ORF307 – Optimization

6. Constrained least squares

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Today's lecture

Constrained least squares

- Linearly constrained least squares
- Solving the constrained least squares problem
- Portfolio optimization

Linearly constrained least squares

Least squares with equality constraints

The (linearly) constrained least squares problem is

$$\begin{array}{ll} \text{minimize} & \boxed{\|Ax - b\|^2} \\ \text{subject to} & \boxed{Cx = d} \end{array}$$

equality constraints

objective function

Problem data

- $m \times n$ matrix A, m-vector b
- $p \times n$ matrix C, p-vector d

Definitions

x is *feasible* if $Cx = d$

$x^\star$ is a *solution* if

- $Cx^\star = d$
- $\|Ax^\star - b\|^2 \leq \|Ax - b\|^2$ for any x satisfying $Cx = d$

Interpretations

- Combine solving linear equations with least squares.
- Like a bi-objective least squares with ∞ weight on second objective, $\|Cx - d\|^2$.

Optimal advertising with budget

m demographic groups
we want to advertise to → v^{des} is the m-vector
of desired views/impressions

n advertising channels
(web publishers, radio, print, etc.) → s is the n-vector
of purchases

$m \times n$ matrix A gives
demographic reach of channels → A_{ij} is the number of views
for group i and dollar spent
on channel j $(1000/\$)$

Views across demographic groups

$$v = As$$

Goal

$$\begin{array}{ll} \text{minimize} & \|As - v^{\text{des}}\|^2 \\ \text{subject to} & \mathbf{1}^T s = \boxed{B} \end{array}$$

