

ORF307 – Optimization

9. Geometry and polyhedra

Bartolomeo Stellato— Spring 2025

Today's lecture

Geometry and polyhedra

- Simple example
- Polyhedra
- Corners: extreme points, vertices, basic feasible solutions
- Constructing basic solutions
- Existence and optimality of extreme points

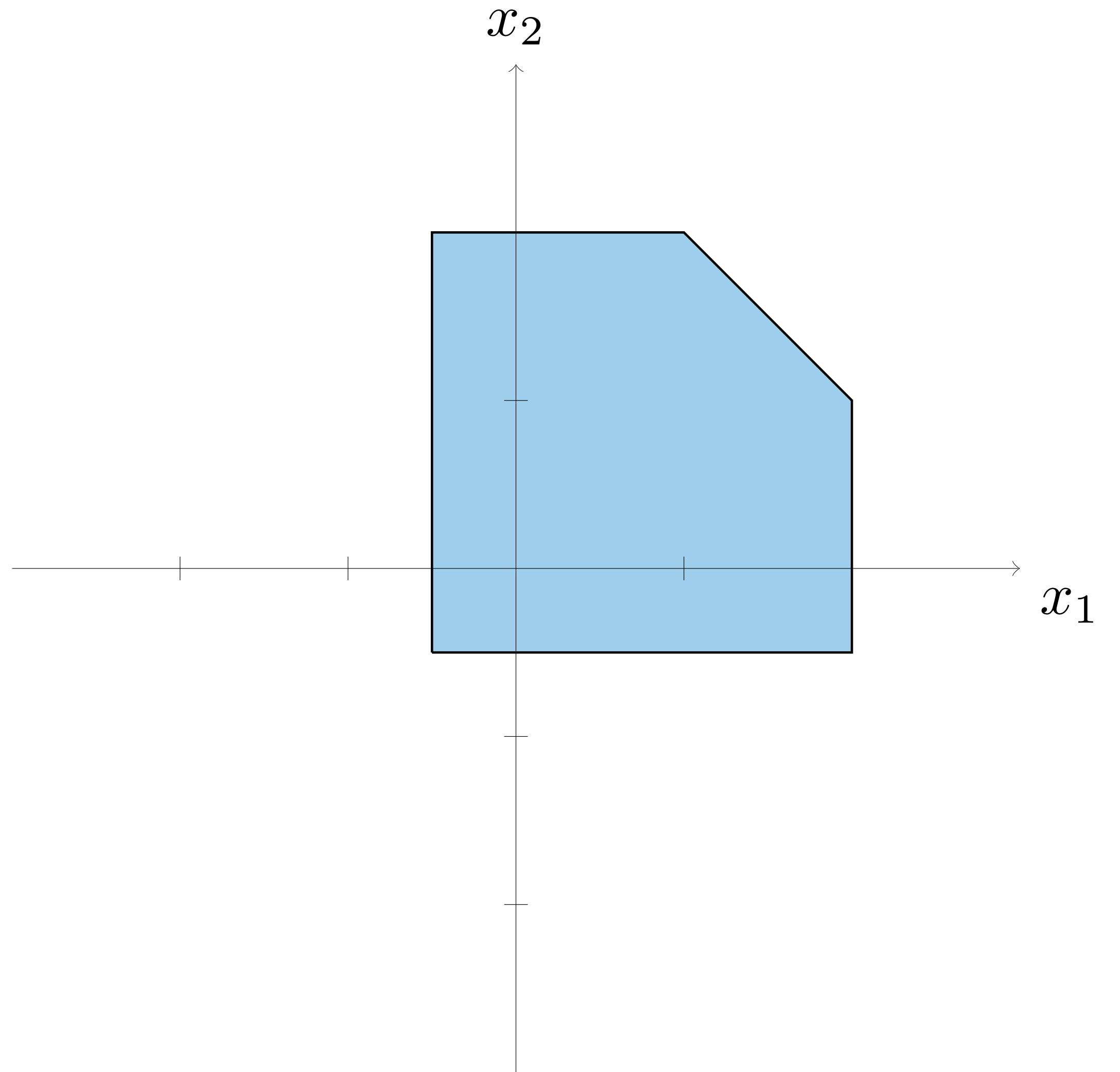
A simple example

minimize $c^T x$

subject to $-1/2 \leq x_1 \leq 2$

$-1/2 \leq x_2 \leq 2$

$x_1 + x_2 \leq 3$



**What kind of optimal
solutions do we get?**

A simple example

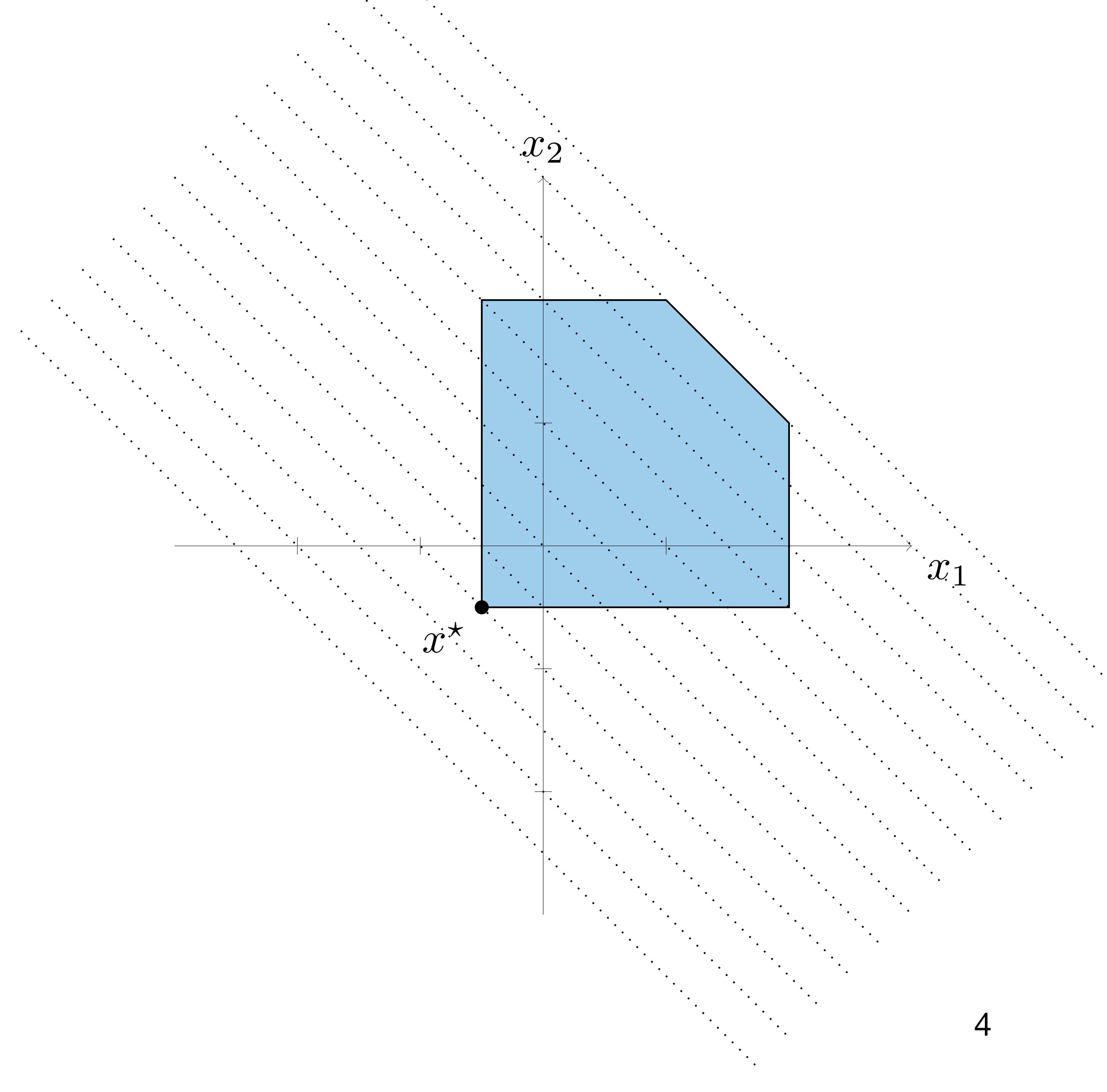
minimize $c^T x$

subject to $-1/2 \leq x_1 \leq 2$

$-1/2 \leq x_2 \leq 2$

$x_1 + x_2 \leq 3$

Suppose $c = (1, 1)$



A simple example

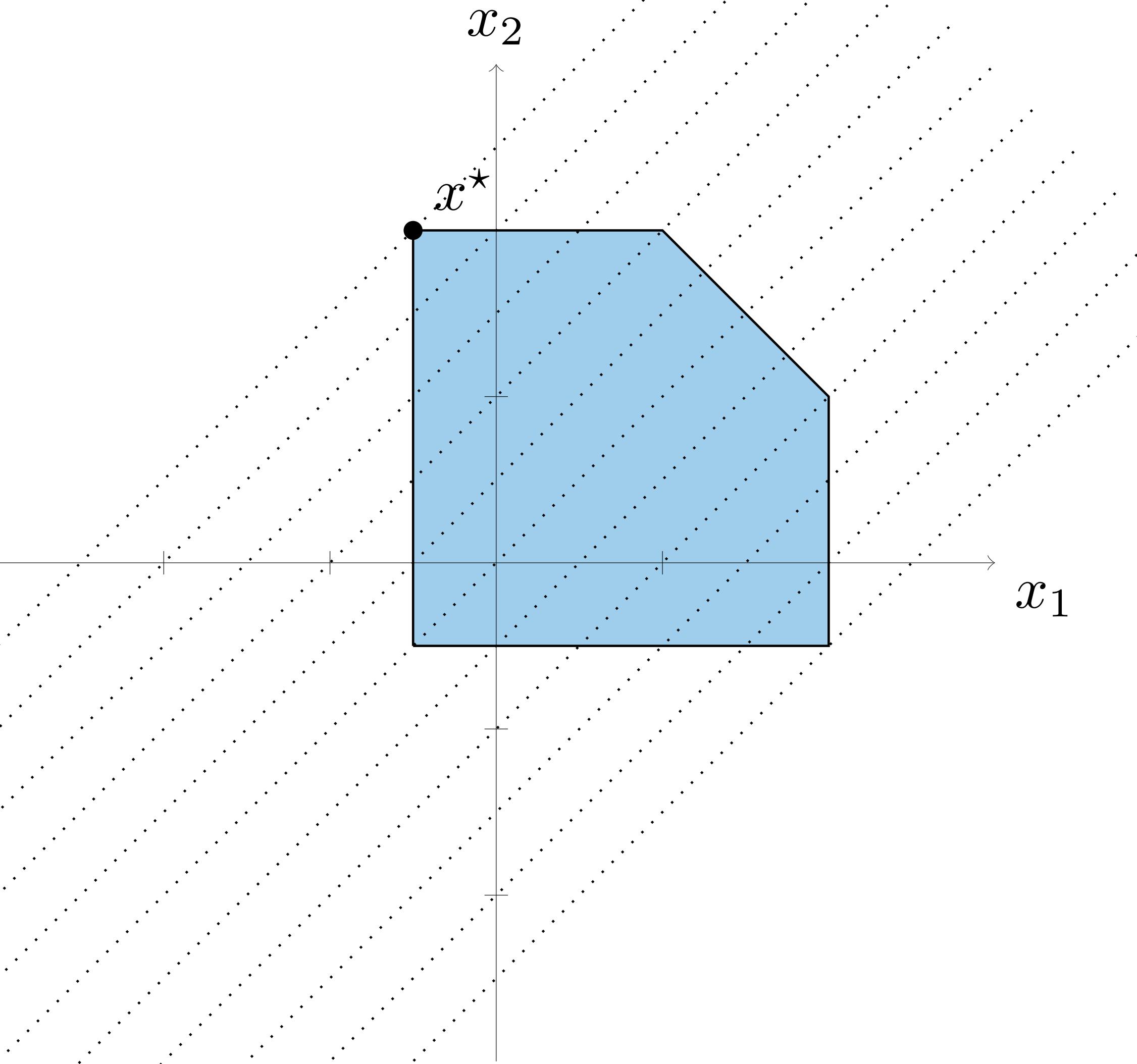
minimize $c^T x$

