

# **ORF307 – Optimization**

## **13. Duality**

# Recap

# Linear optimization formulations

## Standard form LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

## Inequality form LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

# Today's agenda

## Duality

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

# Obtaining lower bounds

# Obtaining lower bounds

## A simple example

$$\begin{array}{ll}\text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + 3x_2 \geq 2\end{array}$$

What is a **lower bound** on the optimal cost?

A lower bound is 2 because  $x_1 + 3x_2 \geq 2$

# Obtaining lower bounds

## Another example

$$\begin{array}{ll}\text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \geq 1\end{array}$$

What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$\begin{aligned} & 1 \cdot (x_1 + x_2 \geq 2) \\ & + 2 \cdot (x_2 \geq 1) \\ & = x_1 + 3x_2 \geq 4 \end{aligned}$$

A lower bound is 4

# Obtaining lower bounds

## A more interesting example

$$\begin{array}{ll}\text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3\end{array}$$

How can we obtain a lower bound?

### Add constraints

$$\begin{array}{l}y_1 \cdot (x_1 + x_2 \geq 2) \\ + y_2 \cdot (x_2 \geq 1) \\ + y_3 \cdot (x_1 - x_2 \geq 3)\end{array}$$

$$\Rightarrow (y_1 + y_3)x_1 + (y_1 + y_2 - y_3)x_2 \geq 2y_1 + y_2 + 3y_3$$

### Match cost coefficients

$$y_1 + y_3 = 1$$

$$y_1 + y_2 - y_3 = 3$$

$$y_1, y_2, y_3 \geq 0$$

**Bound**

### Many options

$$\begin{array}{l}y = (1, 2, 0) \Rightarrow \text{Bound } 4 \\ y = (0, 4, 1) \Rightarrow \text{Bound } 7\end{array}$$

How can we get the **best one**?



# Obtaining lower bounds

## A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$\begin{array}{ll}\text{maximize} & 2y_1 + y_2 + 3y_3 \\ \text{subject to} & y_1 + y_3 = 1 \\ & y_1 + y_2 - y_3 = 3 \\ & y_1, y_2, y_3 \geq 0\end{array}$$

This linear optimization problem is called the **dual problem**

# The dual problem

# Lagrange multipliers

Consider the LP in standard form

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

Lower bound

$$g(y) \leq c^T x^* + y^T (Ax^* - b) = c^T x^*$$

Relax the constraint

$$g(y) = \begin{array}{ll}\text{minimize} & c^T x + y^T (Ax - b) \\ \text{subject to} & x \geq 0\end{array}$$

Best lower bound

$$\text{maximize}_y \quad g(y)$$

# The dual

## Dual function

$$\begin{aligned} g(y) &= \underset{x \geq 0}{\text{minimize}} \left( c^T x + y^T (Ax - b) \right) \\ &= -b^T y + \underset{x \geq 0}{\text{minimize}} \left( c + A^T y \right)^T x \end{aligned}$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

## Dual problem (find the best bound)

$$\begin{aligned} \underset{y}{\text{maximize}} \quad g(y) &= \underset{y}{\text{maximize}} \quad -b^T y \\ &\text{subject to} \quad A^T y + c \geq 0 \end{aligned}$$

# Primal and dual problems

## Primal problem

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

Primal variable  $x \in \mathbf{R}^n$

## Dual problem

$$\begin{array}{ll}\text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0\end{array}$$

Dual variable  $y \in \mathbf{R}^m$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

# Dual of inequality form LP

What if you find an LP with inequalities?

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T (Ax - b)$$

Lower bound

$$g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^*$$

we must have  $y \geq 0$

# Dual of LP with inequalities

## Derivation

### Dual function

$$\begin{aligned} g(y) &= \underset{x}{\text{minimize}} \left( c^T x + y^T (Ax - b) \right) \\ &= -b^T y + \underset{x}{\text{minimize}} \left( c + A^T y \right)^T x \end{aligned}$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y = 0 \quad (\text{and } y \geq 0) \\ -\infty & \text{otherwise} \end{cases}$$

### Dual problem (find the best bound)

$$\begin{aligned} \underset{y}{\text{maximize}} \quad g(y) &= \underset{y}{\text{maximize}} \quad -b^T y \\ &\text{subject to} \quad A^T y + c = 0 \\ &\quad y \geq 0 \end{aligned}$$

# General forms

## Standard form LP

### Primal

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

### Dual

$$\begin{array}{ll}\text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0\end{array}$$

## Inequality form LP

### Primal

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

### Dual

$$\begin{array}{ll}\text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0\end{array}$$

## LP with inequalities and equalities

### Primal

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d\end{array}$$

### Dual

$$\begin{array}{ll}\text{maximize} & -b^T y - d^T z \\ \text{subject to} & A^T y + C^T z + c = 0 \\ & y \geq 0\end{array}$$



# Example from before

$$\begin{array}{ll}\text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3\end{array}$$



## Inequality form LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$b = (-2, -1, -3)$$

## Dual

$$\begin{array}{ll}\text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0\end{array}$$



$$\begin{array}{ll}\text{maximize} & 2y_1 + y_2 + 3y_3 \\ \text{subject to} & -y_1 - y_3 = -1 \\ & -y_1 - y_2 + y_3 = -3 \\ & y_1, y_2, y_3 \geq 0\end{array}$$