← allocated budget

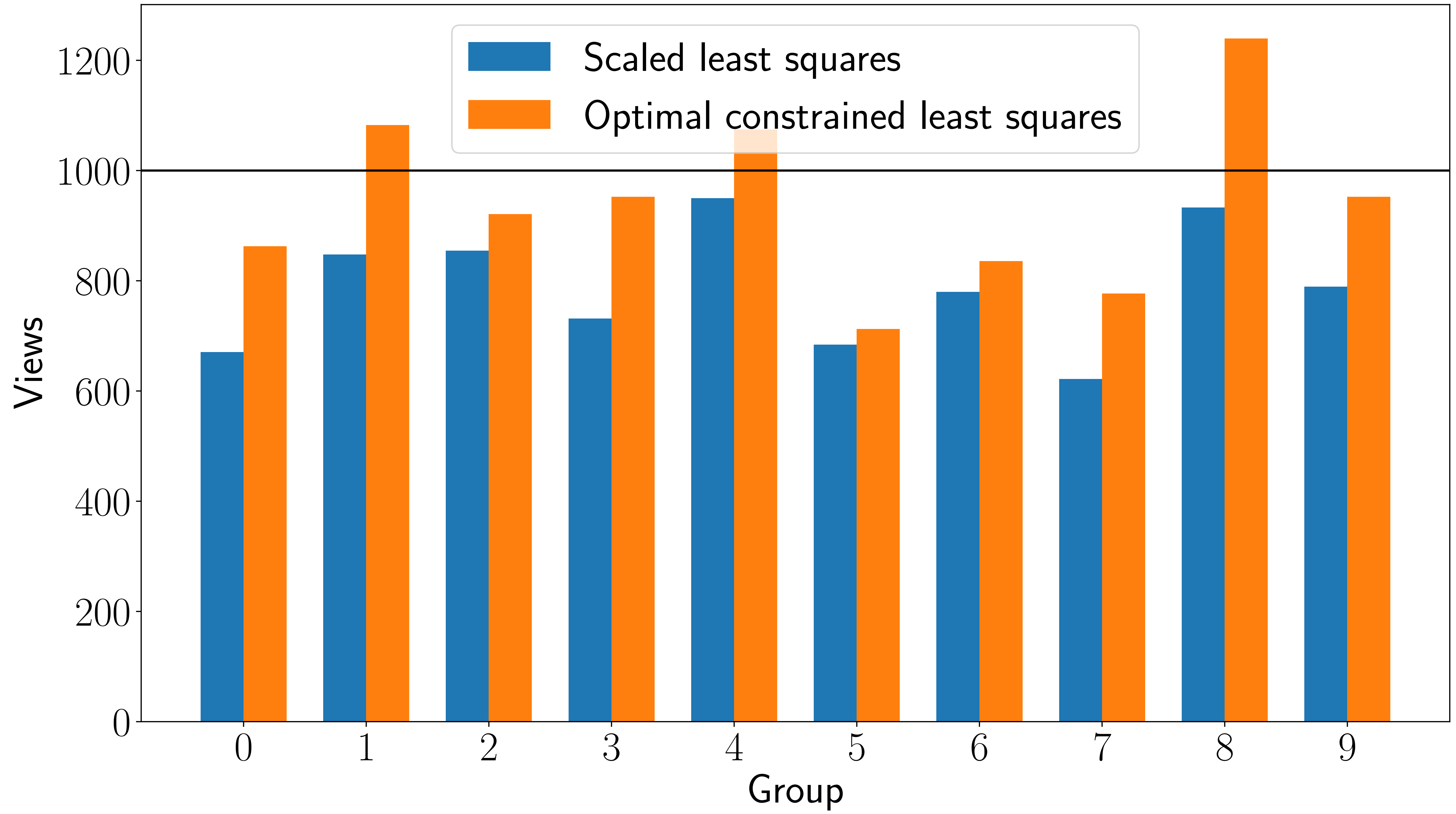
Optimal advertising with budget

Results

$m = 10$ groups, $n = 3$ channels

budget $B = 1284$

desired views vector $v^{\text{des}} = (10^3)\mathbf{1}$



$$\begin{array}{ll} \text{minimize} & \|As - v^{\text{des}}\|^2 \\ \text{subject to} & \mathbf{1}^T s = B \end{array}$$

optimal spending $s^\star = (315, 110, 859)$ $\longrightarrow$ RMS 16.10%

rescaled least squares spending $s^\star = (50, 80, 1154)$ $\longrightarrow$ RMS 23.85%

Least norm problem

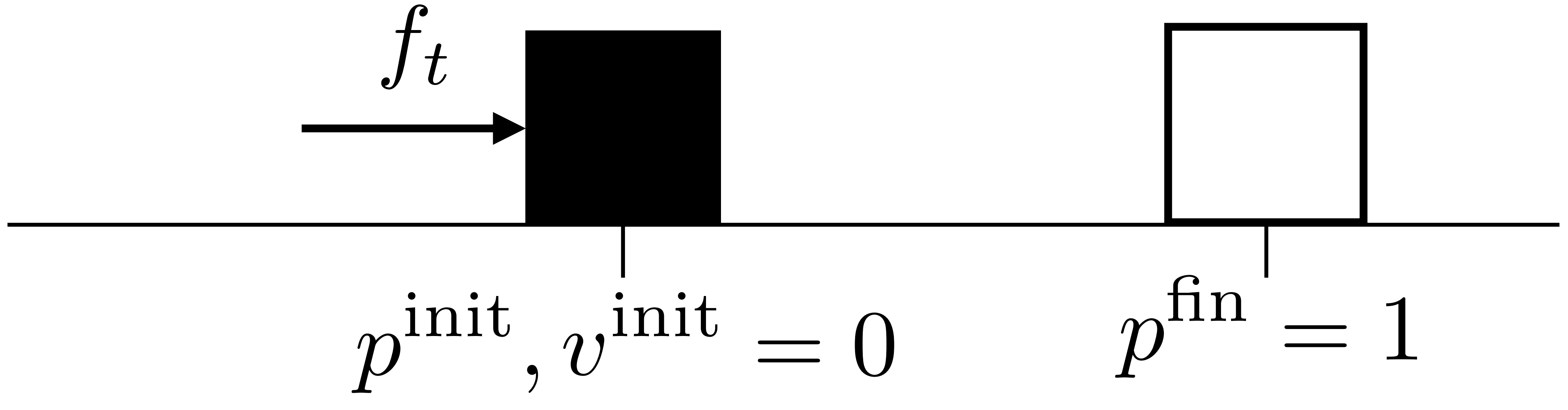
Special case of constrained least squares problem with $A = I$ and $b = 0$