subject to $-1/2 \leq x_1 \leq 2$

$-1/2 \leq x_2 \leq 2$

$x_1 + x_2 \leq 3$

Suppose $c = (1, -1)$



A simple example

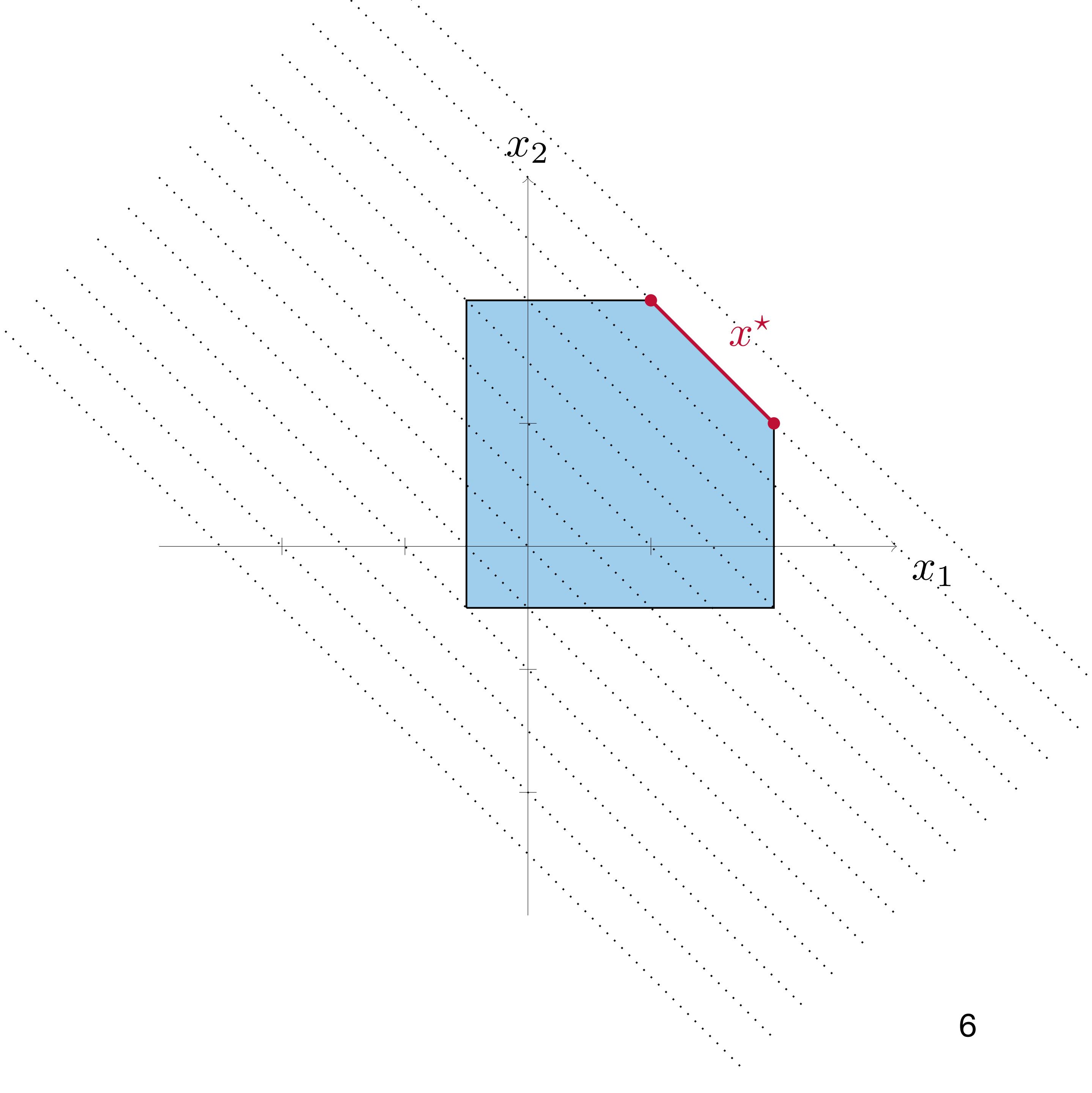
minimize $c^T x$

subject to $-1/2 \leq x_1 \leq 2$

$-1/2 \leq x_2 \leq 2$

$x_1 + x_2 \leq 3$

Suppose $c = (-1, -1)$



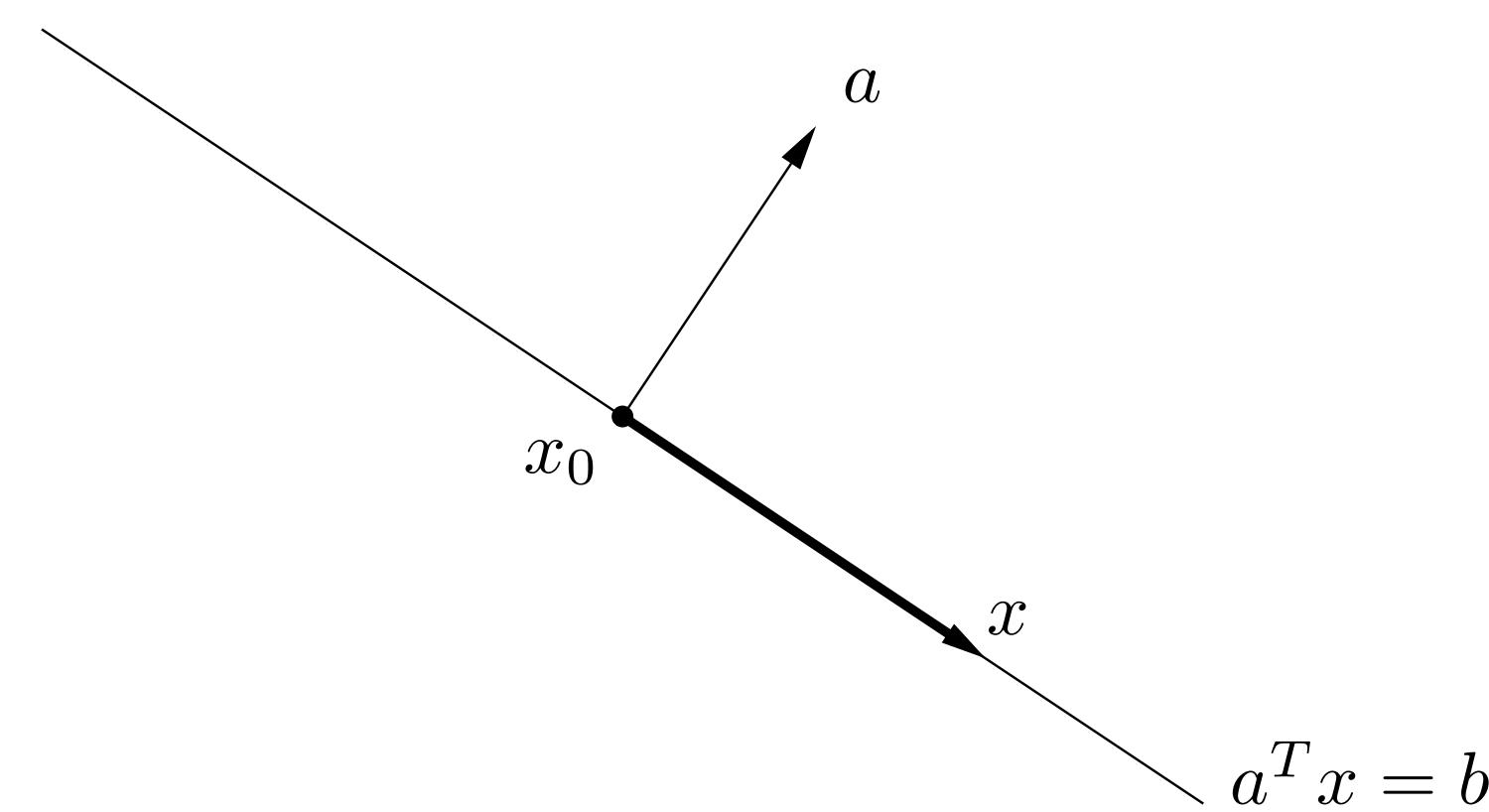
Polyhedra and linear algebra

Hyperplanes and halfspaces

Definitions

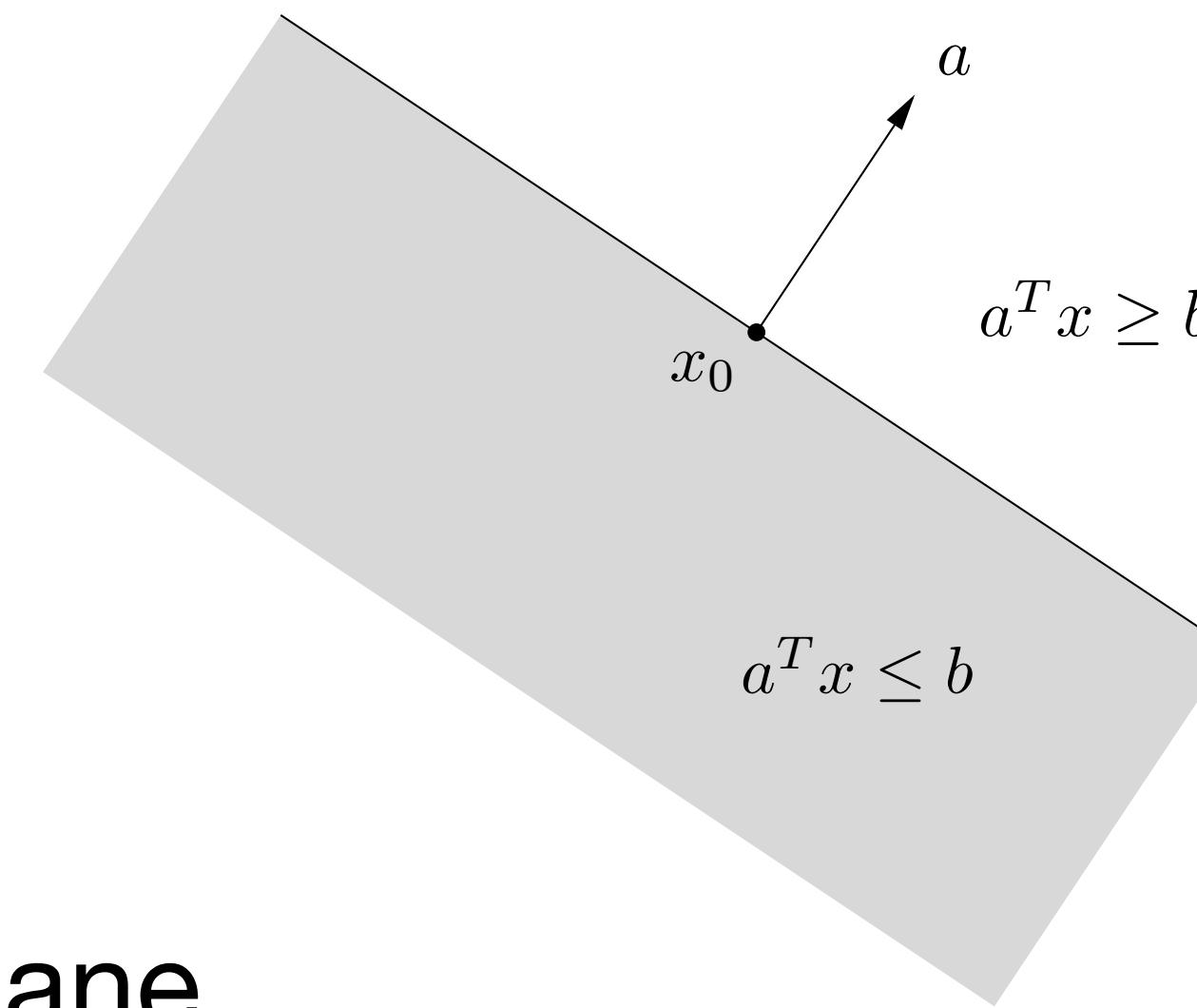
Hyperplane

$$\{x \mid a^T x = b\}$$



Halfspace

$$\{x \mid a^T x \leq b\}$$

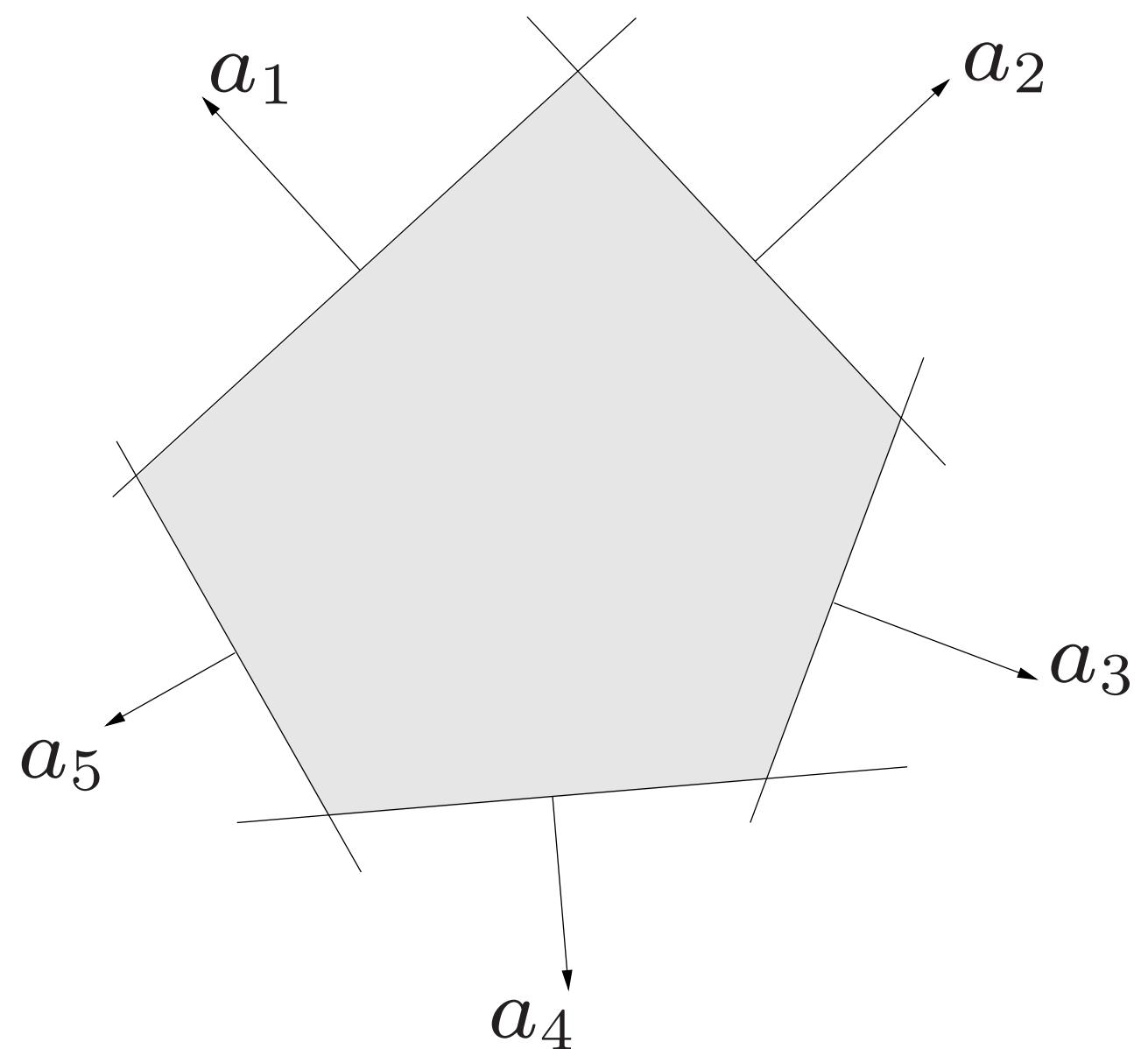


- x_0 is a specific point in the hyperplane
- For any x in the hyperplane defined by $a^T x = b$, $x - x_0 \perp a$
- The halfspace determined by $a^T x \leq b$ extends in the direction of $-a$

