ORF307 – Optimization

12. The simplex method implementation

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Recap

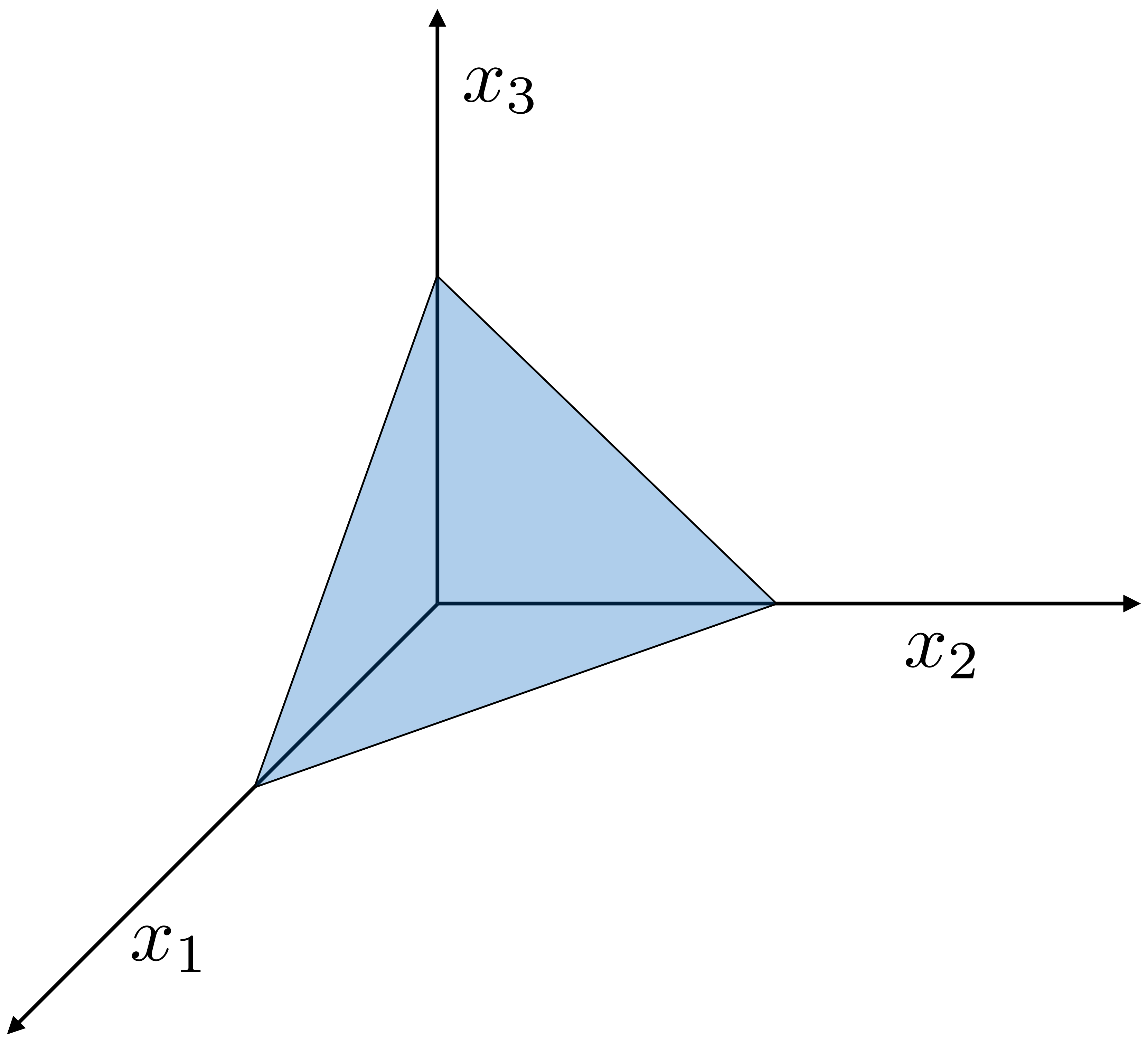
Standard form polyhedra

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Standard form polyhedron

$$P = \{x \mid Ax = b,\ x \geq 0\}$$



$n = 3, m = 1$

Assumption

$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P is an $(n - m)$-dimensional surface

An iteration of the simplex method

Initialization

- a basic feasible solution x
- a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots, A_{B(m)} \end{bmatrix}$

Iteration steps

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal**. **break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ (i exits and j enters)

Example

$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

$$x = (2, 0, 0) \qquad\qquad B = \{1\}$$

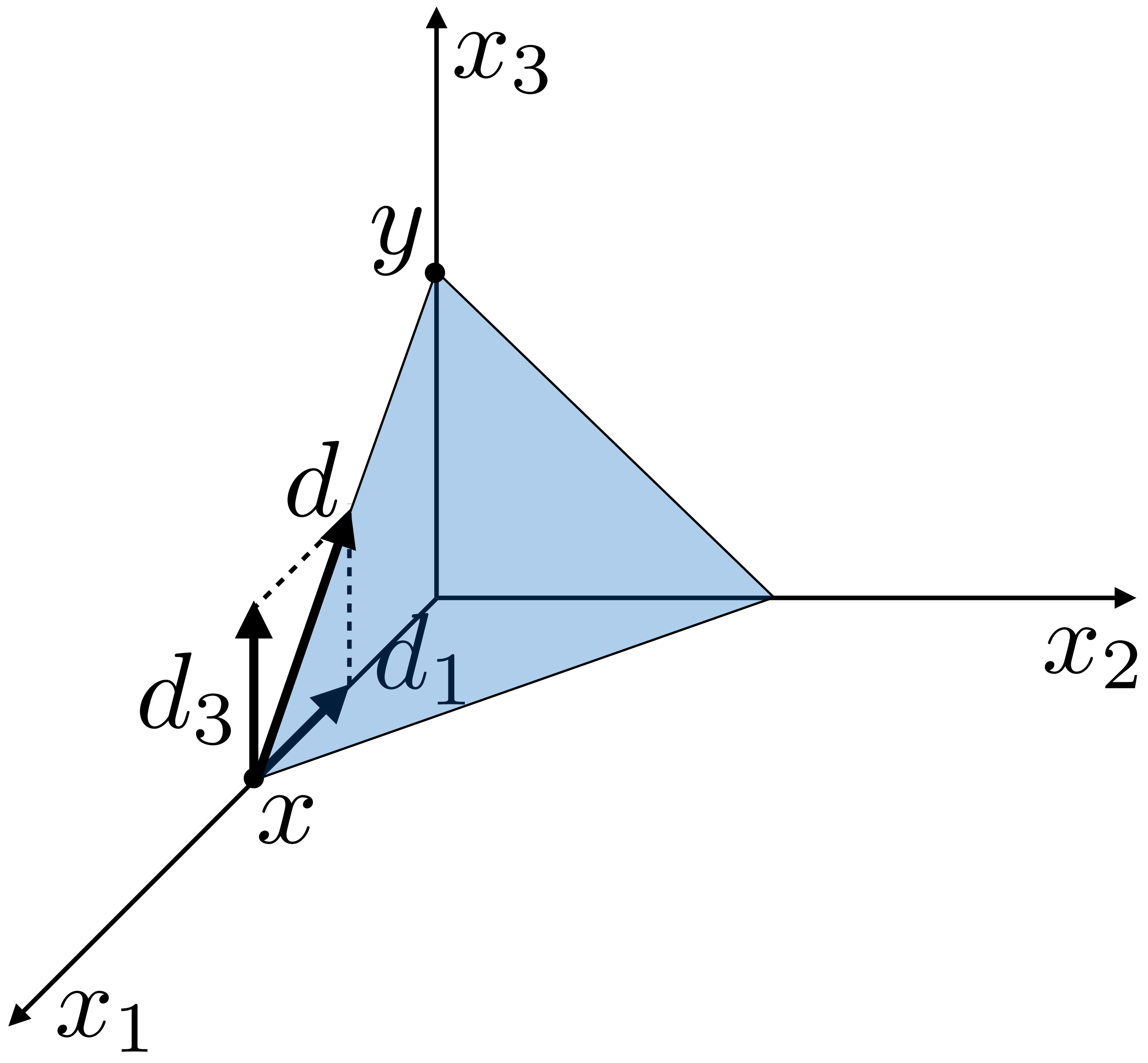
Basic index $j = 3 \longrightarrow d = (-1, 0, 1)$

$$d_j = 1$$

$$A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1$$

Stepsize $\theta^\star = -\dfrac{x_1}{d_1} = 2$

New solution $y = x + \theta^\star d = (0, 0, 2) \qquad \bar{B} = \{3\}$



Finite convergence

Assume that

- $P = \{x \mid Ax = b, x \geq 0\}$ not empty
- Every basic feasible solution **non degenerate**

Then

- The simplex method **terminates after a finite number of iterations**
- At termination we either have one of the following
  - an **optimal basis** B
  - a **direction** d such that $Ad = 0,\ d \geq 0,\ c^T d < 0$ and the optimal cost is $-\infty$

Finite convergence

Proof sketch

At each iteration the algorithm improves

- by a **positive** amount $\theta^\star$
- along the **direction** d such that $c^T d < 0$

Therefore

- The cost strictly decreases
- No basic feasible solution can be visited twice

Since there is a **finite number of basic feasible solutions**

The algorithm **must eventually terminate**

$\blacksquare$

Today's lecture

The simplex method implementation

- Finding an initial basic feasible solution
- Degeneracy
- Full simplex example
- Efficiency

Find an initial point

Initial basic feasible solution

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

How do we get an initial **basic feasible solution** x and a **basis** B ?

