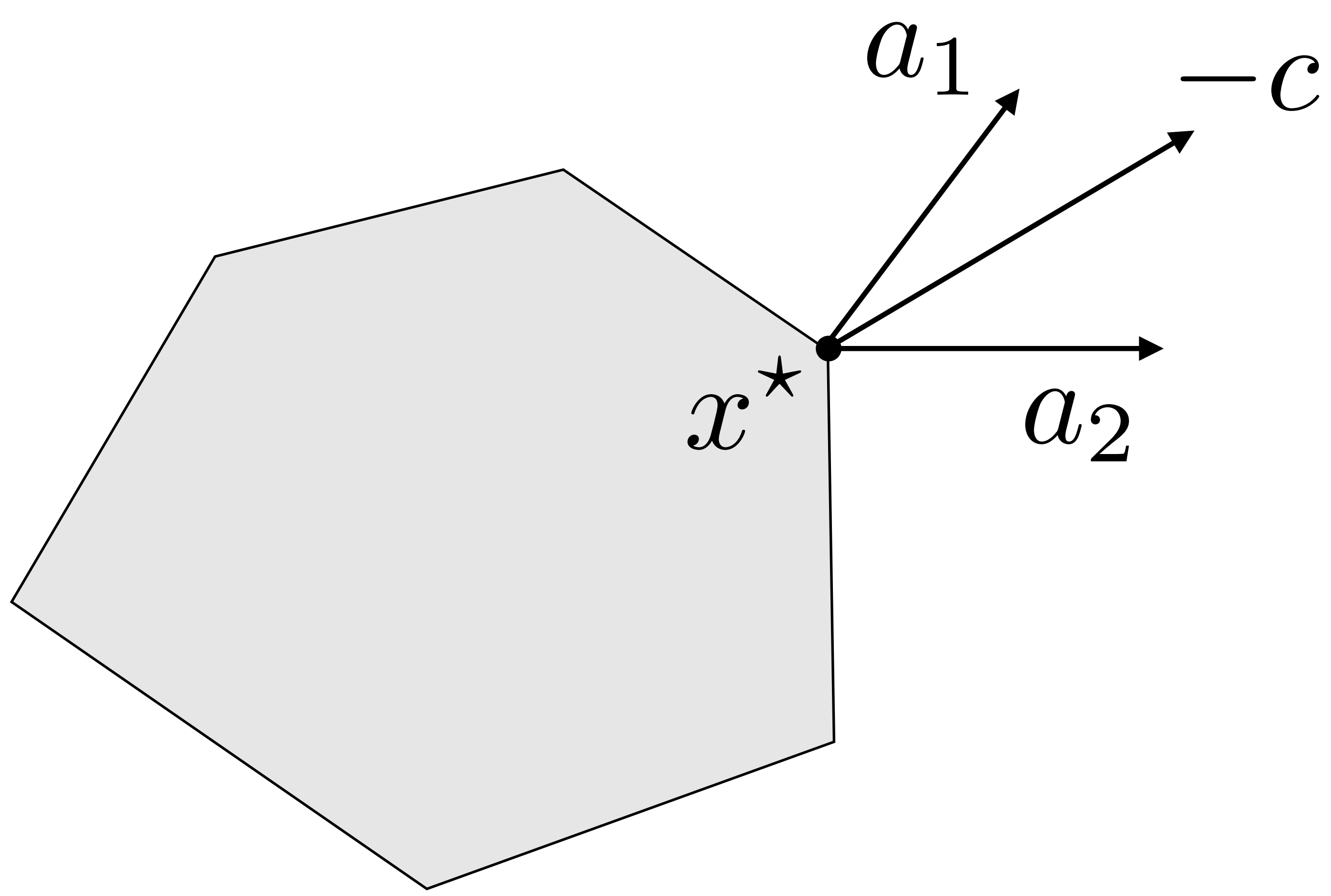
$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \text{and} \qquad y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Complementary slackness is useful to recover $y^\star$ from $x^\star$

Geometric interpretation

Example in $\mathbb{R}^2$



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Optimal dual solution y satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$

KKT Conditions

Lagrangian and duality

Primal

minimize $\quad c^T x$

subject to $\quad Ax \leq b$

Dual

maximize $\quad -b^T y$

subject to $\quad A^T y + c = 0$

$\qquad y \geq 0$

Dual function

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

Karush-Kuhn-Tucker conditions

Optimality conditions for linear optimization

Primal		**Dual**	
minimize	$c^T x$	maximize	$-b^T y$
subject to	$Ax \leq b$	subject to	$A^T y + c = 0$
			$y \geq 0$

Primal feasibility $\quad Ax \leq b$

Dual feasibility $\quad \nabla_x L(x, y) = A^T y + c = 0$ and $y \geq 0$

Complementary slackness $\quad y_i(Ax - b)_i = 0, \quad i = 1, \ldots, m$

Karush-Kuhn-Tucker conditions

Solving linear optimization problems

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

We can solve our optimization problem by solving a system of equations

$$
\begin{aligned}
\nabla_x L(x, y) = A^T y + c &= 0 \\
b - Ax &\geq 0 \\
y &\geq 0 \\
y^T (b - Ax) &= 0
\end{aligned}
$$

Linear optimization duality

Today, we learned to:

- **Interpret** linear optimization duality using game theory
- **Prove** Farkas lemma using duality
- **Geometrically link** primal and dual solutions with complementary slackness
- **Derive** KKT optimality conditions

References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
    - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
    - Chapter 11: Game Theory

Next lecture

- Sensitivity analysis