Polyhedron

Definition

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\} = \{x \mid Ax \leq b\}$$



- Intersection of finite number of halfspaces
- Can include equalities

Polyhedron Example

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\} = \{x \mid Ax \leq b\}$$

minimize $c^T x$

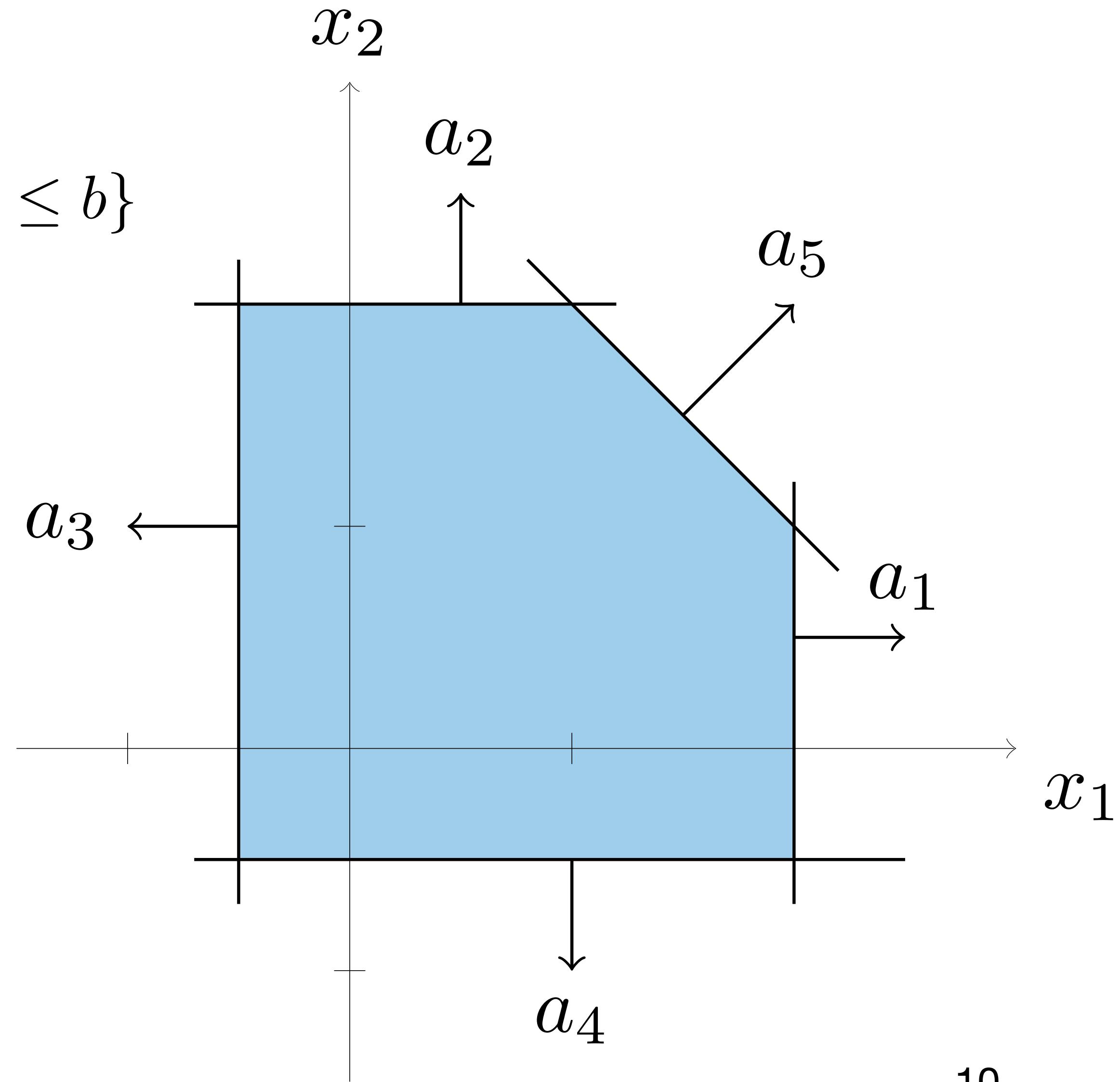
subject to $x_1 \leq 2$

$x_2 \leq 2$

$x_1 \geq -1/2$

$x_2 \geq -1/2$

$x_1 + x_2 \leq 3$



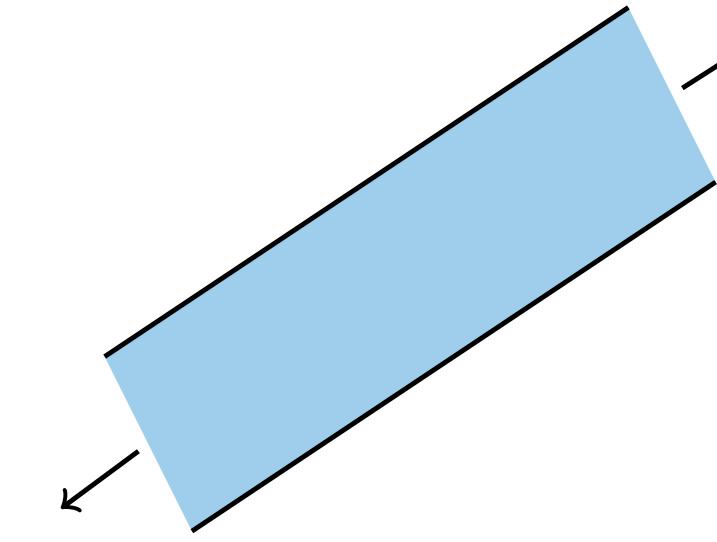
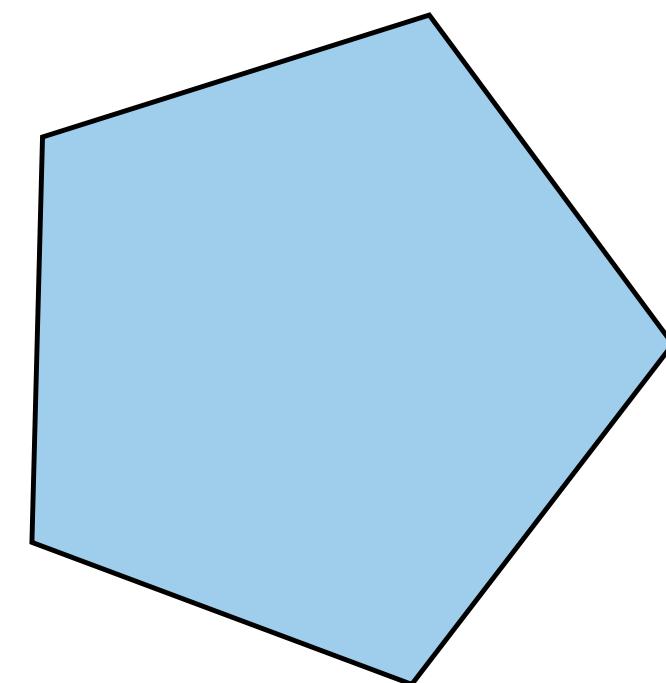
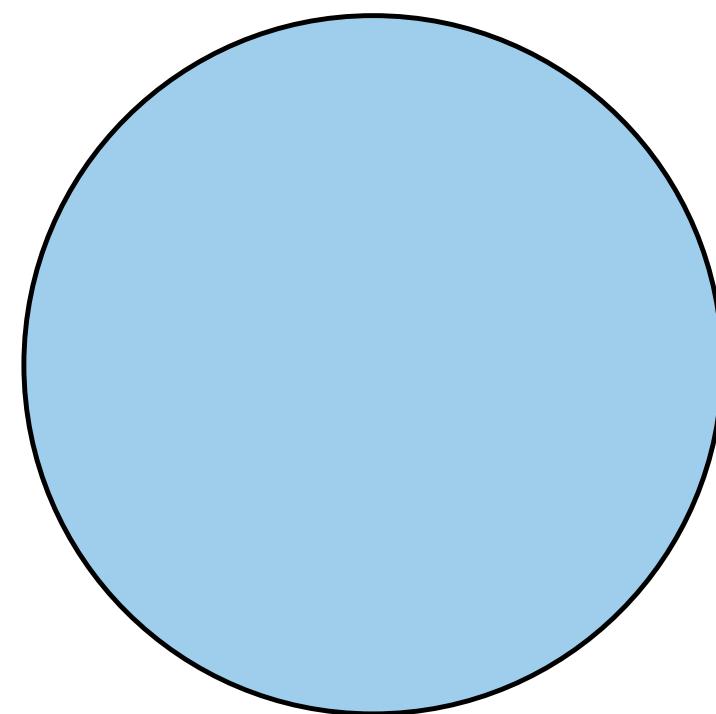
Convex set

Definition

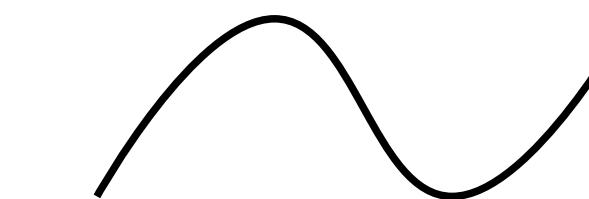
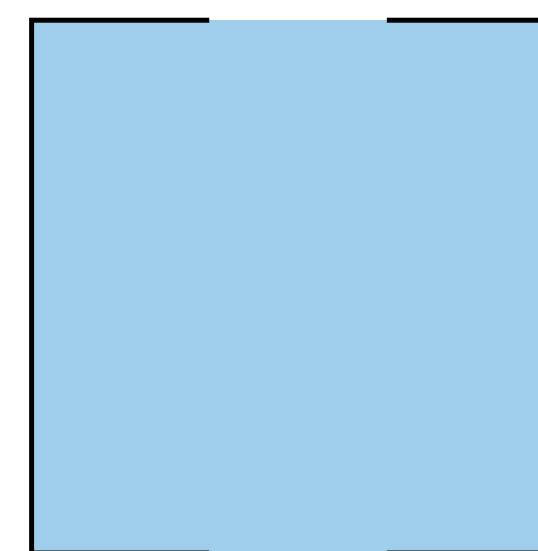
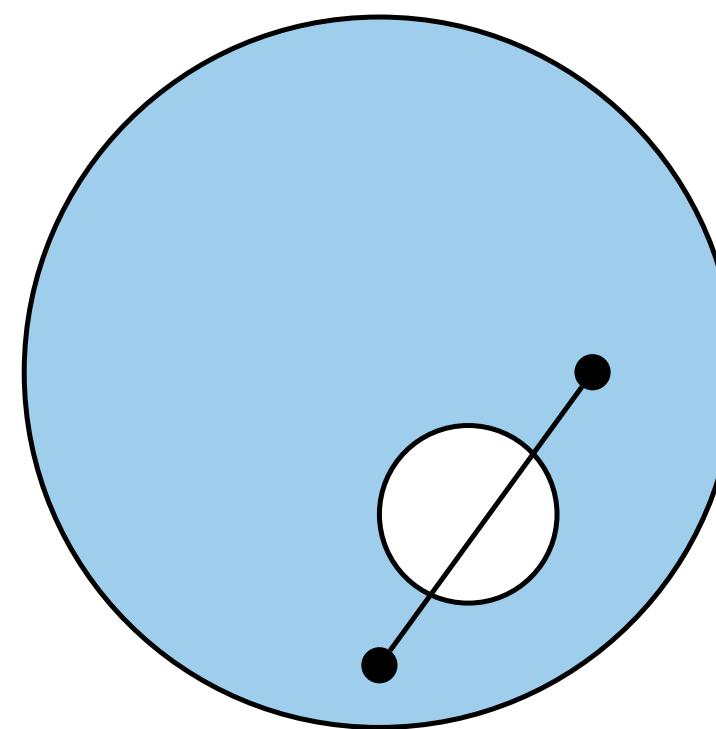
For any $x, y \in C$ and any $\alpha \in [0, 1]$

$\alpha x + (1 - \alpha)y \in C$

Convex



Nonconvex



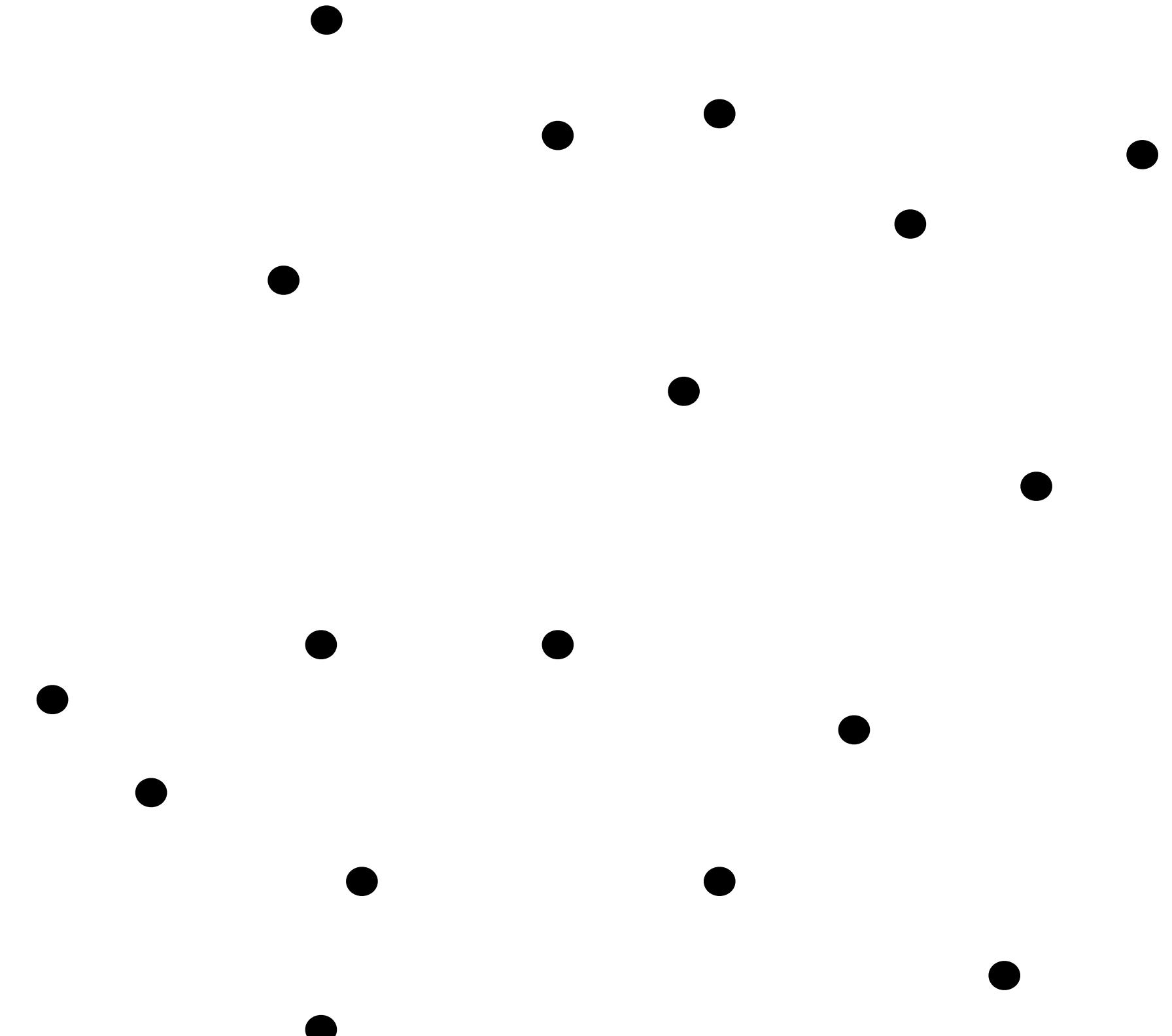
Examples

- \mathbf{R}^n
- Hyperplanes
- Halfspaces
- Polyhedra

Convex combinations

Ingredients :

- A collection of points $C = \{x_1, \dots, x_k\}$
- A collection of non-negative weights α_i
- The weights α_i sum to 1