Does it **exist**?

Finding an initial basic feasible solution

			Auxiliary problem	
minimize	$c^T x$		minimize	$\mathbf{1}^T y$ ← Minimize violations
subject to	$Ax = b$	→	subject to	$Ax + y = b$
	$x \geq 0$			$x \geq 0, y \geq 0$

Assumption $b \geq 0$ w.l.o.g. (if not multiply constraint by -1)

Trivial basic feasible solution: $x = 0, y = b$

Possible outcomes

- **Feasible problem** (cost = 0): $y^\star = 0$ and $x^\star$ is a basic feasible solution
- **Infeasible problem** (cost > 0): $y^\star > 0$ are the violations

Two-phase simplex method

Phase I

1. Construct **auxiliary problem** such that $b \geq 0$
2. Solve auxiliary problem using simplex method starting from $(x, y) = (0, b)$
3. If the optimal value is greater than 0, **problem infeasible**. **break**.

Phase II

1. Recover original problem (drop variables y and restore original cost)
2. Solve original problem starting from the solution x and its basis B.

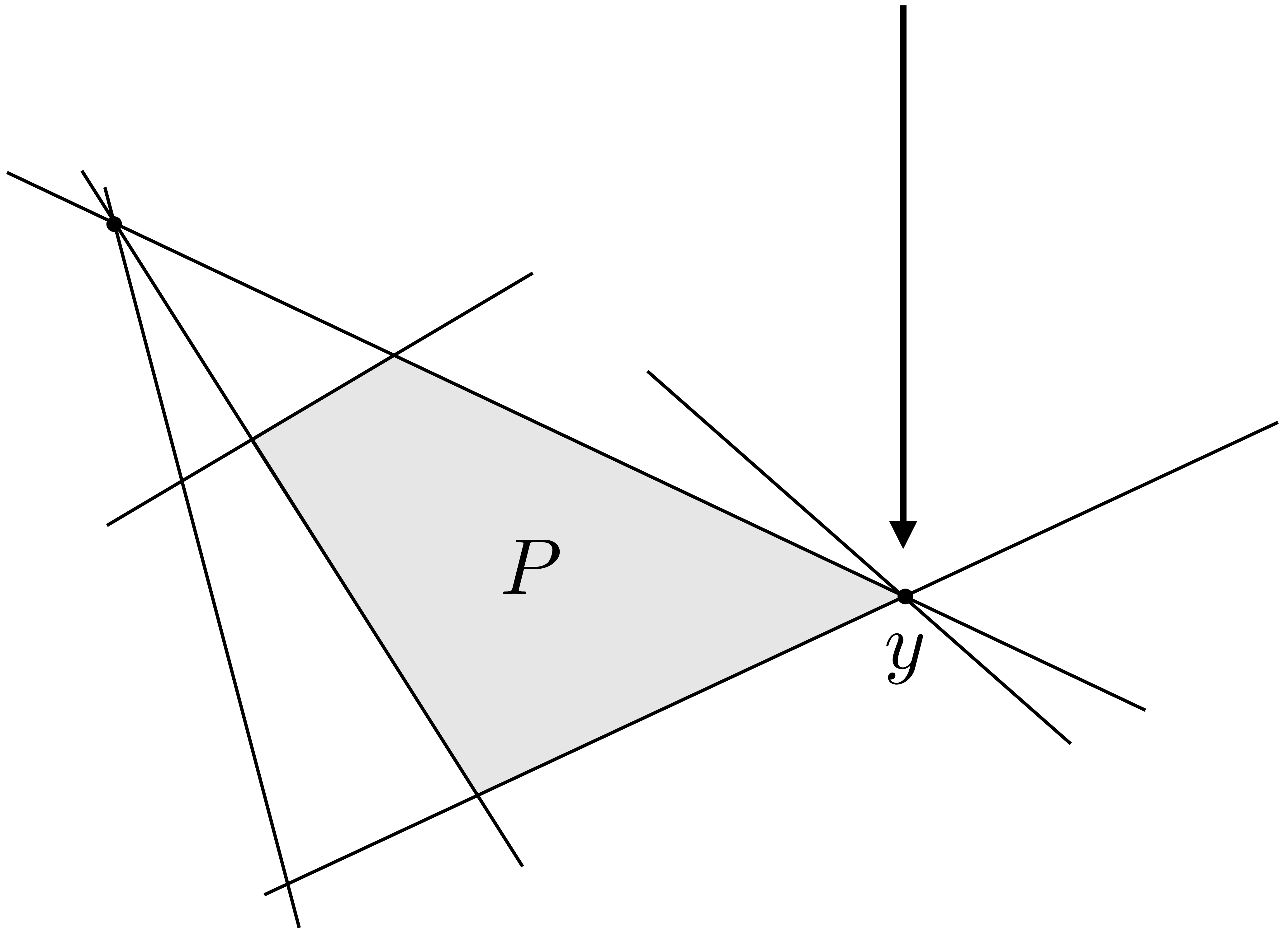
Degeneracy

Degenerate basic feasible solutions

Inequality form polyhedron

A solution y is degenerate if $|\mathcal{I}(y)| > n$

$$P = \{x \mid Ax \leq b\}$$



Degenerate basic feasible solutions

Standard form polyhedron

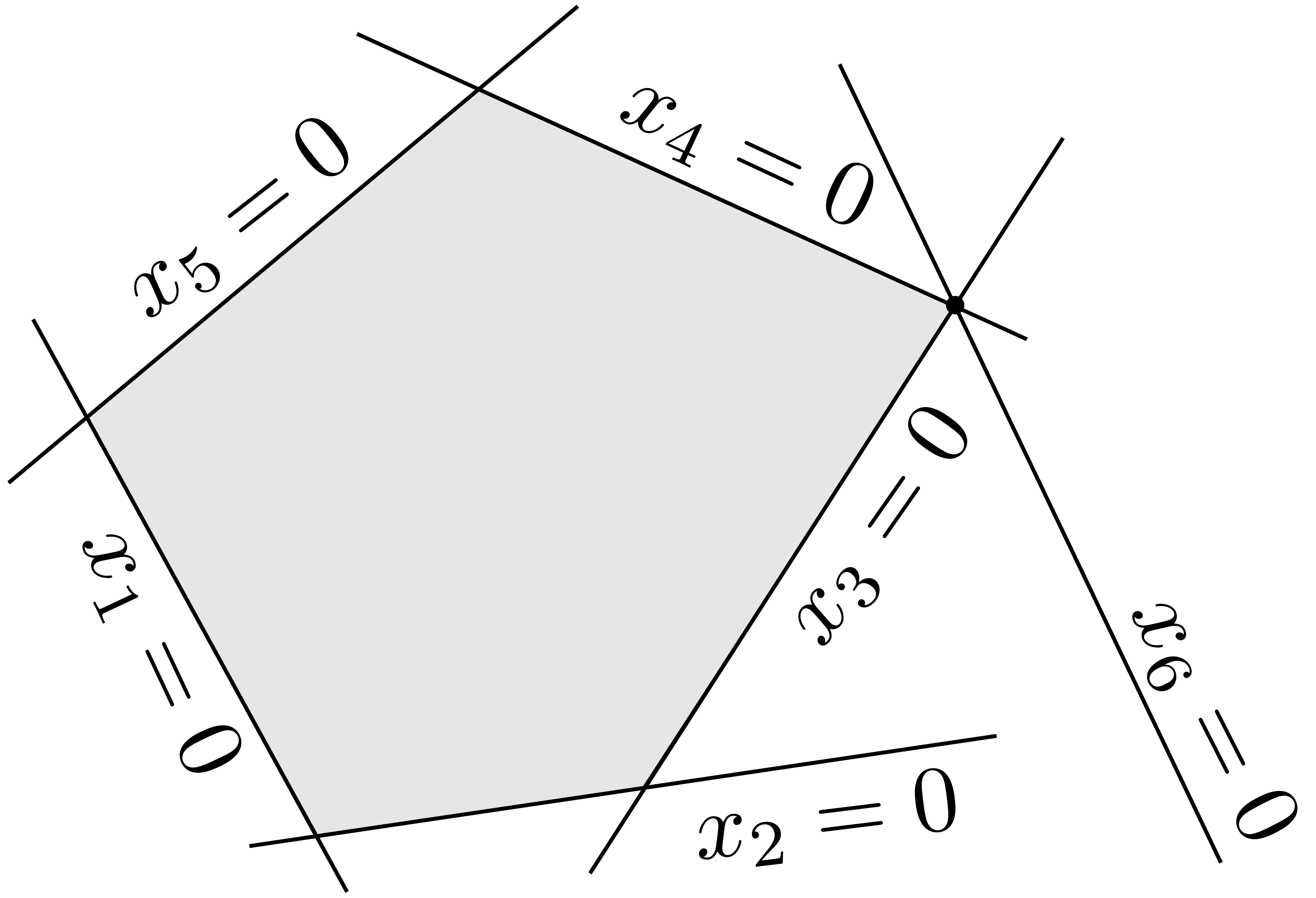
Given a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