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \|x\|^2 \\ \text{subject to} & Cx = d \end{array}$$

Find the smallest vector that satisfies a set of linear equations

Force sequence

Unit mass on frictionless surface, initially at rest



10-vector f gives the forces applied for one second each

Final velocity and position (Newton's laws)

$$v^{\text{fin}} = f_1 + f_2 + \cdots + f_{10}$$

$$p^{\text{fin}} = (19/2)f_1 + (17/2)f_2 + \cdots + (1/2)f_{10}$$

Goal

Let's find f such that $v^{\text{fin}} = 0$ and $p^{\text{fin}} = 1$

Least norm force sequence

Find f that brings to $p^{\mathrm{fin}} = 1$, $v^{\mathrm{fin}} = 0$

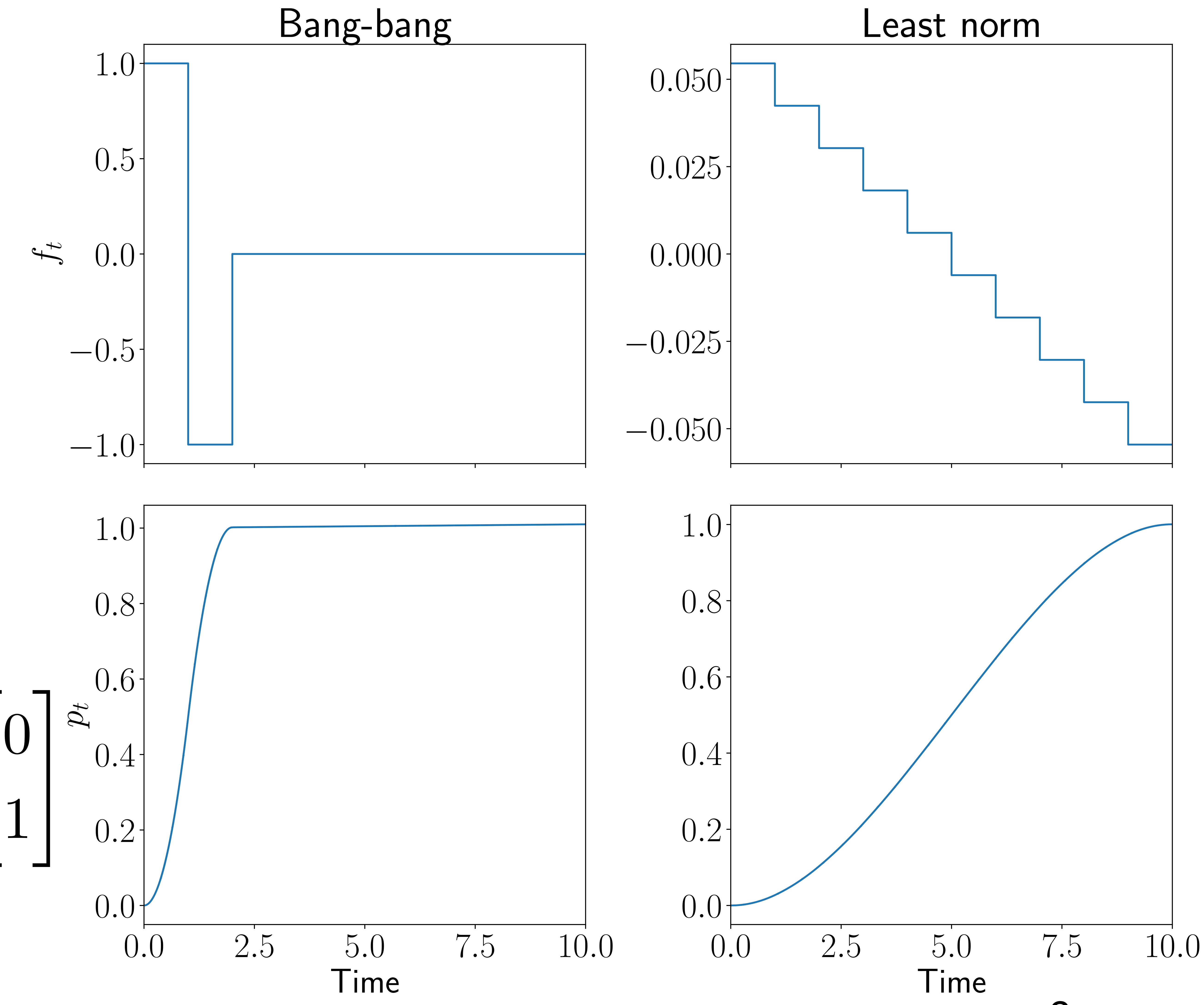
Bang-bang solution

$$f^{\mathrm{bb}} = (1, -1, 0, \ldots, 0) \qquad \|f^{\mathrm{bb}}\|^2 = 2$$

Least norm solution

$$\begin{array}{ll} \text{minimize} & \|f\|^2 \\ \text{subject to} & \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 19/2 & 17/2 & \ldots & 1/2 \end{bmatrix} f = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}$$

$$\|f^{\mathrm{ln}}\|^2 = 0.012$$ **Much cheaper effort!**



Solving the constrained least squares problem

Optimality conditions via calculus

$$\begin{array}{ll} \text{minimize} & f(x) = \|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & f(x) = \|Ax - b\|^2 \\ \text{subject to} & c_i^T x = d_i, \quad i = 1, \dots, p \end{array}$$

Lagrangian function

$$L(x, z) = f(x) + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

Optimality conditions