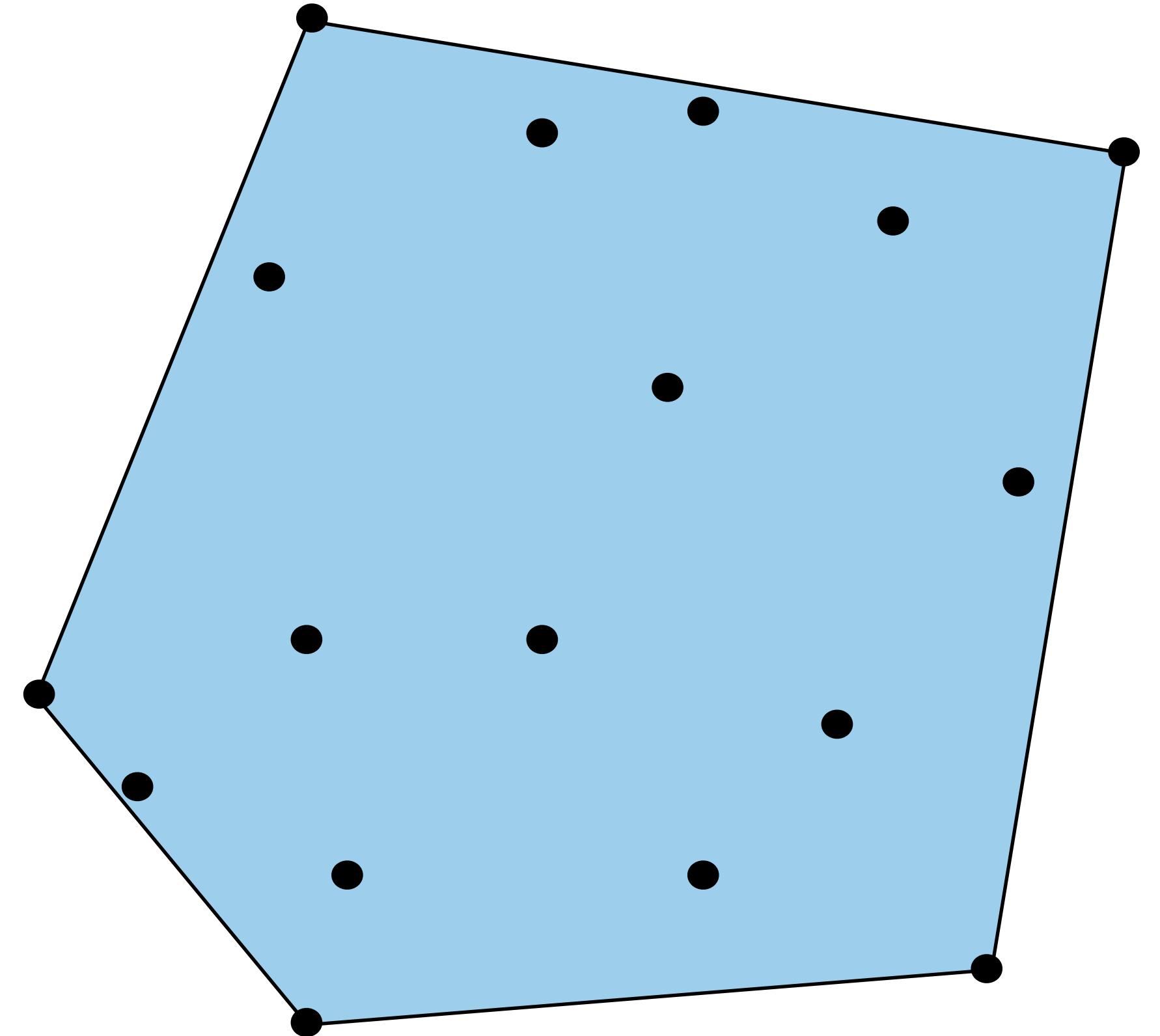


The vector $v = \alpha_1 x_1 + \dots + \alpha_k x_k$ is a **convex combination** of the points.

Convex hull

The **convex hull** is the set of all possible convex combinations of the points.

$$\text{conv } C = \left\{ \sum_{i=1}^n \alpha_i x_i \mid \alpha_i \geq 0, \ i = 1, \dots, n, \ \mathbf{1}^T \alpha = 1 \right\}$$

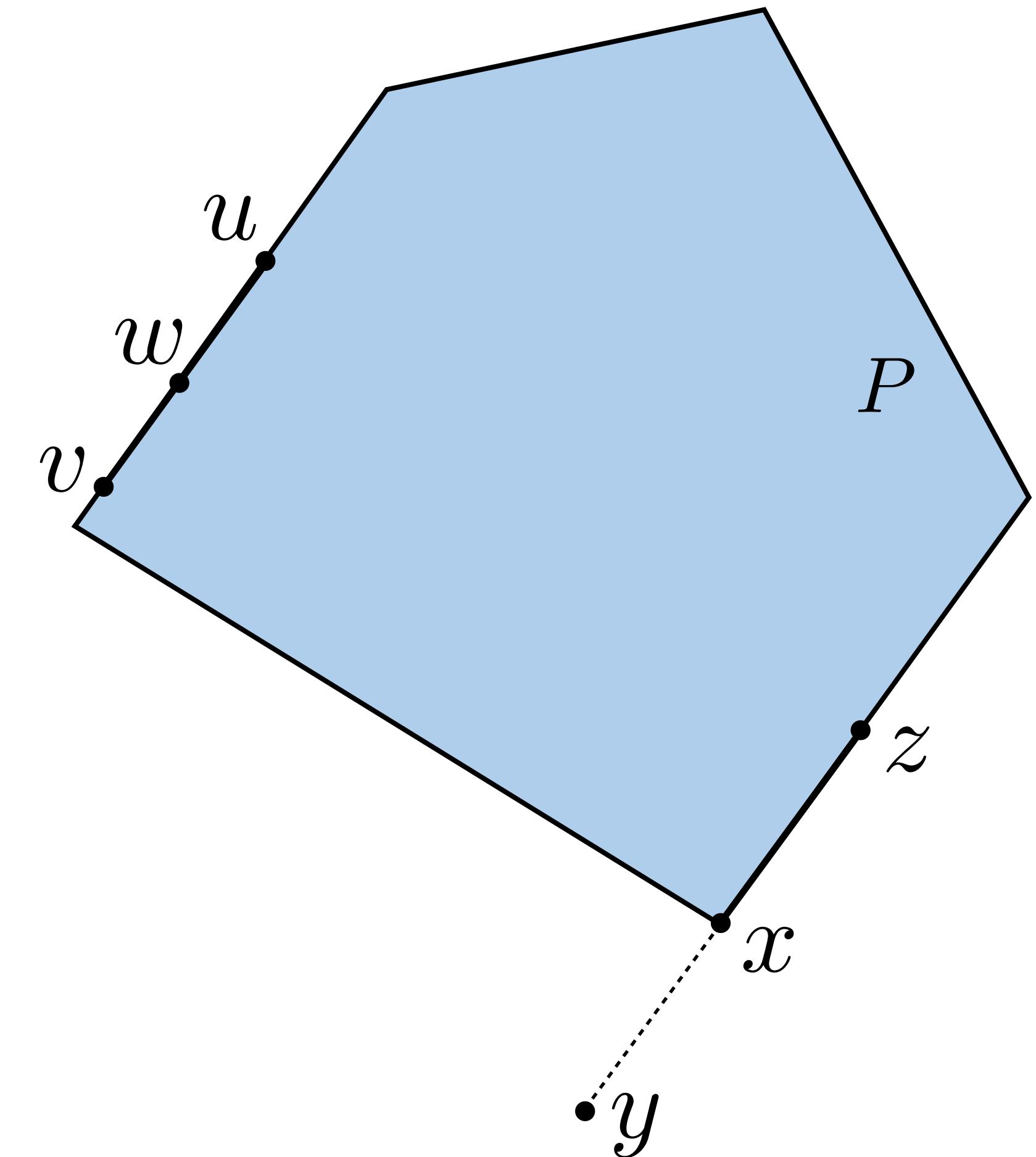


Corners

Extreme points

Definition:

An **extreme point** of a set is one not on a straight line between any other points in the set.

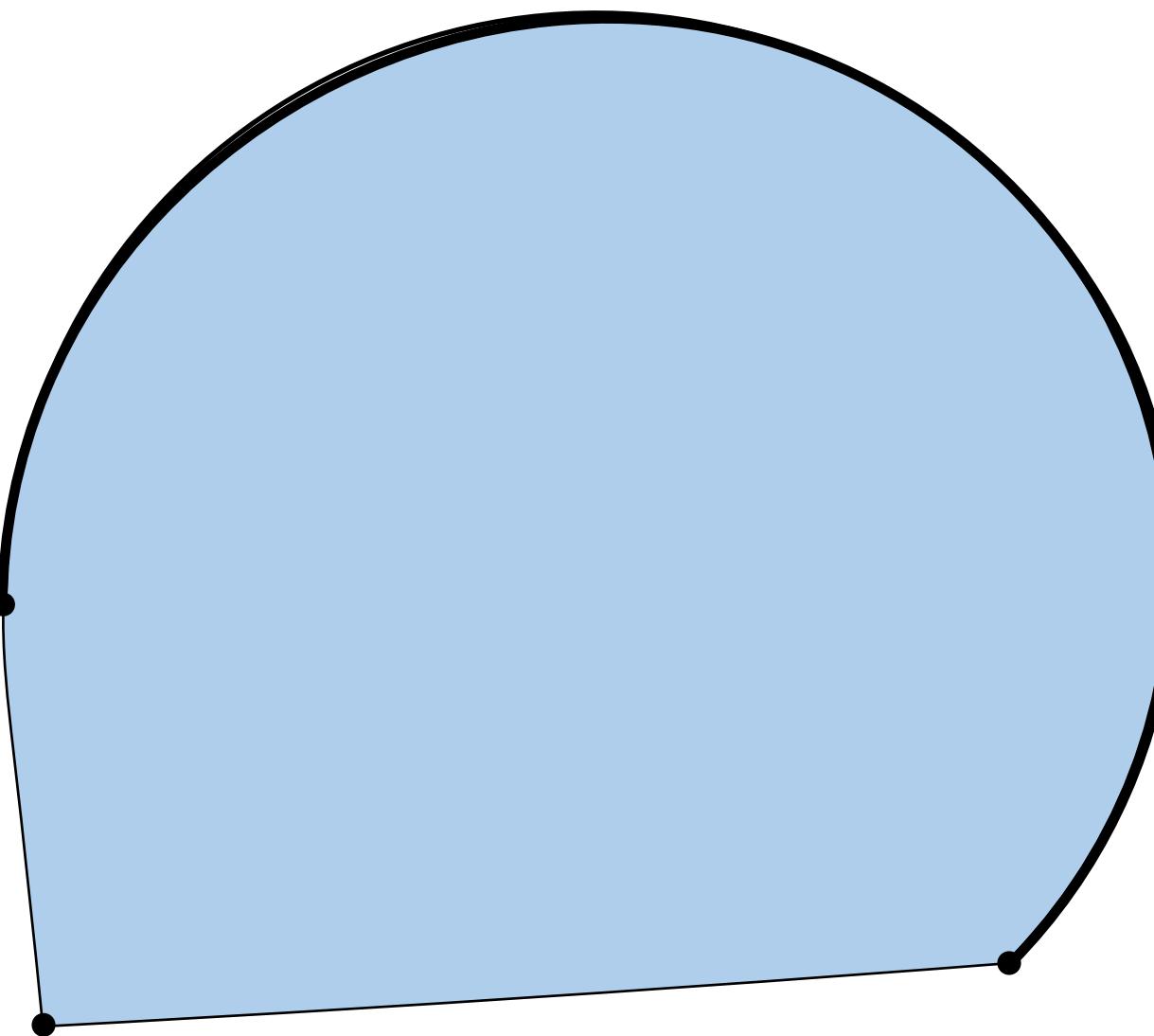


More formal definition:

The point $x \in P$ is an **extreme point** of P if

$\nexists y, z \in P$ ($y \neq x, z \neq x$) and $\alpha \in [0, 1]$ such that $x = \alpha y + (1 - \alpha)z$

Extreme points

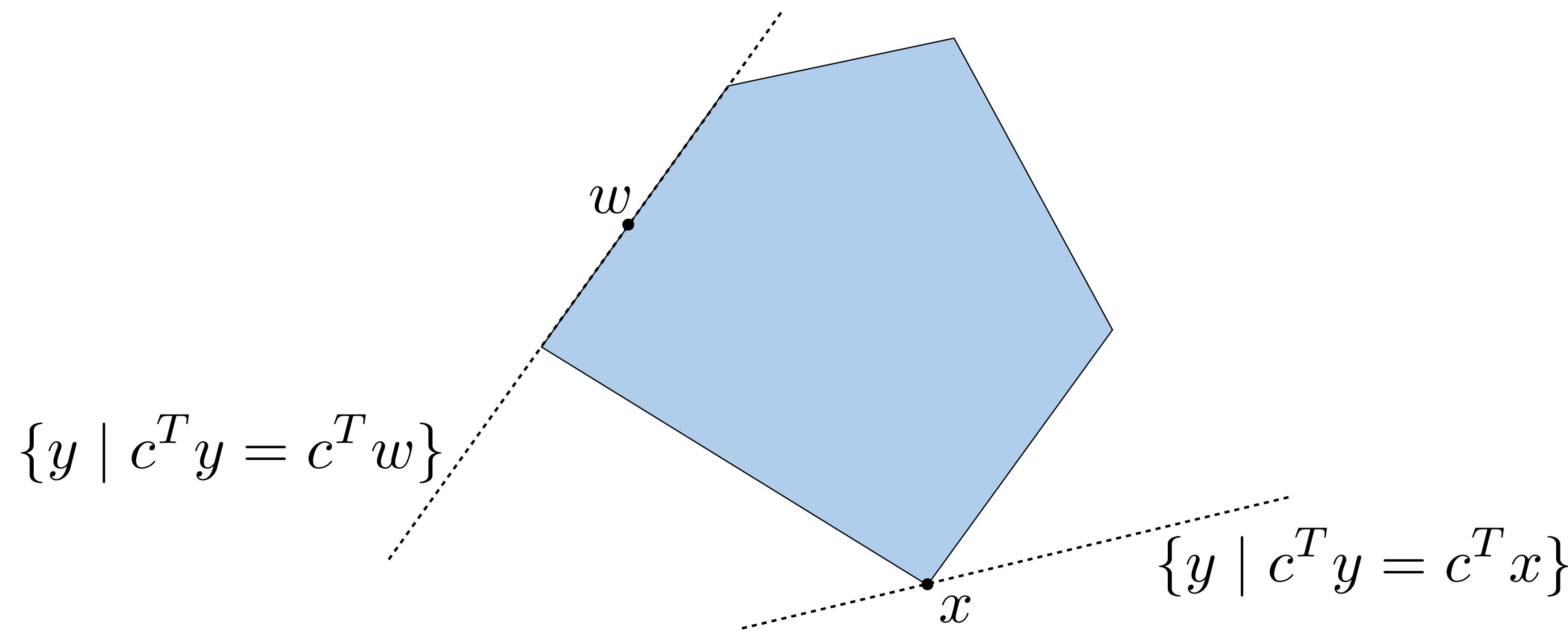


- General convex sets can have an infinite number of extreme points
- **Polyhedra** are convex sets with a finite number of extreme points