we have basic feasible solution x:

- $A_B x_B = b$
- $x_i = 0, \ \forall i \neq B(1), \dots, B(m)$

→ If some of the $x_B = 0$, then it is a **degenerate solution**

$$P = \{x \mid Ax = b, \ x \geq 0\}$$



Degenerate basic feasible solutions

Example

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 1 \\
-x_1 + x_2 - x_3 &= 1 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Degenerate solutions

Basis $B = \{1, 2\}$ $\longrightarrow$ $x = (0, 1, 0)$

Basis $B = \{2, 3\}$ $\longrightarrow$ $y = (0, 1, 0)$

Cycling

Stepsize

6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$

↓

If $i \in B$, $d_i < 0$ and $x_i = 0$ (degenerate)

$$\theta^\star = 0$$

Therefore $y = x + \theta^\star x = x$ and $B \neq \bar{B}$

Same solution and cost
Different basis

Finite termination **no longer guaranteed**!

How can we fix it?

Pivoting rules

Pivoting rules

Choose the index entering the basis

Simplex iterations

3. Choose j such that $\bar{c}_j < 0$ ⟶ **Which j?**

Possible rules

- **Smallest subscript:** smallest j such that $\bar{c}_j < 0$
- **Most negative:** choose j with the most negative $\bar{c}_j$
- **Largest cost decrement:** choose j with the largest $\theta^\star |\bar{c}_j|$

Pivoting rules

Choose index exiting the basis

Simplex iterations

6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i}\right)$ ⟶

We can have more than one i for which $x_i = 0$ (**next** solution is **degenerate**)

Which i?

Smallest index rule

Smallest i such that $\theta^\star = -\frac{x_i}{d_i}$

Bland’s rule to avoid cycles

Theorem
If we use the **smallest index rule** for choosing both the j entering the basis and the i leaving the basis, then **no cycling will occur**.

Proof idea [Vanderbei, Ch 3, Sec 4][Bertsimas and Tsitsiklis, Sec 3.4]

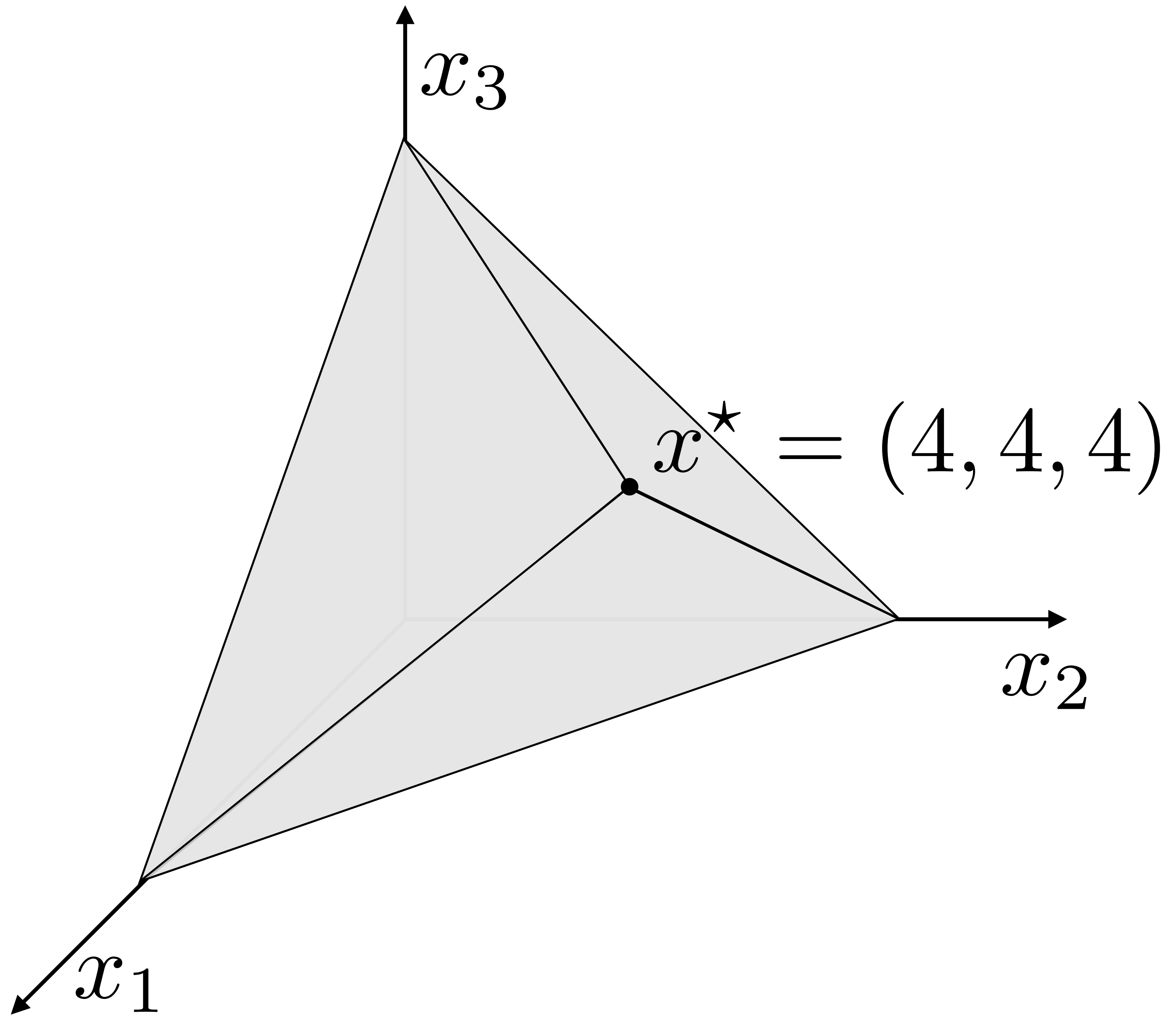
- Assume Bland’s rule is applied and there exists a cycle with different bases.
- Obtain contradiction.