$$\frac{\partial L}{\partial x_i}(x^\star, z) = 0, \quad i = 1, \dots, n,$$

$$\frac{\partial L}{\partial z_i}(x^\star, z) = 0, \quad i = 1, \dots, p$$

Optimality conditions via calculus

$$L(x,z) = x^T A^T A x - 2(A^T b)^T x + b^T b + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

Optimality conditions

$$\frac{\partial L}{\partial z_i}(x^\star, z) = c_i^T x - d_i = 0$$ (we already knew)

$$\frac{\partial L}{\partial x_i}(x^\star, z) = 2\sum_{j=1}^{n}(A^T A)_{ij} x_j^\star - 2(A^T b)_i + \sum_{j=1}^{p} z_j (c_j)_i = 0$$

$\longrightarrow$

Vector form

$$Cx = d$$

$$2A^T A x^\star - 2A^T b + C^T z = 0$$

Karush-Kuhn-Tucker (KKT) conditions

$$\begin{bmatrix} 2A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ z \end{bmatrix} = \begin{bmatrix} 2A^T b \\ d \end{bmatrix}$$

(square set of $n + p$ linear equations)

Note KKT equations are extension of normal equations to constrained least squares

Invertibility of KKT matrix

no longer positive definite in general $\longrightarrow$

$$\begin{bmatrix} 2A^TA & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ z \end{bmatrix} = \begin{bmatrix} 2A^Tb \\ d \end{bmatrix}$$

The KKT matrix is invertible if and only if

- C has linearly independent rows $\longrightarrow$ $p \leq n$ (C is wide)
- $\begin{bmatrix} A \\ C \end{bmatrix}$ has linearly independent columns (true when A has linearly indep. cols) $\longrightarrow$ $m + p \geq n$ $\left(\begin{bmatrix} A \\ C \end{bmatrix} \text{ is tall}\right)$