# To memorize

## Ways to get the dual

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

## Sanity-checks and signs convention

- Consider constraints as  $Ax - b \leq 0$  or  $Ax - b = 0$  (not  $\geq 0$ )
- Each dual variable is associated to a primal constraint
- $y$  free for primal equalities and  $y \geq 0$  for primal inequalities

# Dual of the dual

## Theorem

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

## Exercise

Derive dual and dualize again

### Primal

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d\end{array}$$

### Dual

$$\begin{array}{ll}\text{maximize} & -b^T y - d^T z \\ \text{subject to} & A^T y + C^T z + c = 0 \\ & y \geq 0\end{array}$$

## Theorem

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

# Weak and strong duality

# Optimal objective values

## Primal

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

$p^*$  is the primal optimal value

Primal infeasible:  $p^* = +\infty$

Primal unbounded:  $p^* = -\infty$

## Dual

$$\begin{array}{ll}\text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0\end{array}$$

$d^*$  is the dual optimal value

Dual infeasible:  $d^* = -\infty$

Dual unbounded:  $d^* = +\infty$

# Weak duality

## Theorem

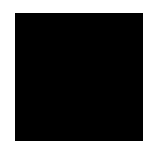
If  $x, y$  satisfy:

- $x$  is a feasible solution to the primal problem
  - $y$  is a feasible solution to the dual problem
- $\longrightarrow -b^T y \leq c^T x$

## Proof

We know that  $Ax \leq b$ ,  $A^T y + c = 0$  and  $y \geq 0$ . Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$



## Remark

- Any dual feasible  $y$  gives a **lower bound** on the primal optimal value
- Any primal feasible  $x$  gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$  is the **duality gap**

# Weak duality

## Corollaries

### Unboundedness vs feasibility

- Primal unbounded ( $p^* = -\infty$ )  $\Rightarrow$  dual infeasible ( $d^* = -\infty$ )
- Dual unbounded ( $d^* = +\infty$ )  $\Rightarrow$  primal infeasible ( $p^* = +\infty$ )

### Optimality condition

If  $x, y$  satisfy:

- $x$  is a feasible solution to the primal problem
- $y$  is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*,  $c^T x + b^T y = 0$

Then  $x$  and  $y$  are **optimal solutions** to the primal and dual problem respectively

# Strong duality

## Theorem

If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^* = p^*$$



# Strong duality

## Constructive proof

Given a primal optimal solution  $x^*$  we will construct a dual optimal solution  $y^*$

Apply simplex to problem in **standard form**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{l} \bullet \text{ optimal basis } B \\ \bullet \text{ optimal solution } x^* \text{ with } A_B x_B^* = b \\ \bullet \text{ reduced costs } \bar{c} = c - A^T A_B^{-T} c_B \geq 0 \end{array}$$

Define  $y^*$  such that  $y^* = -A_B^{-T} c_B$ . Therefore,  $A^T y^* + c \geq 0$  ( $y^*$  dual feasible).

$$-b^T y^* = -b^T (-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^* = c^T x^*$$

By weak duality theorem corollary,  $y^*$  is an optimal solution of the dual.

Therefore,  $d^* = p^*$ .



# Exception to strong duality

## Primal

$$\begin{array}{ll}\text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1\end{array}$$

Optimal value is  $p^* = +\infty$

## Dual

$$\begin{array}{ll}\text{maximize} & y \\ \text{subject to} & 0 \cdot y + 1 = 0 \\ & y \geq 0\end{array}$$

Optimal value is  $d^* = -\infty$

Both **primal** and **dual infeasible**

# Relationship between primal and dual

	$p^* = +\infty$	$p^*$ finite	$p^* = -\infty$
$d^* = +\infty$	primal inf. dual unb.		
$d^*$ finite		optimal values equal	
$d^* = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- (1, 1) and (3, 3) proven by **weak duality**
- (3, 1) and (2, 2) proven by **strong duality**

# Example

# Production problem

maximize  $x_1 + 2x_2$  ← Profits  
subject to  $x_1 \leq 100$   
 $2x_2 \leq 200$  ← Resources  
 $x_1 + x_2 \leq 150$   
 $x_1, x_2 \geq 0$

## Dualize

1. Transform in inequality form

minimize  $c^T x$   
subject to  $Ax \leq b$

2. Derive dual

maximize  $-b^T y$   
subject to  $A^T y + c = 0$   
 $y \geq 0$

$$c = (-1, -2)$$
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$b = (100, 200, 150, 0, 0)$$

# Production problem

## Dualized

$$\begin{array}{ll}\text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0\end{array}$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

## Fill-in data

$$\begin{array}{ll}\text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 - y_4 = 1 \\ & 2y_2 + y_3 - y_5 = 2 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0\end{array}$$



## Eliminate variables

$$\begin{array}{ll}\text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0\end{array}$$

# Production problem

## The dual

$$\begin{array}{ll}\text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0\end{array}$$

## Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs  $1 \times$  resource 1 and 3):  $y_1 + y_3 \geq 1$
  - Product 2 (price 2, needs  $2 \times$  resource 2 and  $1 \times$  resource 3):  $2y_2 + y_3 \geq 2$

# Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Interpret** simple dual optimization problems



# References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 5: Duality theory

# Next lecture

More on duality:

- Game theory
- Complementary slackness
- Farkas lemma