Example

Example

Inequality form

$$\begin{array}{ll} \text{minimize} & -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} & x_1 + 2x_2 + 2x_3 \leq 20 \\ & 2x_1 + x_2 + x_3 \leq 20 \\ & 2x_1 + 2x_2 + x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$



Standard form

$$\begin{array}{ll} \text{minimize} & -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} & \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix} \\ & x \geq 0 \end{array}$$

Example

Start

minimize $c^T x$

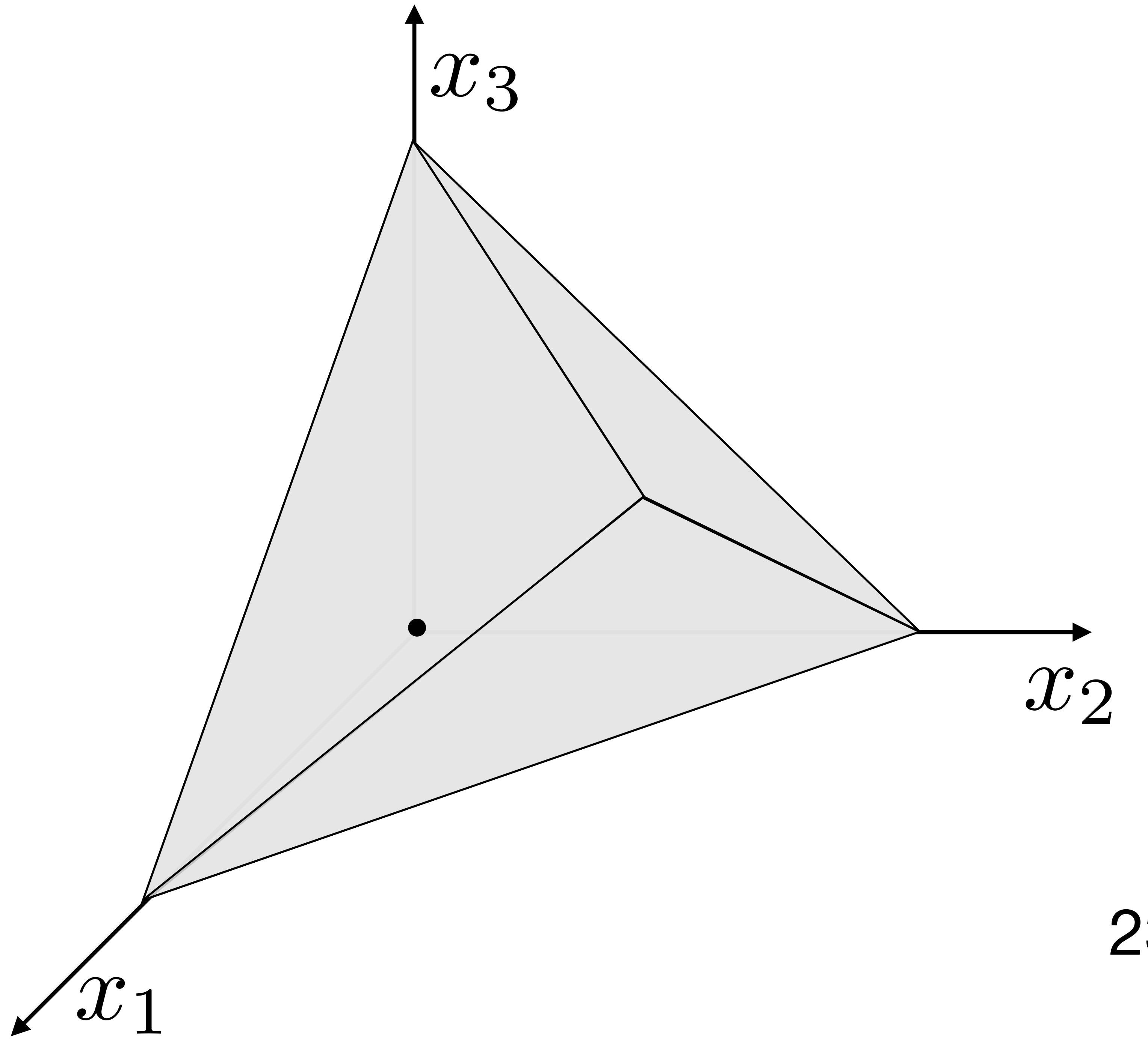
subject to $Ax = b$

$x \geq 0$

$$c = (-10, -12, -12, 0, 0, 0)$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (20, 20, 20)$$



Initialize

$$x = (0, 0, 0, 20, 20, 20) \qquad A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example
Iteration 1

Current point

$x = (0, 0, 0, 20, 20, 20)$
$c^T x = 0$

Basis: $\{4, 5, 6\}$

$$A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c = (-10, -12, -12, 0, 0, 0)$

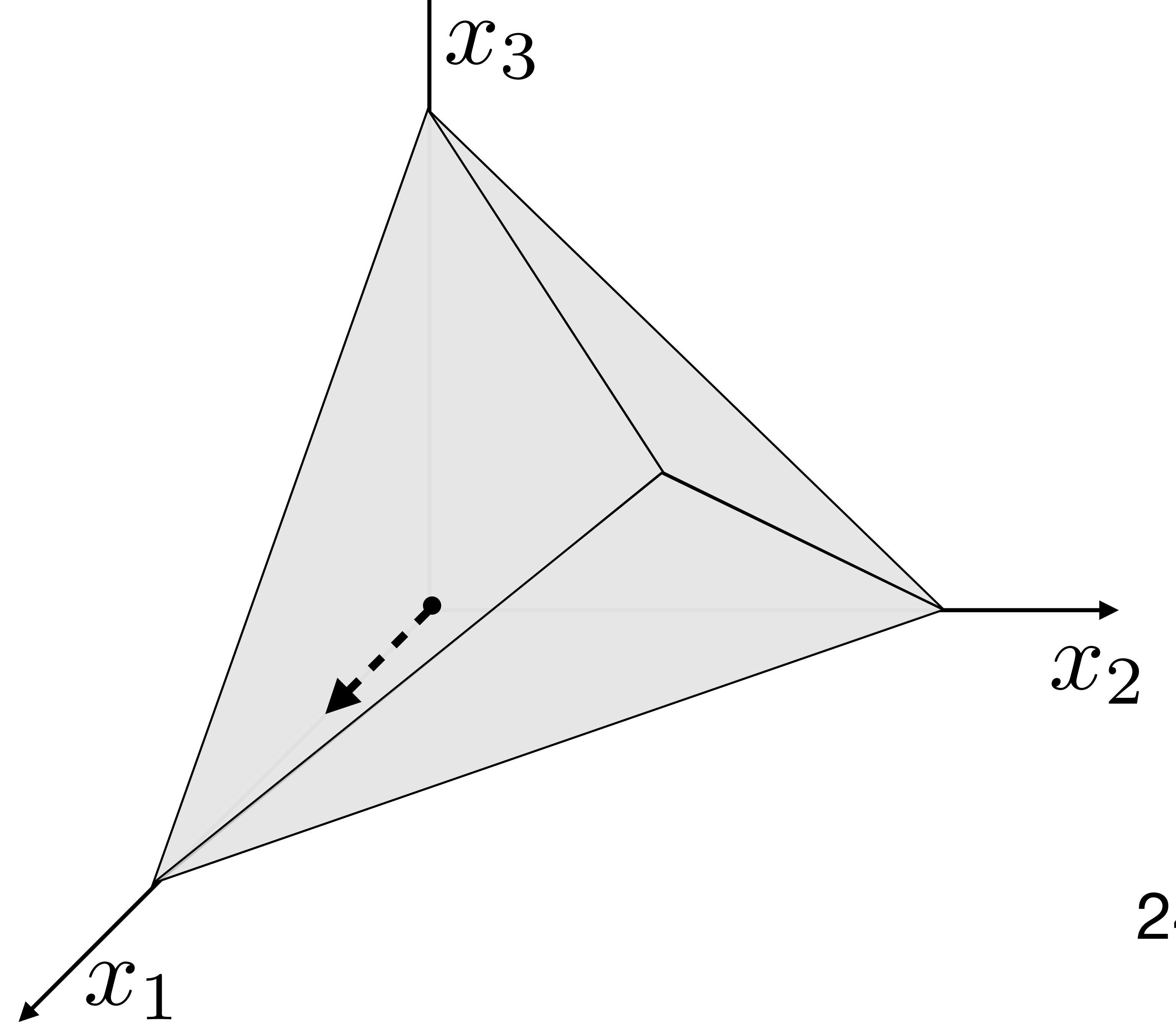
$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$b = (20, 20, 20)$

Reduced costs $\bar{c} = c$
Solve $A_B^T p = c_B \quad \Rightarrow \quad p = c_B = 0$
$\bar{c} = c - A^T p = c$

Direction $d = (1, 0, 0, -1, -2, -2), \quad j = 1$
Solve $A_B d_B = -A_j \quad \Rightarrow \quad d_B = (-1, -2, -2)$

Step $\theta^\star = 10, \quad i = 5$
$\theta^\star = \min_{\{i | d_i < 0\}} (-x_i / d_i) = \min\{20, 10, 10\}$
New $x \leftarrow x + \theta^\star d = (10, 0, 0, 10, 0, 0)$