Complexity (with $p \leq n \leq m$)

- Factor + solve: $2mn^2 + (2/3)(n+p)^3 + 2(n+p)^2 \approx 2mn^2$
- Solve given a new b (prefactored): $2mn + 2(n+p)^2 \approx 2mn$

same as unconstrained

Optimality from KKT solution

For $x^\star$ and $z^\star$ such that

$$2A^T Ax^\star + C^T z^\star = 2A^T b, \quad Cx^\star = d$$

Given a feasible x and z, we can write the objective (just as least squares)

$$\begin{aligned}\|Ax - b\|^2 &= \|(Ax - Ax^\star) + (Ax^\star - b)\|^2 \\ &= \|A(x - x^\star)\|^2 + \|Ax^\star - b\|^2 + 2(x - x^\star)^T A^T (Ax^\star - b)\end{aligned}$$

We can expand last term, using $2A^T(Ax^\star - b) = -C^T z^\star$ and $Cx = Cx^\star = d$

$$2(x - x^\star)^T A^T (Ax^\star - b) = -(x - x^\star)^T C^T z^\star = -(C(x - x^\star))^T z^\star = 0$$

$$\|Ax - b\|^2 = \|A(x - x^\star)\|^2 + \|Ax^\star - b\|^2 \geq \|Ax^\star - b\|^2$$

$x^\star$ is **optimal**

Portfolio optimization

Portfolio allocation weights

We want to invest V dollars in n different *assets* (stocks, bonds, ...) over periods $t = 1, \ldots, T$

Portfolio allocation weights

n-vector w gives the fraction of our total portfolio held in each asset

Properties

- Vw_j dollar value hold in asset j
- $\mathbf{1}^T w = 1$ (normalized)
- $w_j < 0$ means short positions (you borrow) (must be returned at time T)
- Example: $w = (-0.2, 0.0, 1.2)$

Short position of $0.2V$ on asset 1

Don't hold any of asset 2

Hold $1.2V$ in asset 3

Leverage, long-only portfolios, and cash

Leverage

$$L = |w_1| + \cdots + |w_n| = \|w\|_1$$

$L = 1$ when all weights are nonnegative ("long only portfolio")

Uniform portfolio

$$w = \mathbf{1}/n$$

Risk free asset

We often assume asset n is "risk-free" (e.g., cash)

if $w = e_n$, it means the portfolio is all cash

Return over a period

Asset returns

$\tilde{r}_t$ is the (fractional) return of each asset over period t

example: $\tilde{r}_t = (0.01, -0.023, 0.02)$
(often expressed as percentage)

Portfolio return

$$r_t = \tilde{r}_t^T w$$

It is the (fractional) return for the entire portfolio over period t

Total portfolio value after a period

$$V_{t+1} = V_t + V_t \tilde{r}_t^T w = V_t(1 + r_t)$$

Return matrix

R is the $T \times n$ matrix of **asset returns**

R_{tj} is the return of asset j in period t

$$
\begin{array}{cc}
 & \begin{array}{cccc} \text{AAPL} & \text{GOOG} & \text{MMM} & \text{US \$} \end{array} \\
R = & \begin{bmatrix} 0.00219 & 0.0006 & -0.00113 & 0.00005 \\ 0.00744 & -0.00894 & -0.00019 & 0.00005 \\ 0.01488 & -0.00215 & 0.00433 & 0.00005 \end{bmatrix} \begin{array}{l} \text{Mar 1, 2016} \\ \text{Mar 2, 2016} \\ \text{Mar 3, 2016} \end{array}
\end{array}
$$

Hold constant portfolio
with weights w over T periods

Columns interpretation
Column j is time series
of asset j returns

Rows interpretation
Row t is $\tilde{r}_t$ is the asset
return vector over period t

Note. If nth asset risk-free,
the last column of R is $\mu^{\mathrm{rf}}\mathbf{1}$,
where μ^{rf} is the risk-free
per-period interest reate