Vertices

The point $x \in P$ is a **vertex** if $\exists c$ such that x is the unique optimum of

$$\begin{array}{ll}\text{minimize} & c^T y \\ \text{subject to} & y \in P\end{array}$$



Basic feasible solution

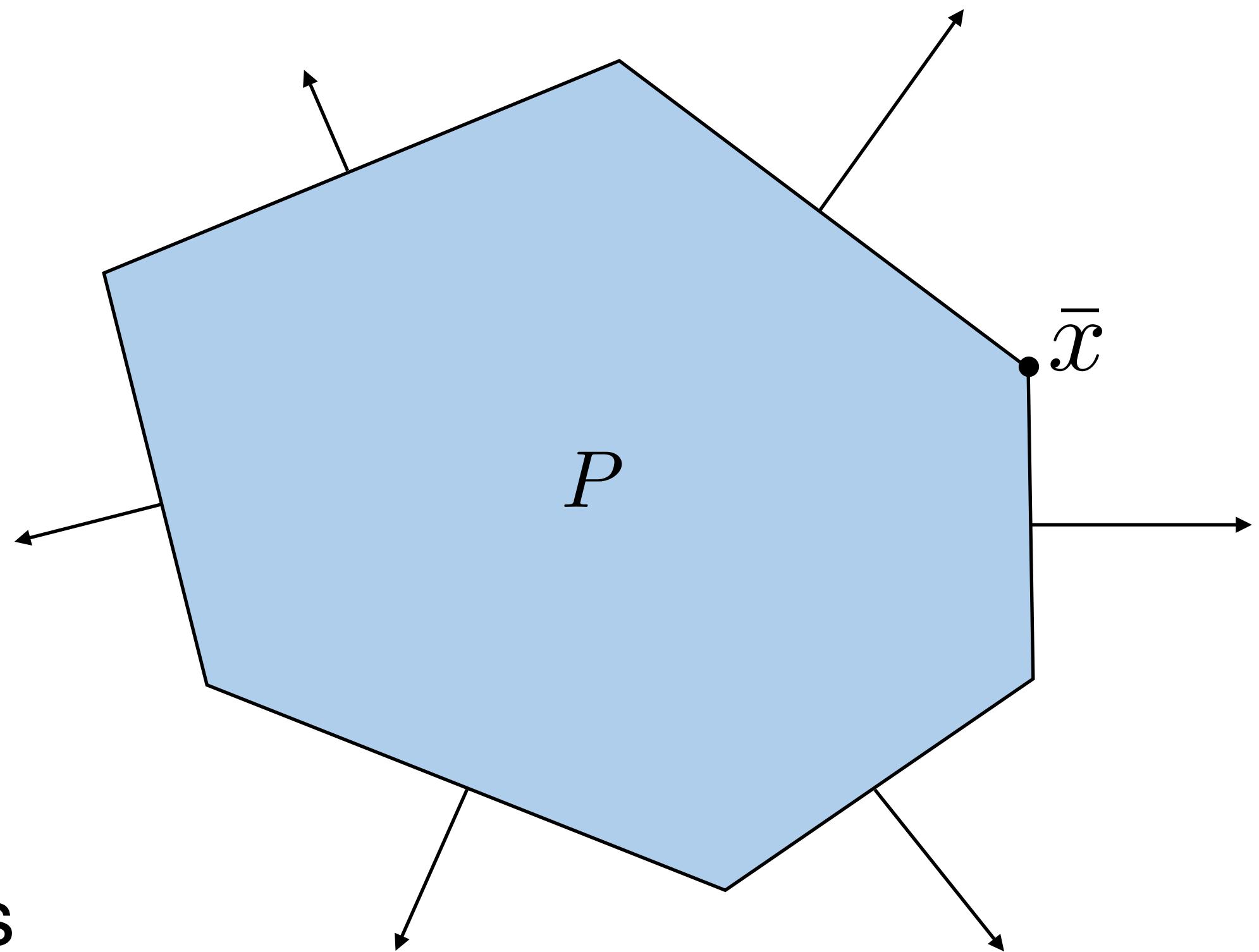
Assume we have a polytope $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$

Active constraints at \bar{x}

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \dots, m\} \mid a_i^T \bar{x} = b_i\}$$

Basic feasible solution $\bar{x} \in P$

$\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

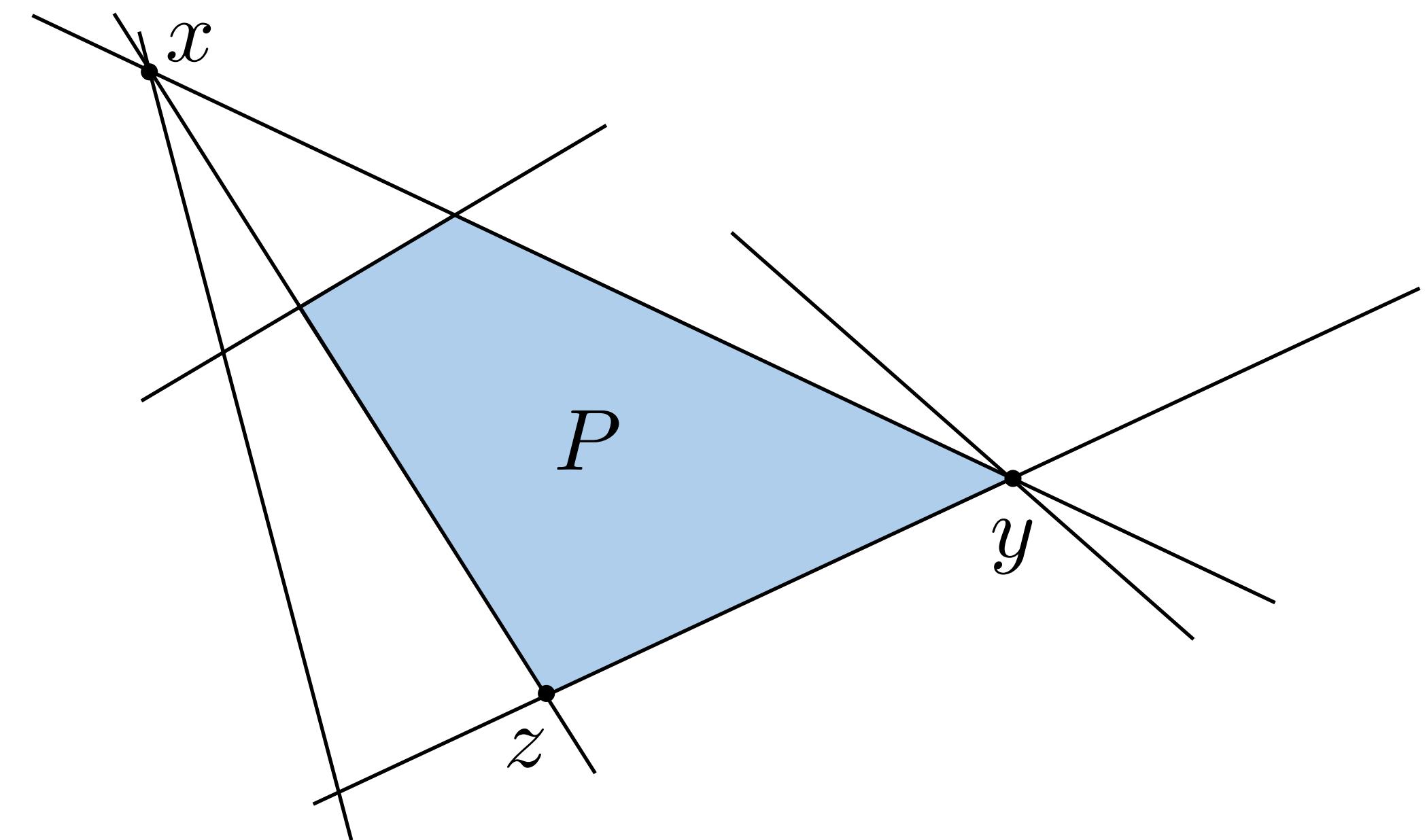


Degenerate basic feasible solutions

A solution \bar{x} is **degenerate** if $|\mathcal{I}(\bar{x})| > n$

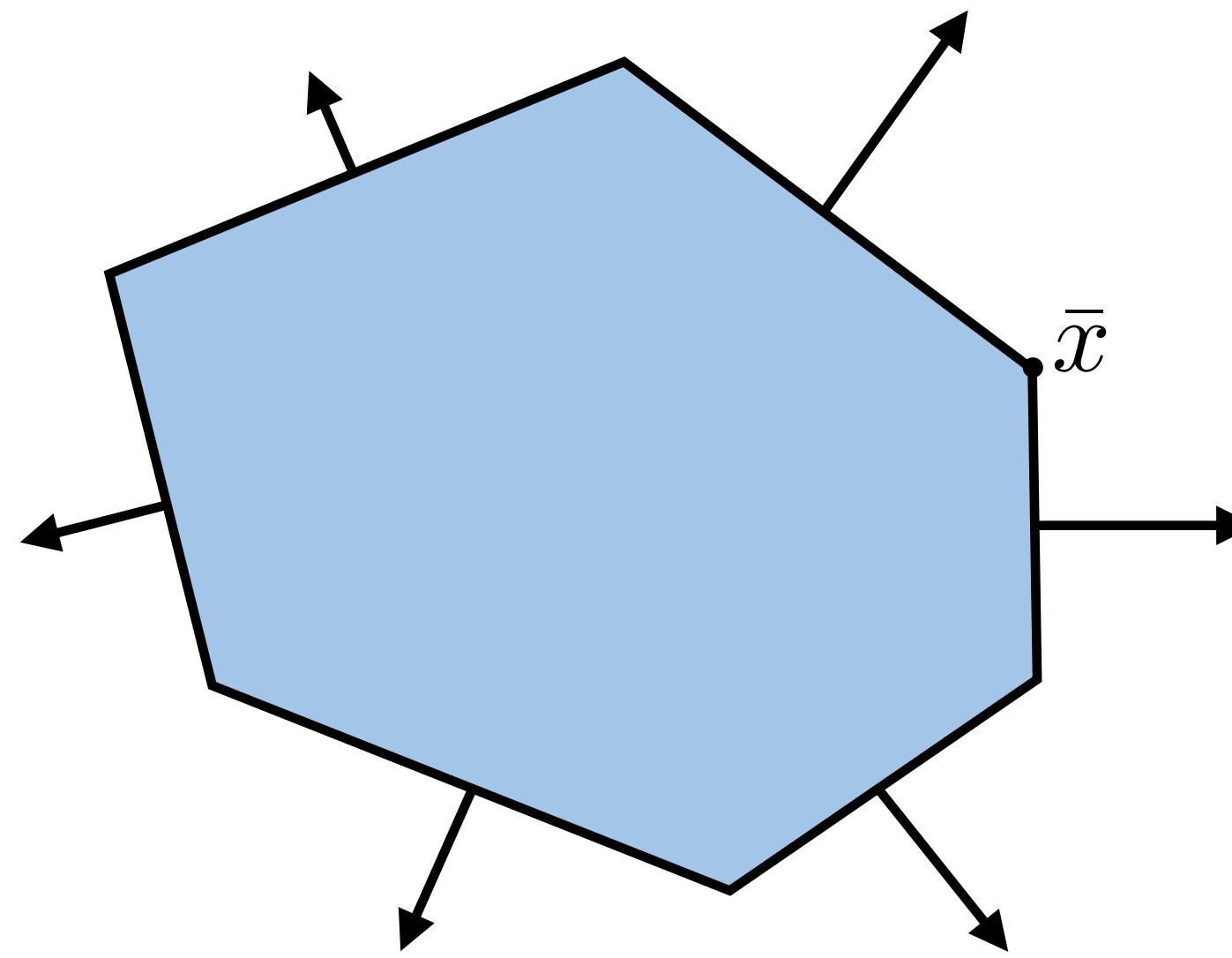
True or False?

Basic	Feasible	Degenerate
x		
y		
z		



An Equivalence Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



x is a **vertex** \iff x is an **extreme point** \iff x is a **basic feasible solution**

Equivalent theorem proof

Vertex → Extreme point

If x is a vertex, $\exists c$ such that $c^T x < c^T y, \quad \forall y \in P, y \neq x$

Let's assume x is not an extreme point:

$\exists y, z \neq x$ such that $x = \lambda y + (1 - \lambda)z$

Since x is a vertex, $c^T x < c^T y$ and $c^T x < c^T z$

Therefore, $c^T x = \lambda c^T y + (1 - \lambda)c^T z > \lambda c^T x + (1 - \lambda)c^T x = c^T x$