Example
Iteration 2

Current point

$x = (10, 0, 0, 10, 0, 0)$
$c^T x = -100$

Basis: $\{4, 1, 6\}$

$$A_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$c = (-10, -12, -12, 0, 0, 0)$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$b = (20, 20, 20)$

Reduced costs $\bar{c} = (0, -7, -2, 0, 5, 0)$

Solve $A_B^T p = c_B \quad \Rightarrow \quad p = (0, -5, 0)$

$\bar{c} = c - A^T p = (0, -7, -2, 0, 5, 0)$

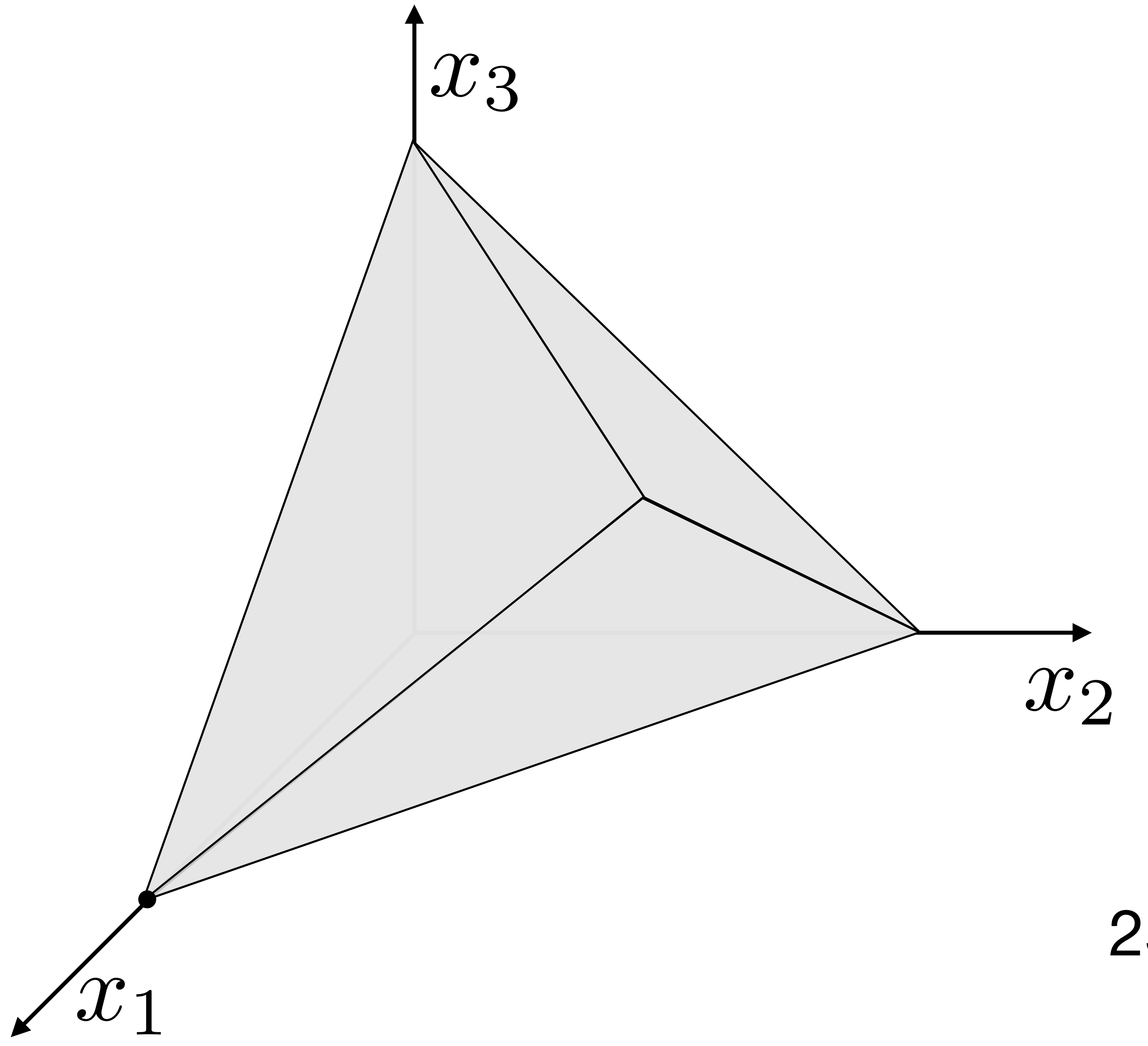
Direction $d = (-0.5, 1, 0, -1.5, 0, -1), \quad j = 2$

Solve $A_B d_B = -A_j \quad \Rightarrow \quad d_B = (-1.5, -0.5, -1)$

Step $\theta^\star = 0, \quad i = 6$

$$\theta^\star = \min_{\{i | d_i < 0\}} (-x_i / d_i) = \min\{6.66, 20, 0\}$$

New $x \leftarrow x + \theta^\star d = (10, 0, 0, 10, 0, 0)$



Example
Iteration 3

Current point

$x = (10, 0, 0, 10, 0, 0)$
$c^T x = -100$

Basis: $\{4, 1, 2\}$

$$A_B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$c = (-10, -12, -12, 0, 0, 0)$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$b = (20, 20, 20)$

Reduced costs $\bar{c} = (0, 0, -9, 0, -2, 7)$

Solve $A_B^T p = c_B \quad \Rightarrow \quad p = (0, 2, -7)$

$\bar{c} = c - A^T p = (0, 0, -9, 0, -2, 7)$

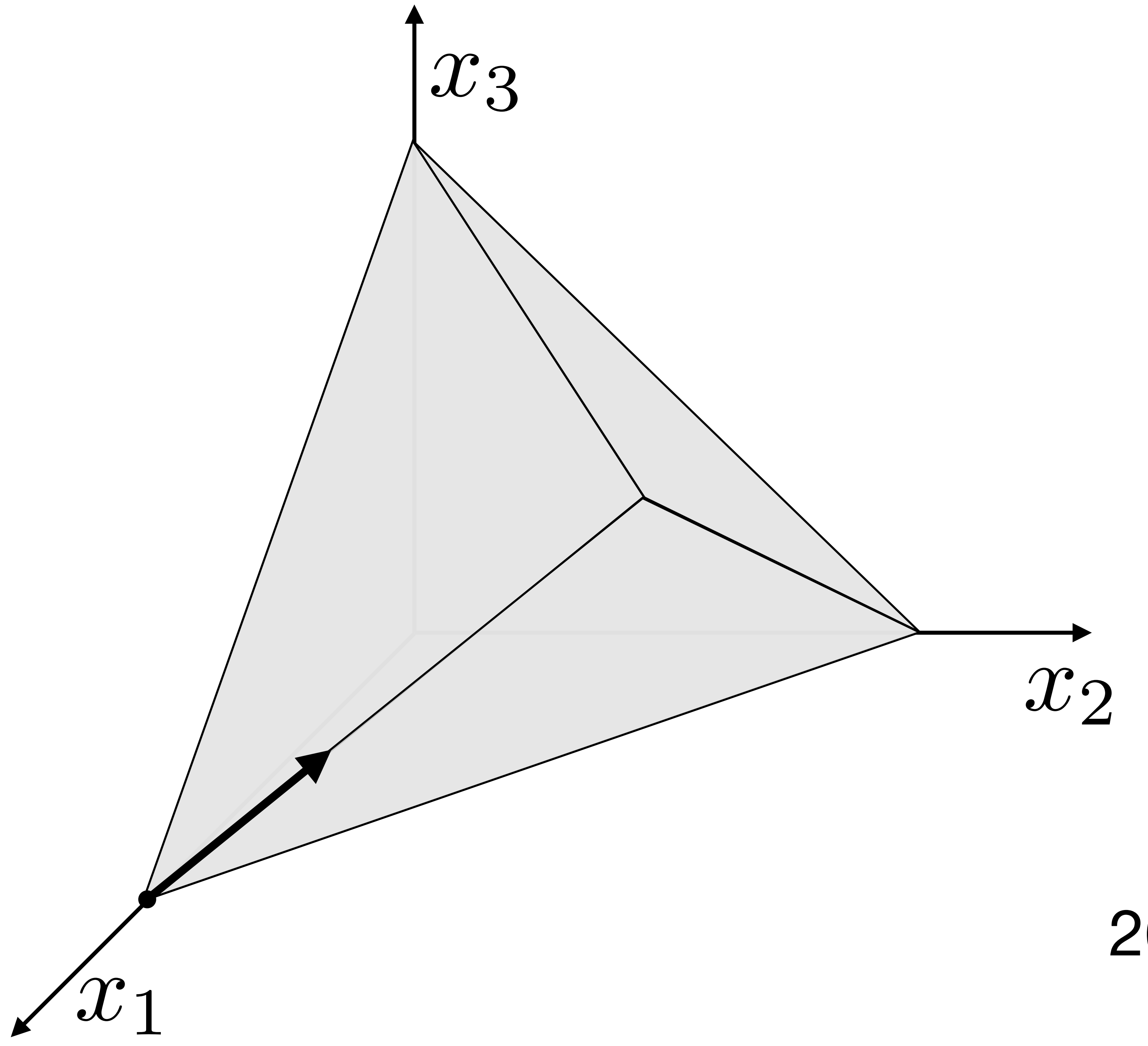
Direction $d = (-1.5, 1, 1, -2.5, 0, 0), \quad j = 3$