Portfolio returns (time series)

$$r = Rw \quad (T\text{-vector})$$

Returns over multiple periods

r is time series T-vector of portfolio returns

average return
(or just return)

$$\mathbf{avg}(r) = \mathbf{1}^T r / T$$

risk
(standard deviation)

$$\mathbf{std}(r) = \|r - \mathbf{avg}(r)\mathbf{1}\| / \sqrt{T}$$

Total portfolio value

$$\begin{aligned} V_{T+1} &= V_1(1+r_1)\cdots(1+r_T) \\ &\approx V_1 + V_1(r_1 + \cdots + r_T) \\ &= V_1 + T\mathbf{avg}(r)V_1 \end{aligned}$$

(for $|r_t|$ small, e.g., ≤ 0.01 ignore higher order terms)

For high portfolio value we need large $\mathbf{avg}(r)$

Annualized return and risk

Mean return and and risk are often expressed in **annualized form** (per year)

Given P trading periods per year (i.e., 250 days)

annualized return $= P\mathbf{avg}(r)$, annualized risk $= \sqrt{P}\mathbf{std}(r)$

Portfolio optimization

How shall we choose the portfolio weight vector w?

Goals

High (mean) return
$\mathbf{avg}(r)$

Low risk
$\mathbf{std}(r)$

Data

- We know **realized asset returns** but not future ones
- **Optimization.** We choose w that would have worked well in the past
- **True goal.** Hope it will work well in the future (just like data fitting)

Portfolio optimization

Minimize risk given a target return

Chose n-vector w to solve

$$\begin{array}{ll} \text{minimize} & \mathbf{std}(Rw)^2 = (1/T)\|Rw - \rho\mathbf{1}\|^2 \\ \text{subject to} & \mathbf{1}^T w = 1 \\ & \mathbf{avg}(Rw) = \rho \end{array}$$

(past) target mean return

(past) portfolio returns time series

Solutions w are **Pareto optimal**

Our question

what would have been the best constant allocation w,
had we known future returns?

Example allocations

Annual return 1% (risk-free asset has 1% return)

$$w = (0.00, 0.00, 0.00, \ldots, 0.00, 0.00, 1.00)$$

Annual return 13%

$$w = (0.02, -0.07, -0.05, \ldots, -0.03, 0.06, 0.56)$$

Annual return 25%

$$w = (0.05, -0.143, -0.09, \ldots, -0.07, 0.12, 0.12)$$

Asking for higher annual returns yields

- More invested in risky, but high return assets
- Larger short positions (”leveraging”)

Portfolio optimization

As constrained least squares

$$\begin{array}{ll} \text{minimize} & \|Rw - \rho\mathbf{1}\|^2 \\ \text{subject to} & \begin{bmatrix}\mathbf{1}^T \\ \mu^T\end{bmatrix} w = \begin{bmatrix}1 \\ \rho\end{bmatrix} \end{array}$$

μ is the n-vector of average returns per asset

$$\begin{aligned} \mathbf{avg}(r) &= (1/T)\mathbf{1}^T(Rw) \\ &= (1/T)(R^T\mathbf{1})^T w = \mu^T w \end{aligned}$$

Solution via KKT linear system

$$\begin{bmatrix} 2R^TR & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2\rho T\mu \\ 1 \\ \rho \end{bmatrix}$$

Optimal portfolios

Rewrite right-hand side

$$\begin{bmatrix} 2\rho T\mu \\ 1 \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 2T\mu \\ 0 \\ 0 \end{bmatrix}$$

Two fund theorem

Optimal portfolio w is an affine function of ρ

$$\begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2R^T R & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 2R^T R & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2T\mu \\ 0 \\ 1 \end{bmatrix}$$