⇒ contradiction



Equivalent theorem proof

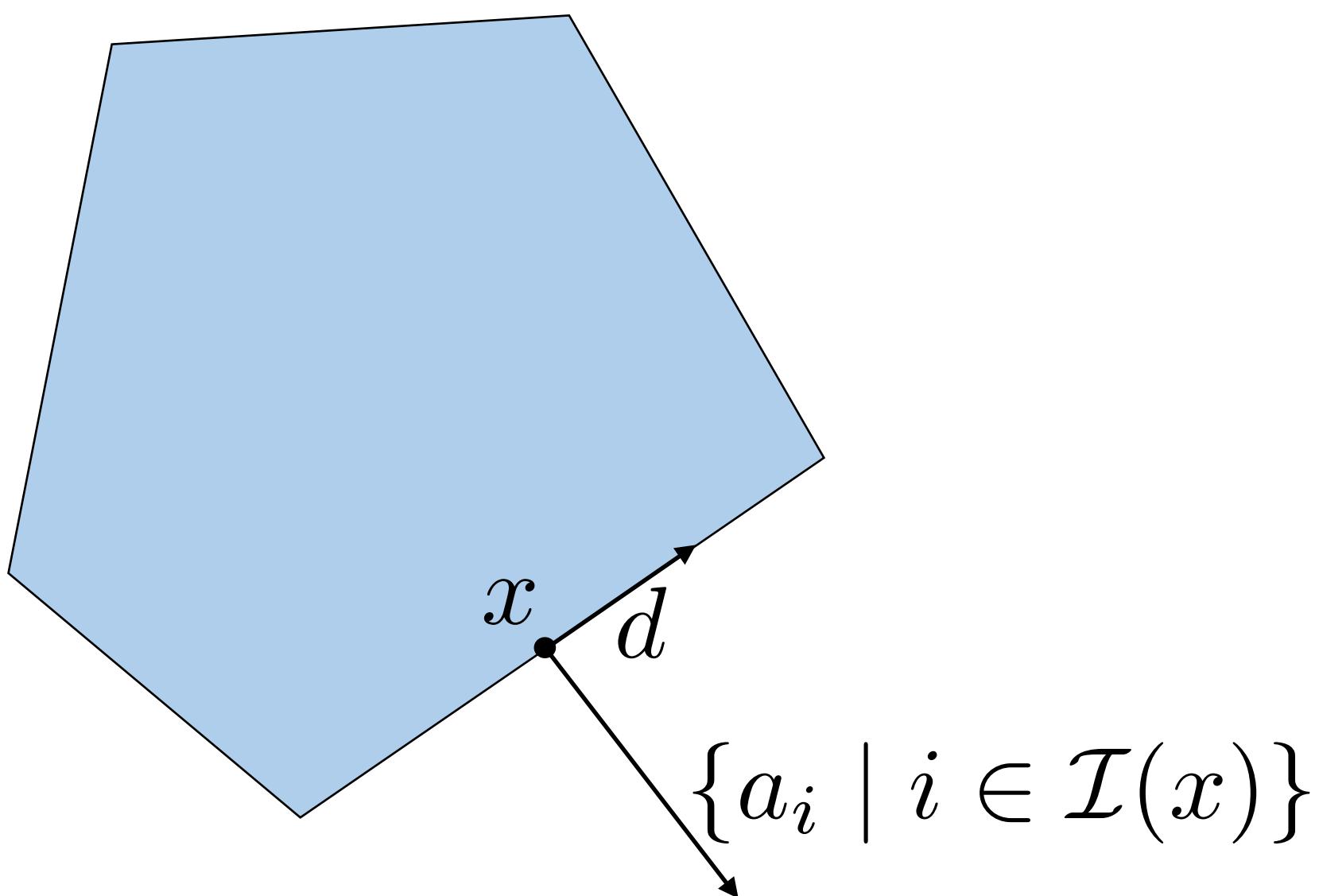
Extreme point → Basic feasible solution

(proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span \mathbf{R}^n

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$



Equivalent theorem proof

Extreme point → Basic feasible solution

(proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span \mathbf{R}^n

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$

Let $\epsilon > 0$ and define $y = x + \epsilon d$ and $z = x - \epsilon d$

For $i \in \mathcal{I}(x)$ we have $a_i^T y = b_i$ and $a_i^T z = b_i$

For $i \notin \mathcal{I}(x)$ we have $a_i^T x < b_i \Rightarrow a_i^T (x + \epsilon d) < b_i$ and $a_i^T (x - \epsilon d) < b_i$

Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda)z$ with $\lambda = 0.5$.

$\implies x$ is **not an extreme point**

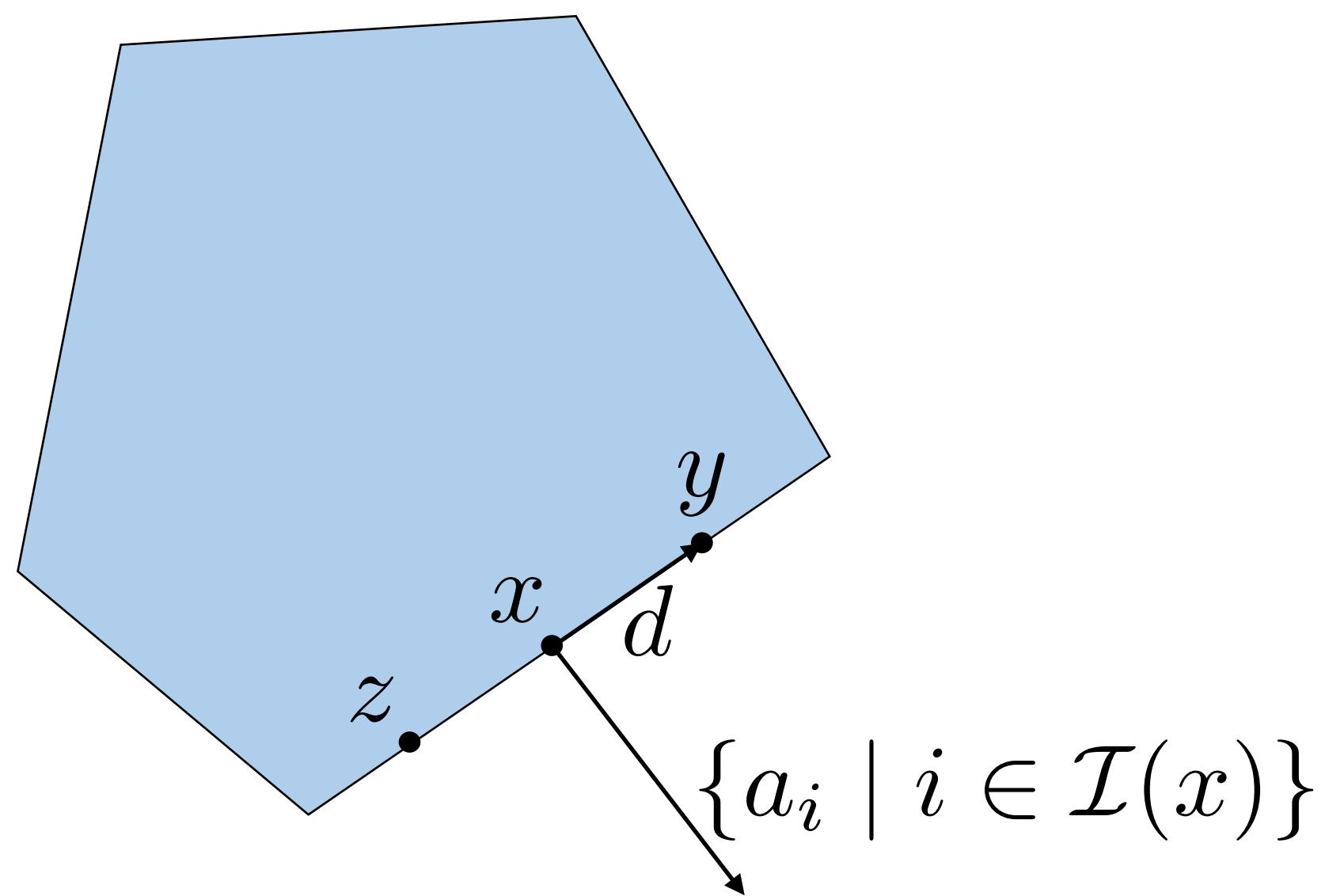


Equivalent theorem proof

Extreme point \rightarrow Basic feasible solution

(proof by contraposition)

Suppose $x \in P$ is not basic feasible solution



Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda)z$ with $\lambda = 0.5$.

$\implies x$ is not an extreme point



Equivalence theorem proof

Basic feasible solution → Vertex

Left as exercise

Hint

Define $c = \sum_{i \in \mathcal{I}(x)} a_i$

Constructing basic solutions

3D example

One equality ($m = 1, n = 3$)

minimize $c^T x$

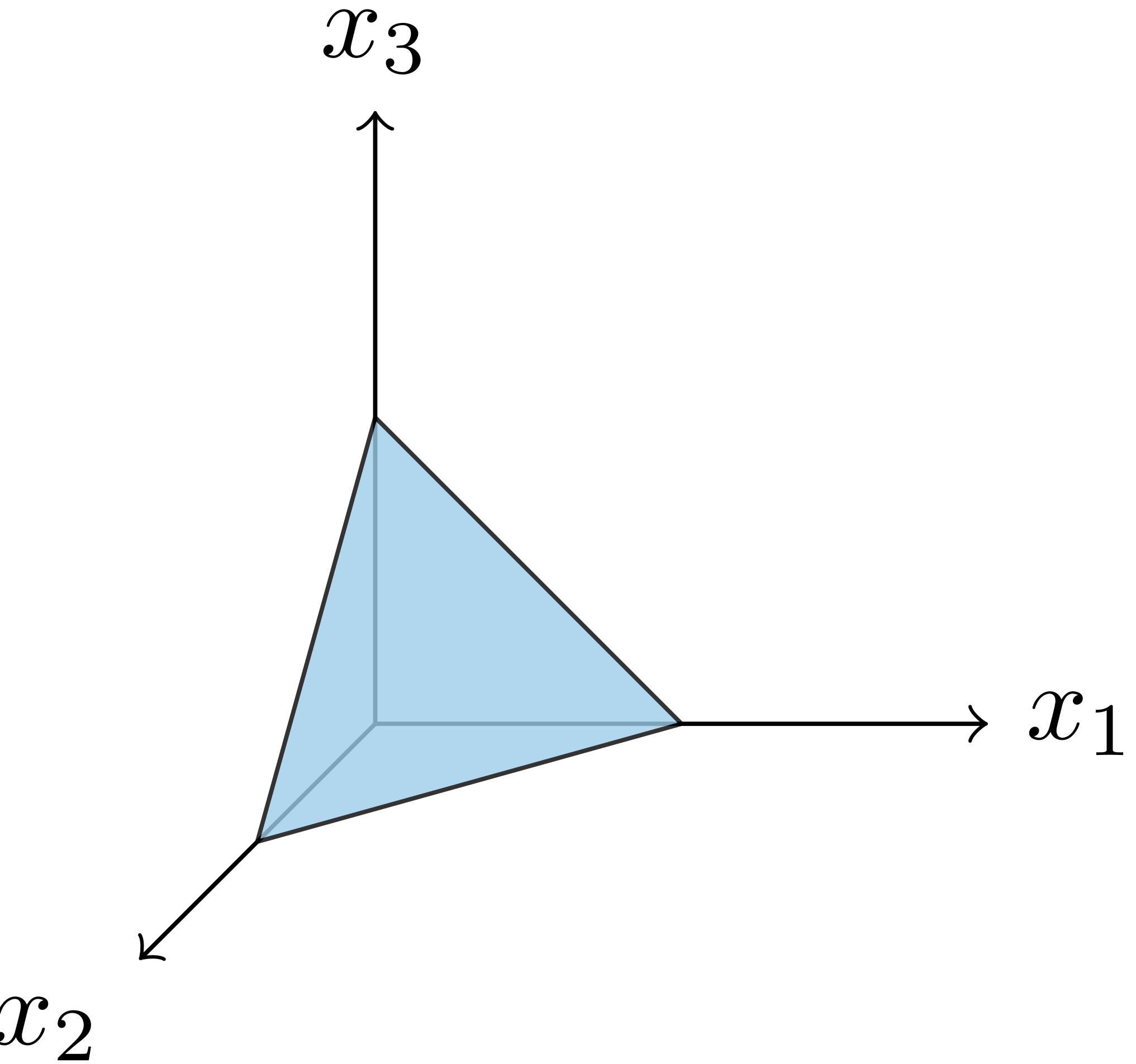
subject to $x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

Basic feasible solution \bar{x} has n linearly independent active constraints.



$n - m = 2$ inequalities have to be tight: $x_i = 0$



3D example

Two equalities $(m = 2, n = 3)$

minimize $c^T x$

subject to $x_1 + x_3 = 1$

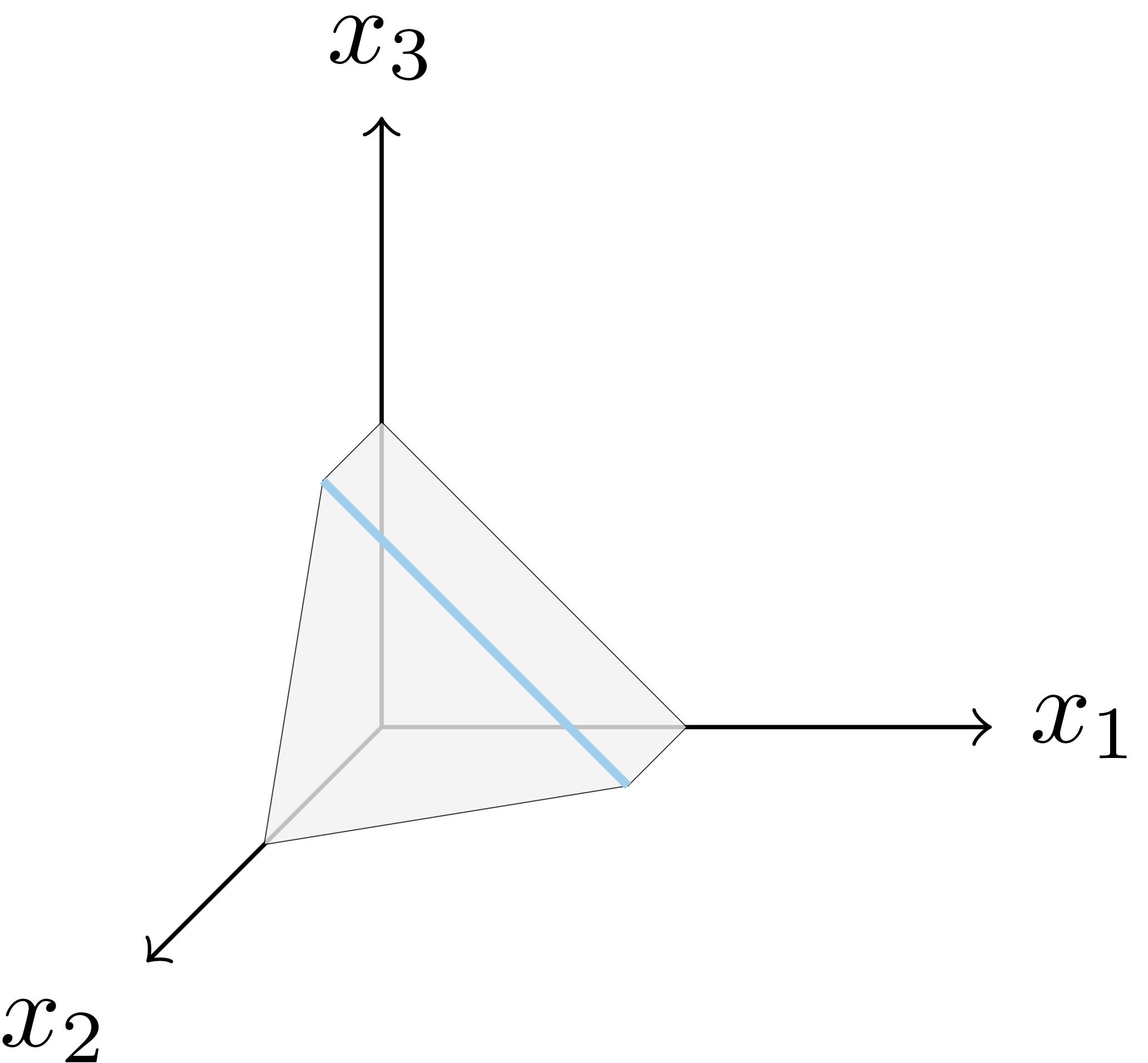
$$(1/2)x_1 + x_2 + (1/2)x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Basic feasible solution \bar{x} has n linearly independent active constraints.