Solve $A_B d_B = -A_j \quad \Rightarrow \quad d_B = (-2.5, -1.5, 1)$

Step $\theta^\star = 4, \quad i = 4$

$$\theta^\star = \min_{\{i \mid d_i < 0\}} (-x_i / d_i) = \min\{4, 6.67\}$$

New $x \leftarrow x + \theta^\star d = (4, 4, 4, 0, 0, 0)$



Example

Iteration 4

Current point

$x = (4, 4, 4, 0, 0, 0)$

$c^T x = -136$

Basis: $\{3, 1, 2\}$

$$A_B = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$c = (-10, -12, -12, 0, 0, 0)$

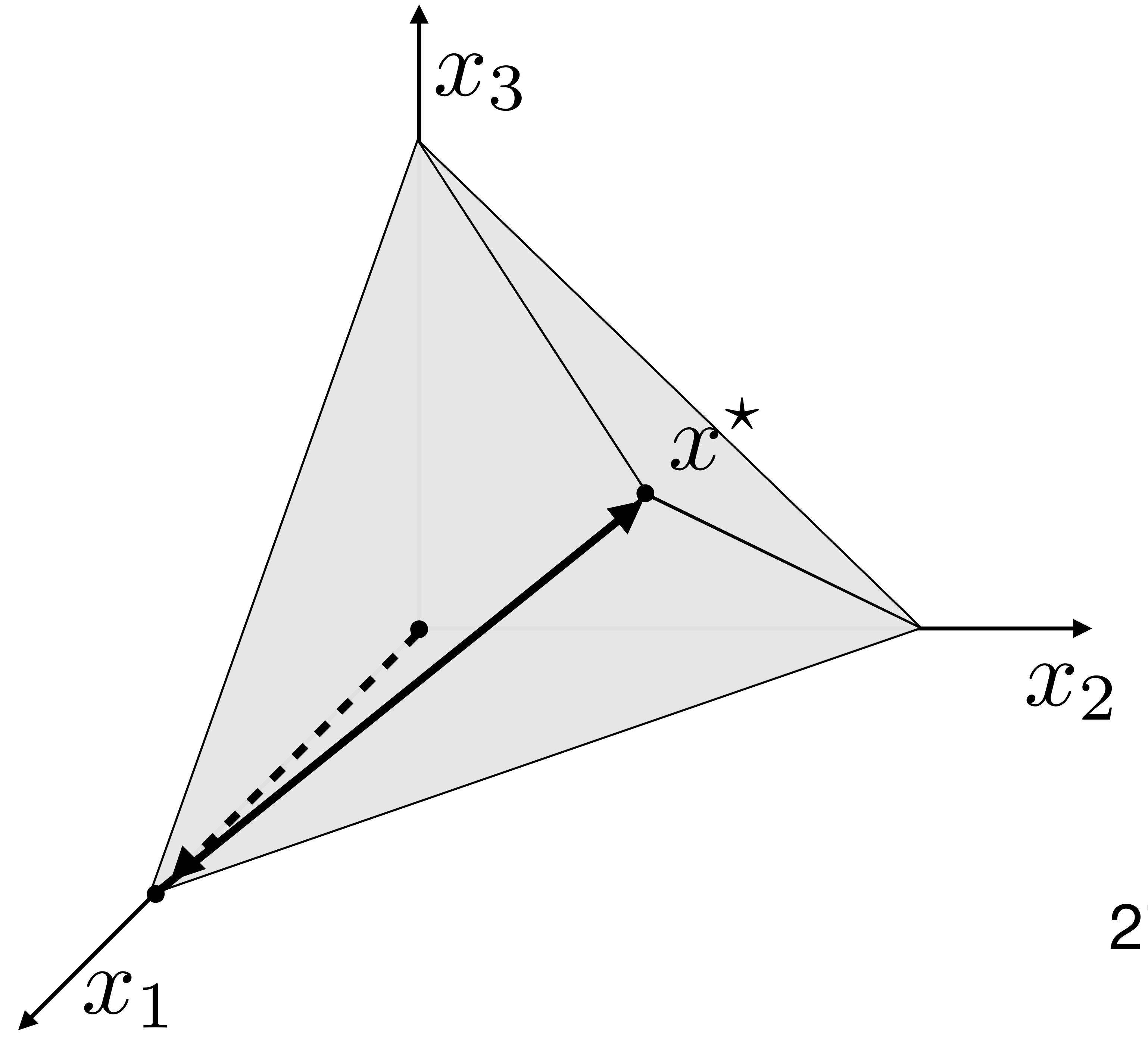
$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$b = (20, 20, 20)$

Reduced costs $\bar{c} = (0, 0, 0, 3.6, 1.6, 1.6)$

Solve $A_B^T p = c_B \quad \Rightarrow \quad p = (-3.6, -1.6, -1.6)$

$\bar{c} = c - A^T p = (0, 0, 0, 3.6, 1.6, 1.6)$



Optimal

$\bar{c} \geq 0 \longrightarrow x^\star = (4, 4, 4, 0, 0, 0)$

Complexity

Complexity of a single simplex iteration

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ (i exits and j enters)

Bottleneck

Two linear systems

Linear system solutions

Very similar linear systems		LU factorization $(2/3)n^3$ flops		Easy linear systems $4n^2$ flops
$A_B^T p = c_B$ $A_B d_B = -A_j$	→	$A_B = PLU$	→	$U^T L^T P^T p = c_B$ $PLU d_B = -A_j$

Factorization is expensive

Do we need to recompute it at every iteration?

Basis update

Index update

- j enters (x_j becomes $\theta^\star$)
- $i = B(\ell)$ exists (x_i becomes 0)

$\longrightarrow$

Basis matrix change

$$A_{\bar{B}} = A_B + (A_i - A_j)e_\ell^T$$

Example

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \{4, 1, 6\} \quad \rightarrow \quad \bar{B} = \{4, 1, 2\}$$

- 2 enters
- $6 = B(3)$ exists

$$A_{\bar{B}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{A_B} + \underbrace{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{A_2 e_3^T} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_6 e_3^T} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

Smarter linear system solution

Basis matrix change

$$A_{\bar{B}} = A_B + \overbrace{(A_i - A_j)}^{v} e_\ell^T$$

$\longrightarrow$

Matrix inversion lemma
(from homework 2)

$$(A_B + v e_\ell^T)^{-1} = \left(I - \frac{1}{1 + e_\ell^T A_B^{-1} v} A_B^{-1} v e_\ell^T \right) A_B^{-1}$$

Solve $A_{\bar{B}} d_{\bar{B}} = -A_j$

1. Solve $A_B z^1 = e_\ell$ ($2n^2$ flops)
2. Solve $A_B z^2 = -A_j$ ($2n^2$ flops)
3. Solve $d_{\bar{B}} = z^2 - \frac{v^T z^2}{1 + v^T z^1} z^1$

Remarks

- Same complexity for $A_B^T p = c_B$ ($4n^2$ flops)
- k-th next iteration ($4kn^2$ flops, derive as exercise...)
- Once in a while (e.g., $k = 100$), better to refactor A_B

Complexity of a single simplex iteration

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i}\right)$
7. Define y such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ (i exits and j enters)

Bottleneck
Two linear systems

→

Matrix inversion lemma trick
$\approx n^2$ per iteration
(very cheap)

How many iterations do we need?

Complexity of the simplex method

Example of worst-case behavior

Innocent-looking problem

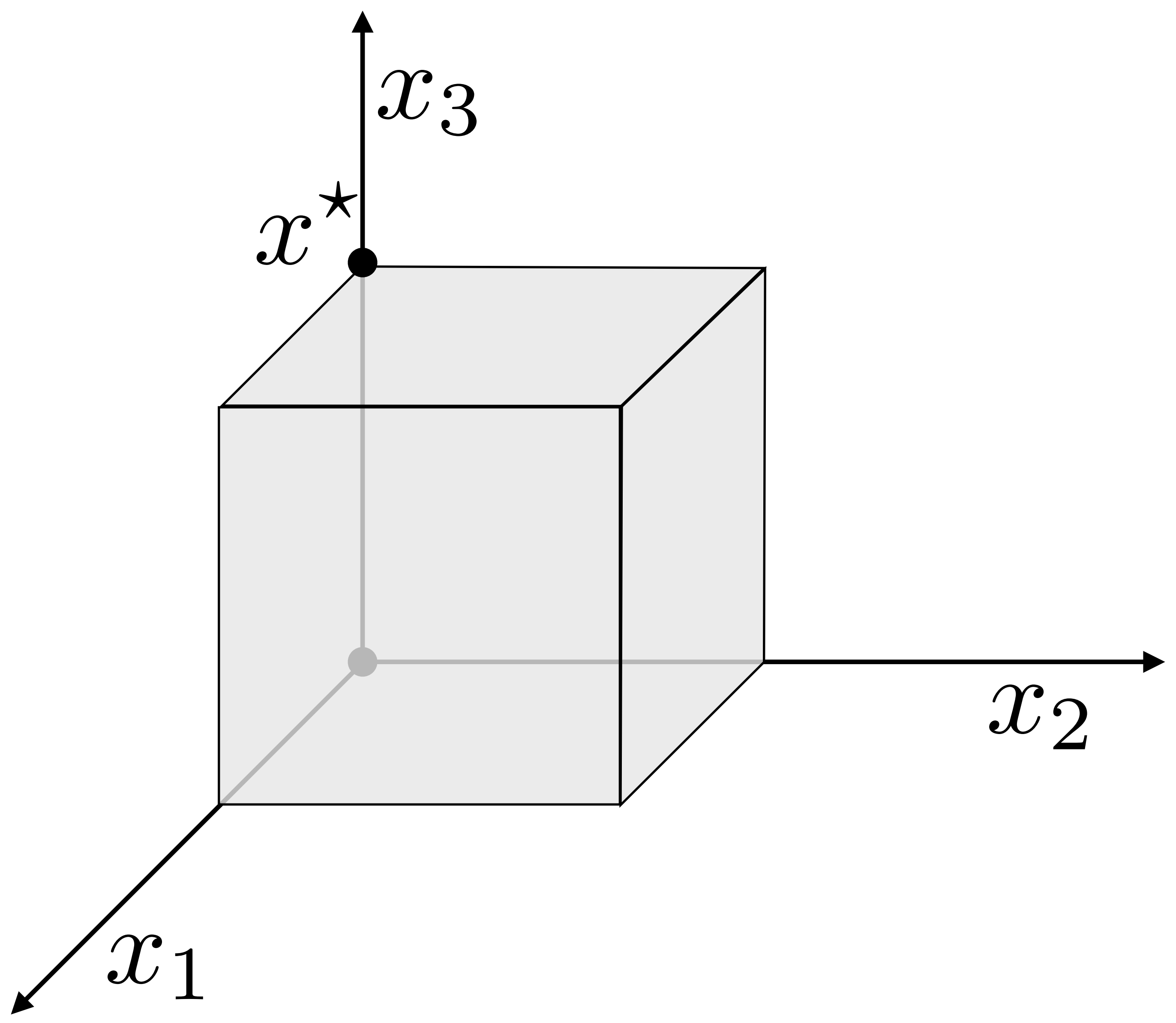
minimize $\quad -x_n$

subject to $\quad 0 \leq x \leq 1$

2^n **vertices**

$2^n/2$ vertices: cost = 1

$2^n/2$ vertices: cost = 0



Perturb unit cube

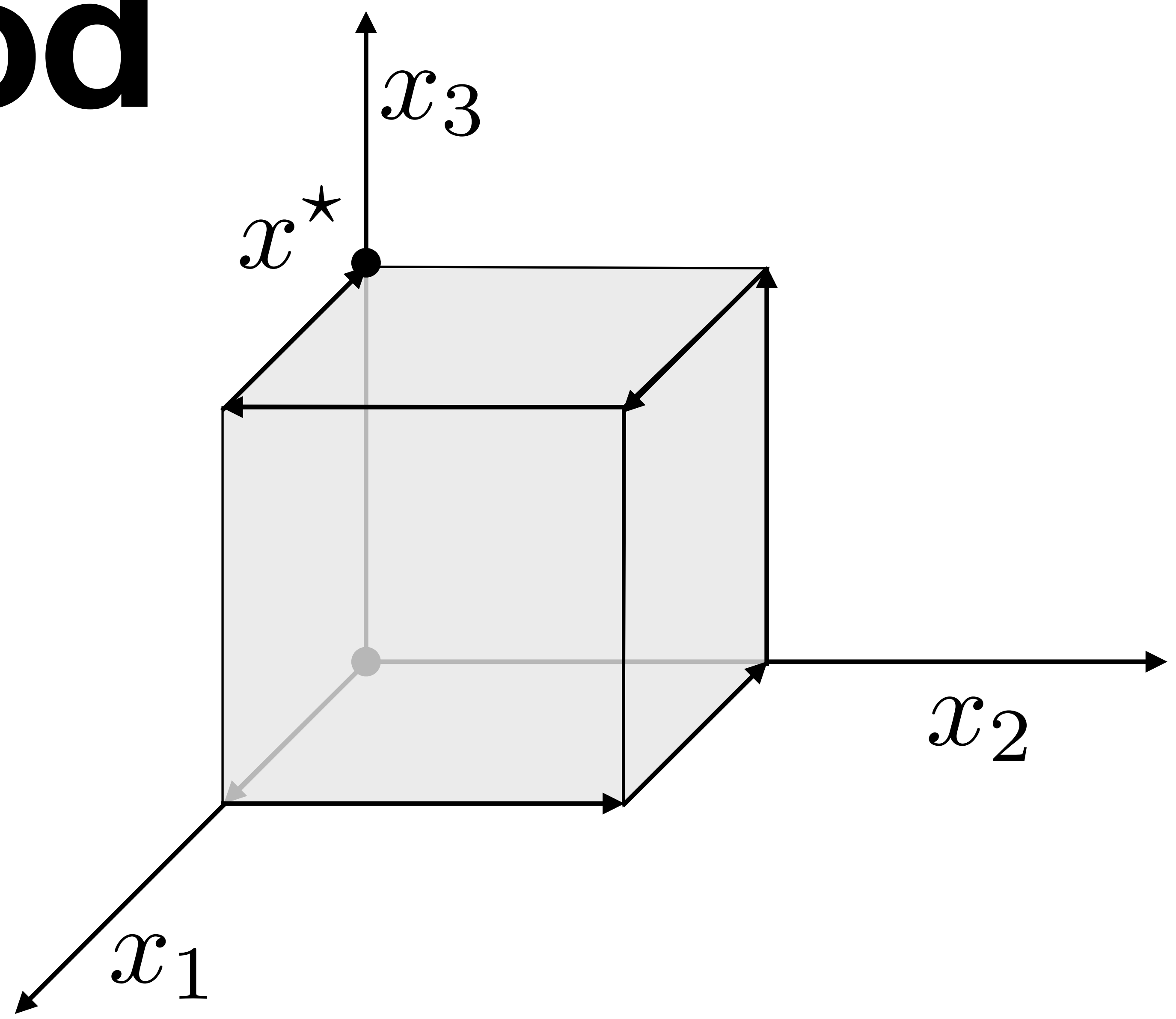
minimize $\quad -x_n$

subject to $\quad \epsilon \leq x_1 \leq 1$

$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n$$

Complexity of the simplex method

Example of worst-case behavior



$$
\begin{array}{ll}
\text{minimize} & -x_n \\
\text{subject to} & \epsilon \leq x_1 \leq 1 \\
& \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n
\end{array}
$$

Theorem

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations**

Remark

- A **different pivot rule** would have converged in one iteration.
- We have a bad example for every pivot rule.