$n - m = 1$ inequalities have to be tight: $x_i = 0$



Constructing a basic solution

Two equalities $(m = 2, n = 3)$

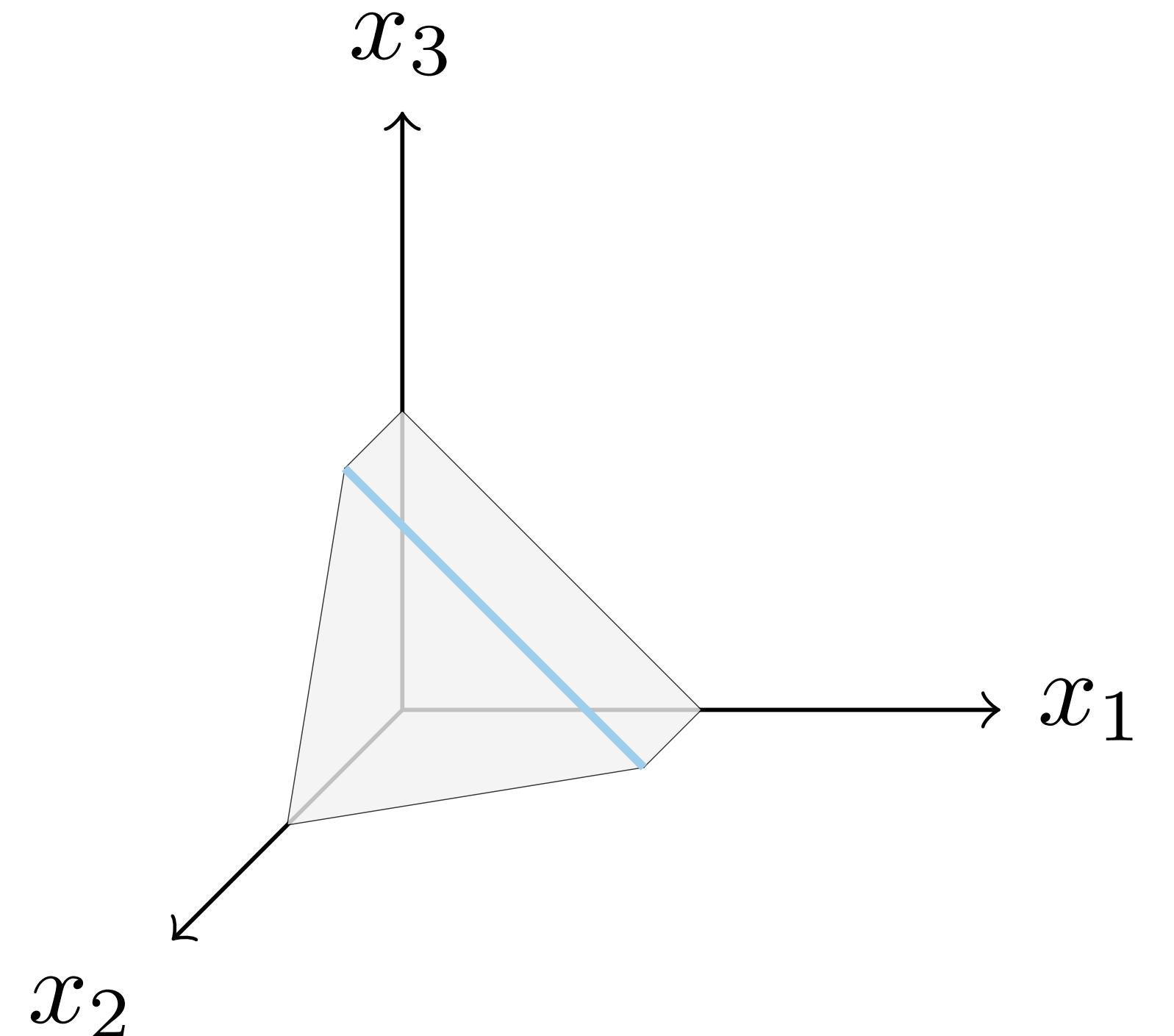
minimize $c^T x$

subject to $x_1 + x_3 = 1$

$$(1/2)x_1 + x_2 + (1/2)x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

$n - m = 1$ inequalities have to be tight: $x_i = 0$



Set $x_1 = 0$ and solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow (x_2, x_3) = (0.5, 1)$$

3D example

Three equalities $(m = 3, n = 3)$

minimize $c^T x$

subject to $x_1 + x_3 = 1$

$$(1/2)x_1 + x_2 + (1/2)x_3 = 1$$

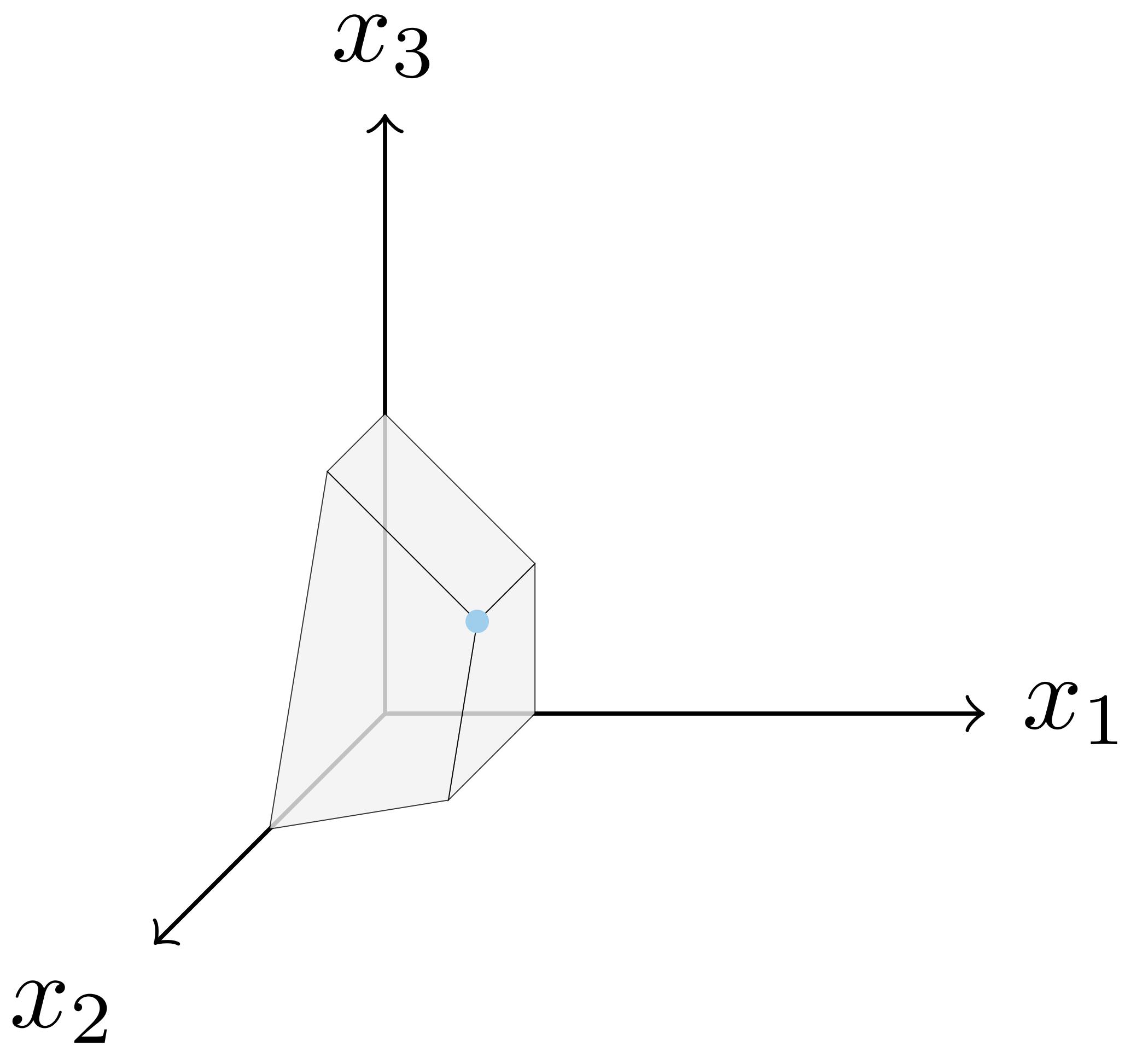
$$2x_1 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Basic feasible solution \bar{x} has n linearly independent active constraints.



$n - m = 0$ inequalities have to be tight: $x_i = 0$



Standard form polyhedra

Standard form LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Assumption

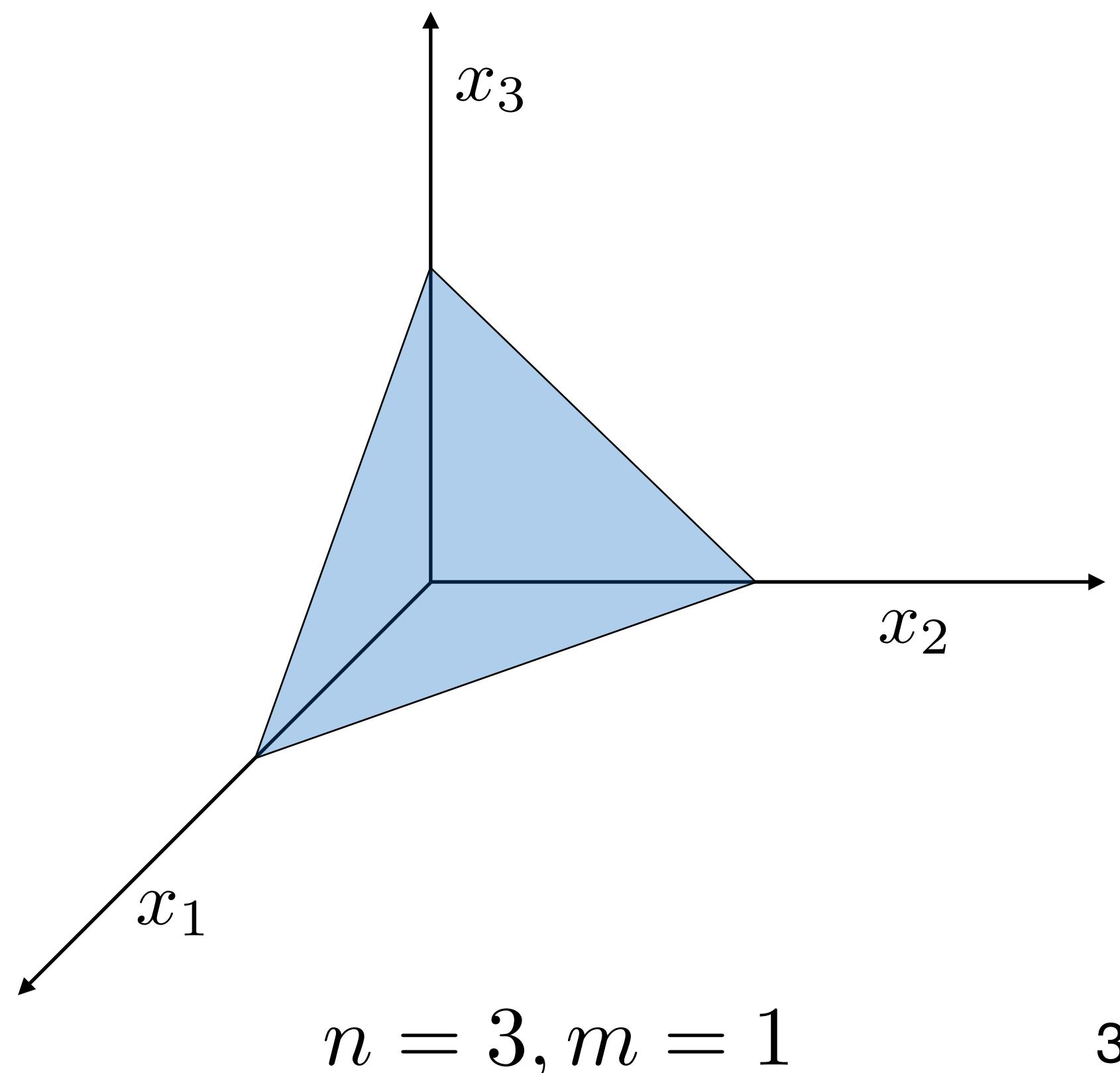
$A \in \mathbb{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P is an $(n - m)$ -dimensional surface

Standard form polyhedron

$$P = \{x \mid Ax = b, x \geq 0\}$$



Basic solutions

Standard form polyhedra

$$P = \{x \mid Ax = b, x \geq 0\} \quad \text{with} \quad A \in \mathbf{R}^{m \times n} \text{ has full row rank } m \leq n$$

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
 - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
 - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