Complexity of the simplex method

We do **not know any polynomial version of the simplex method**, no matter which pivoting rule we pick. → Still **open research question!**

Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions, e.g. 2^n

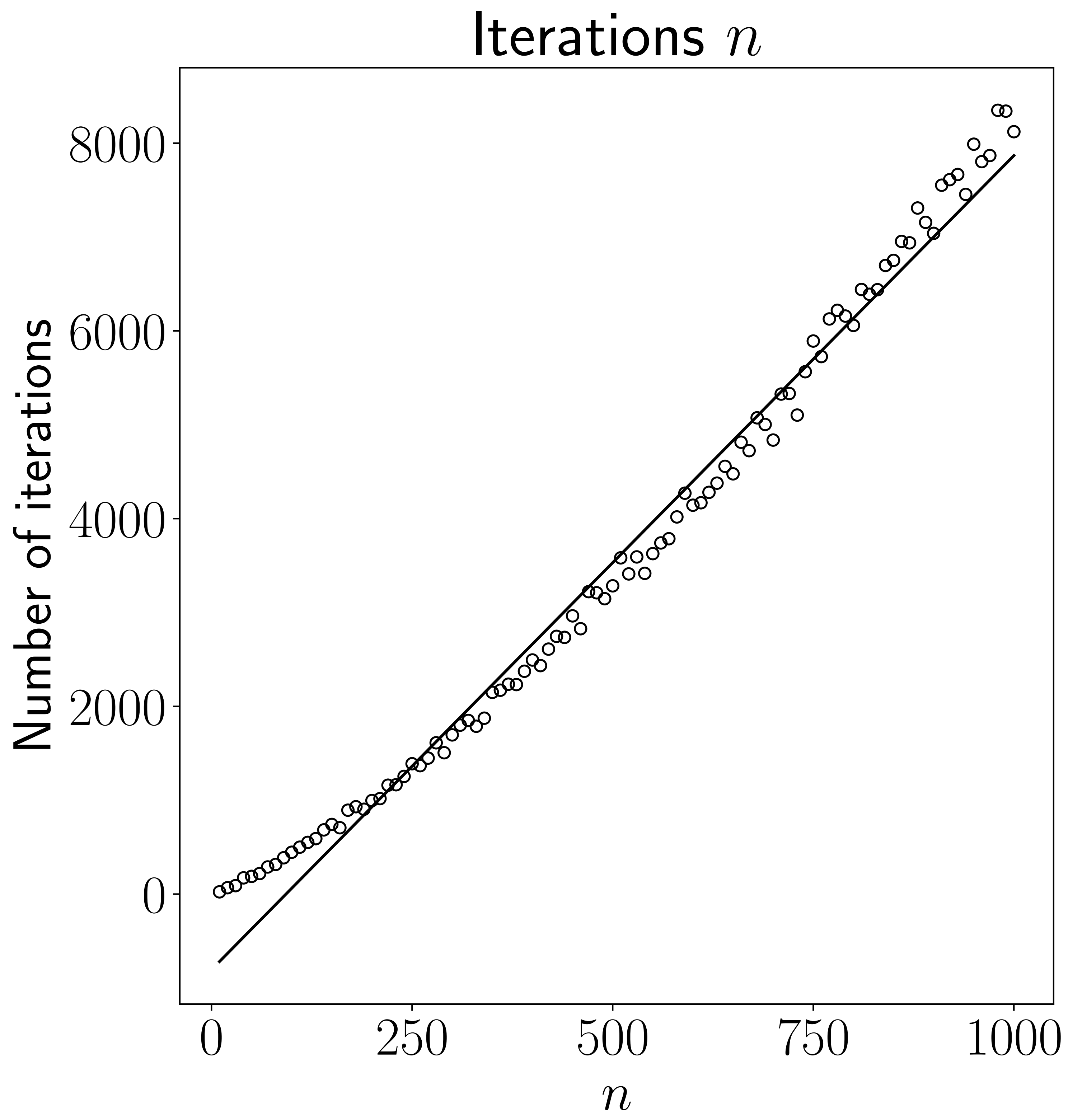
Good news: average-case
Practical performance is very good. On average, it stops in n iterations.

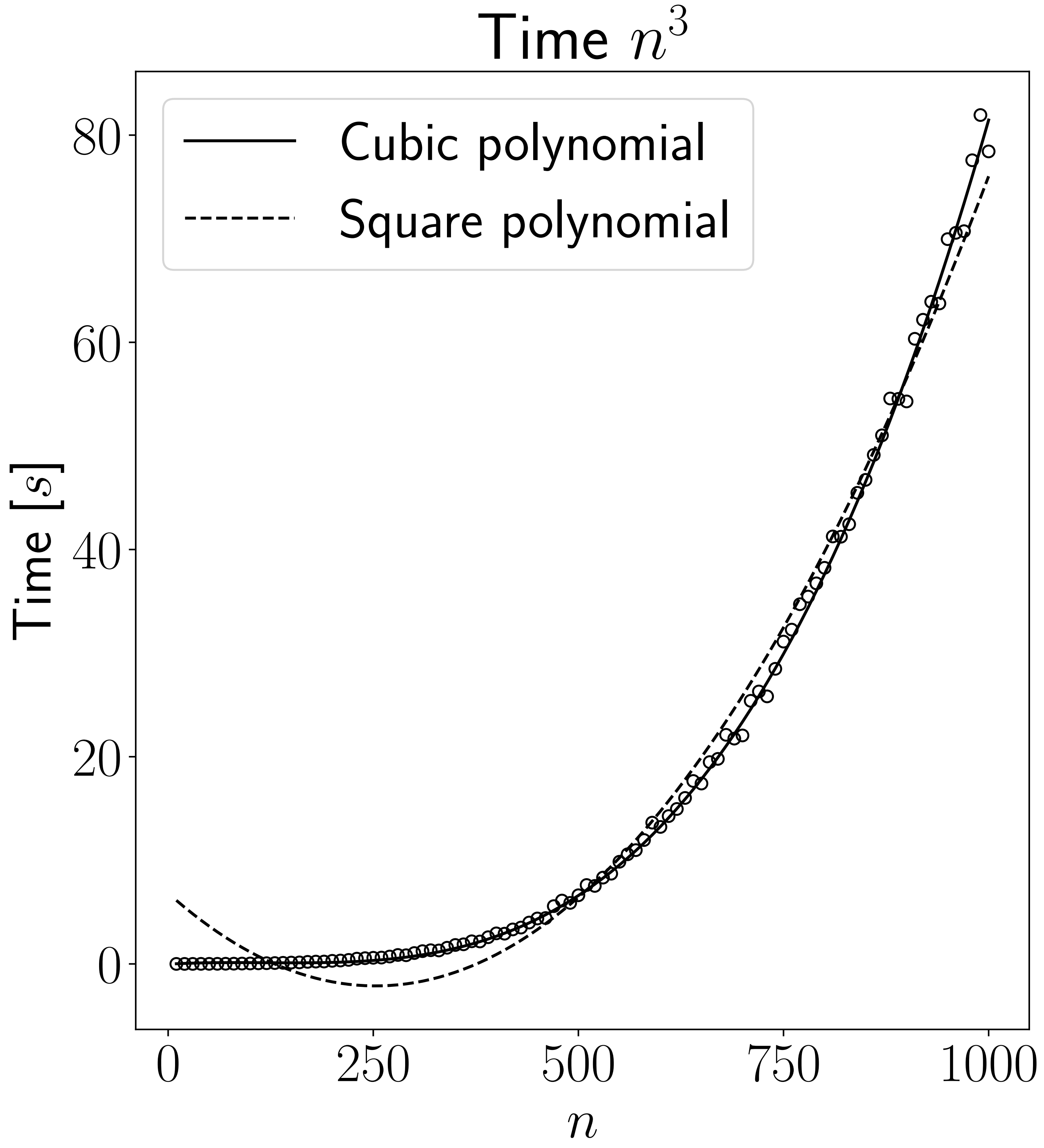
Average simplex complexity

Random LPs

minimize $c^T x$
subject to $Ax \leq b$

n variables
$3n$ constraints





The simplex method implementation

Today, we learned to:

- **Find** an initial basic feasible solution (Phase-I/II Simplex)
- **Deal** with degenerate basic feasible solution (Bland’s rule)
- **Compute** the simplex method complexity (**per iteration** and **overall)**

References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 3: The simplex method
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 3: Degeneracy
  - Chapter 4: Efficiency of the simplex method
  - Chapter 8: Implementation issues

Next lecture

- Duality