Constructing basic solution

1. Choose any m independent columns of A : $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

Basis matrix Basis columns Basic variables

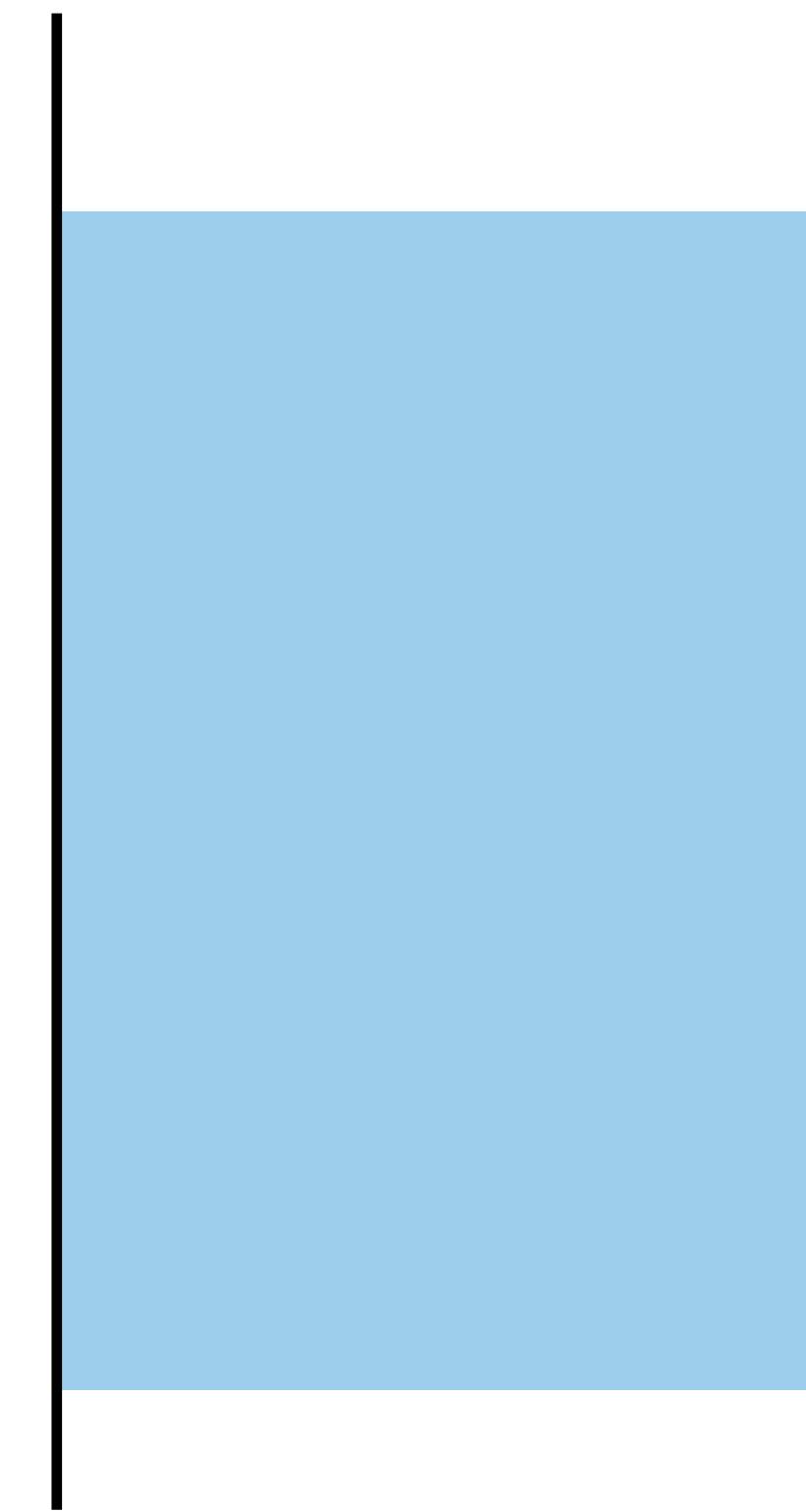
$$A_B = \left[\begin{array}{c|c|c|c|c} & & & & \\ \hline A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} & \\ & & & & \end{array} \right], \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

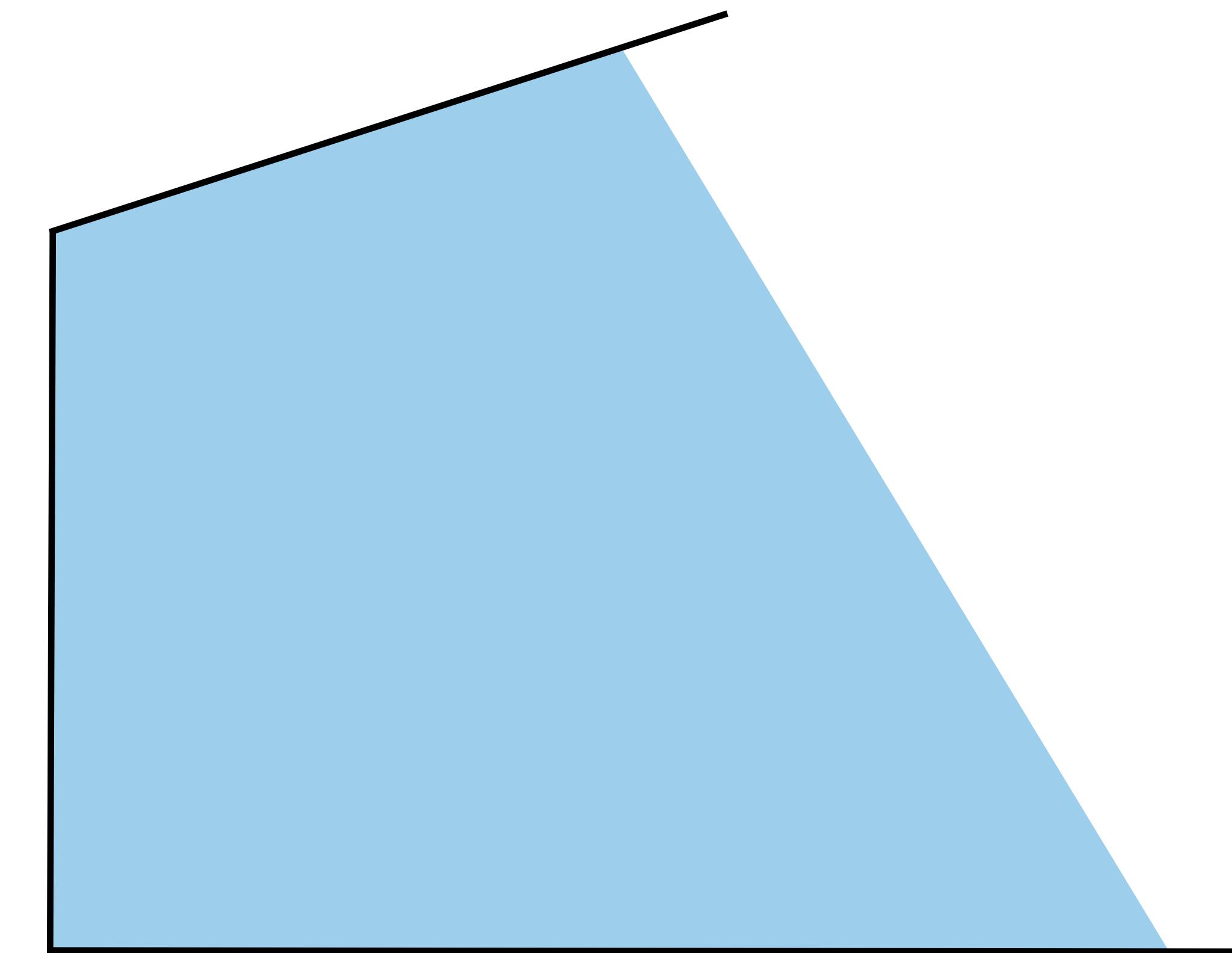
Existence and optimality of extreme points

Existence of extreme points

Example



No extreme points



Extreme points

Existence of extreme points

Characterization

A polyhedron P **contains a line** if
 $\exists x \in P$ and a nonzero vector d such that $x + \lambda d \in P, \forall \lambda \in \mathbf{R}$.

Given a polyhedron $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$, the following are **equivalent**

- P does not contain a line
- P has at least one extreme point
- n of the a_i vectors are linearly independent

Corollary

Every nonempty bounded polyhedron has
at least one basic feasible solution

Optimality of extreme points

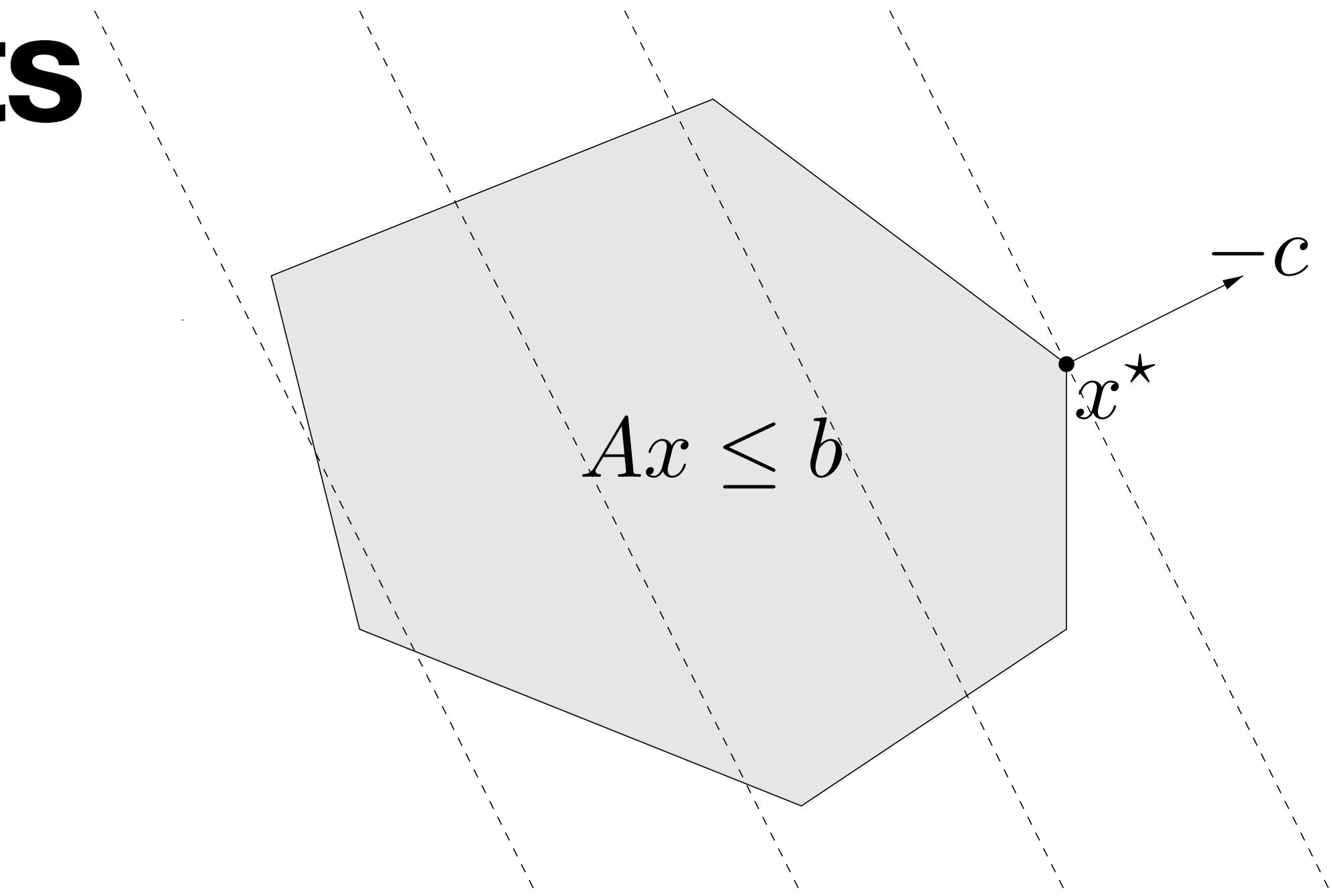
$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

If

- P has at least one extreme point
- There exists an optimal solution x^*

Then, there exists an optimal solution that is an **extreme point** of P .

Solution method: restrict search to **extreme points**.



How to search among basic feasible solutions?

Idea

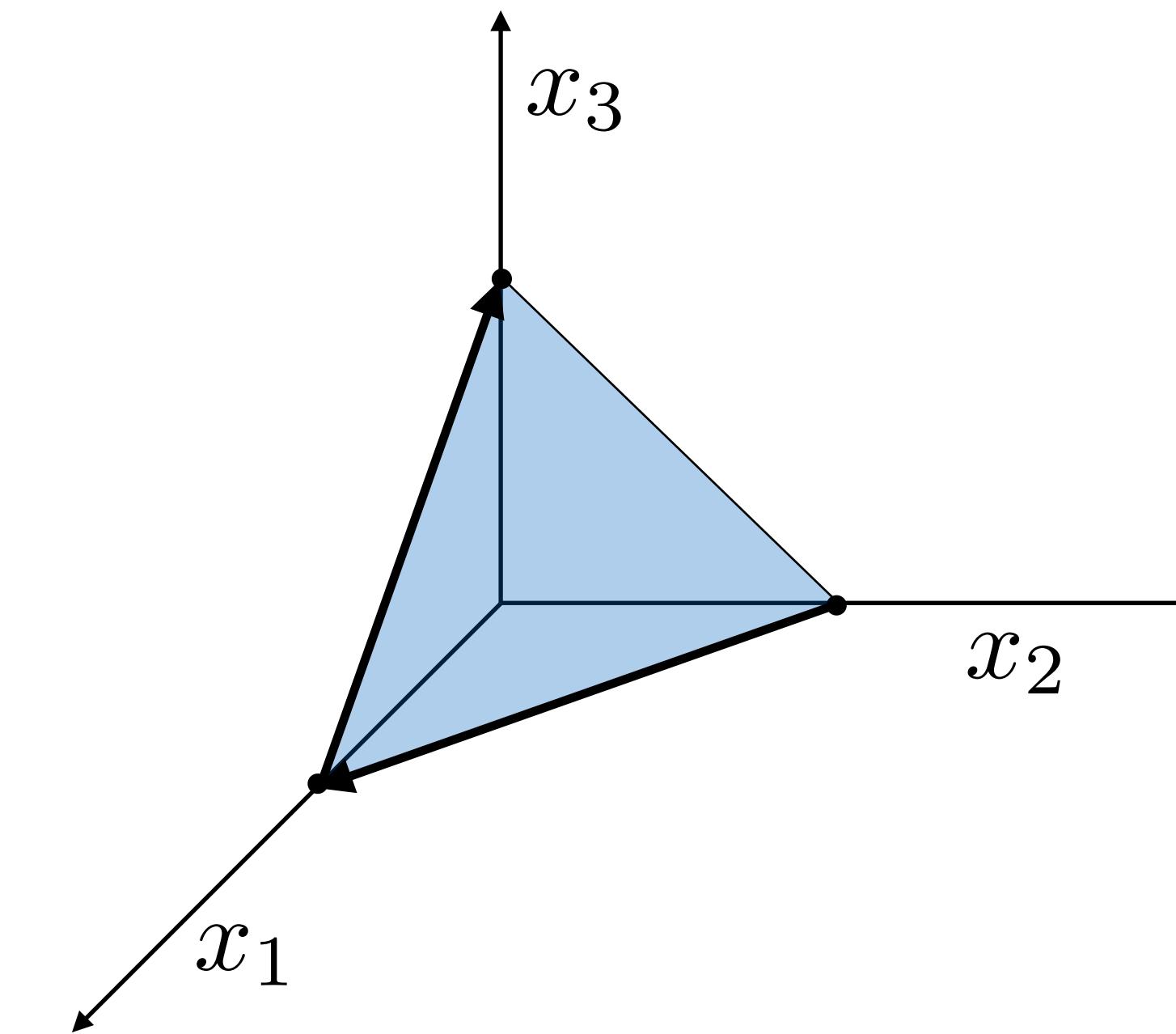
List all the basic feasible solutions, compare objective values and pick the best one.

Intractable!

If $n = 1000$ and $m = 100$, we have 10^{143} combinations!

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Geometry of linear optimization

Today, we learned to:

- **Apply geometric and algebraic properties** of polyhedra to characterize the “corners” of the feasible region.
- **Construct basic feasible solutions** by solving a linear system.
- **Recognize existence and optimality** of extreme points.

References

- Bertsimas and Tsitsiklis: Introduction to Linear Programming
 - Chapter 2.1 – 2.6 : geometry of linear programming

Next topics

More applications

The simplex method