We can rewrite the first n-components as the combination of two portfolios (funds)

$$w = w_0 + \rho v$$

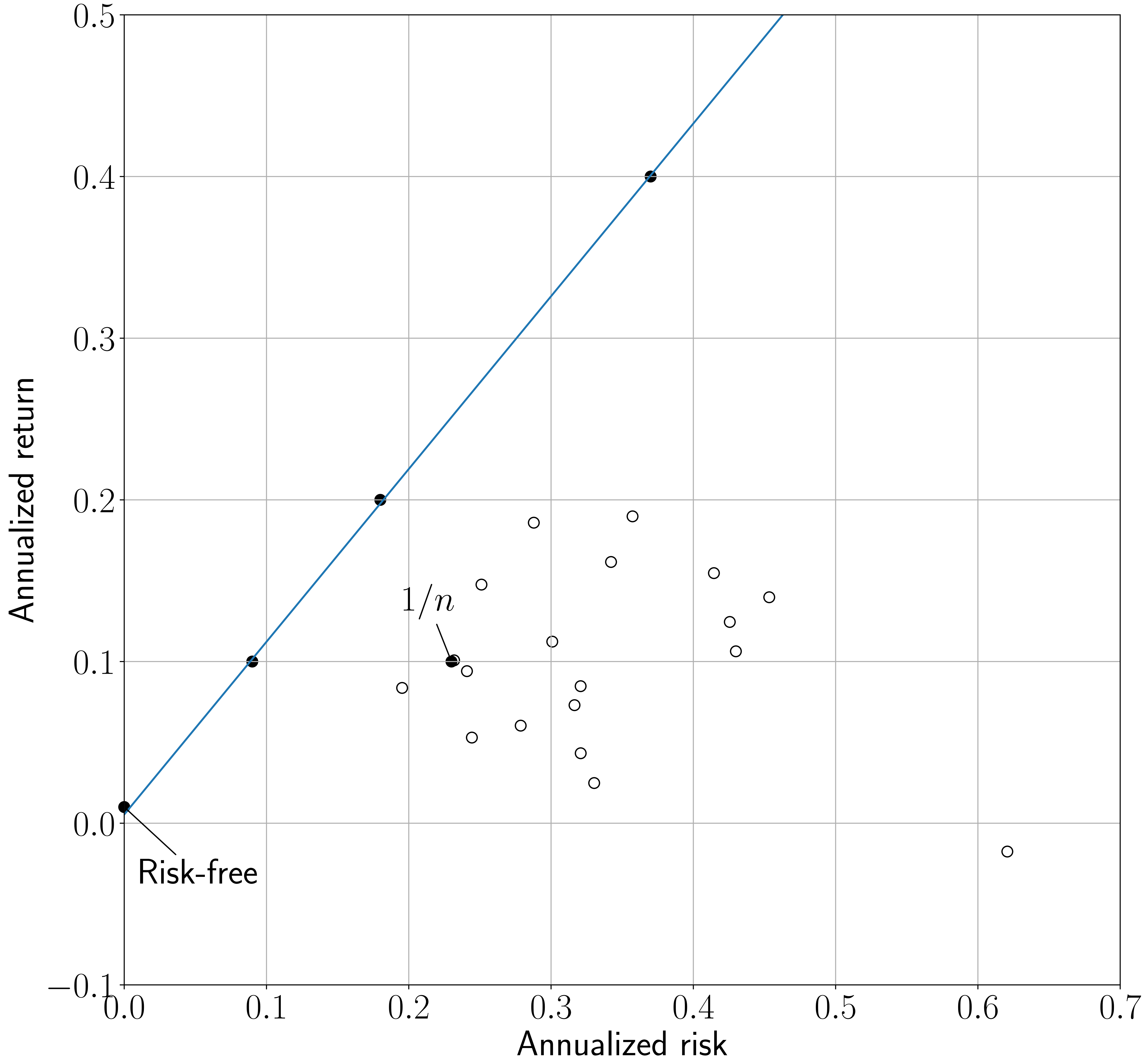
Risk-free ($\rho = 0$)

Other optimal portfolio

Example

20 assets over 2000 days (past)

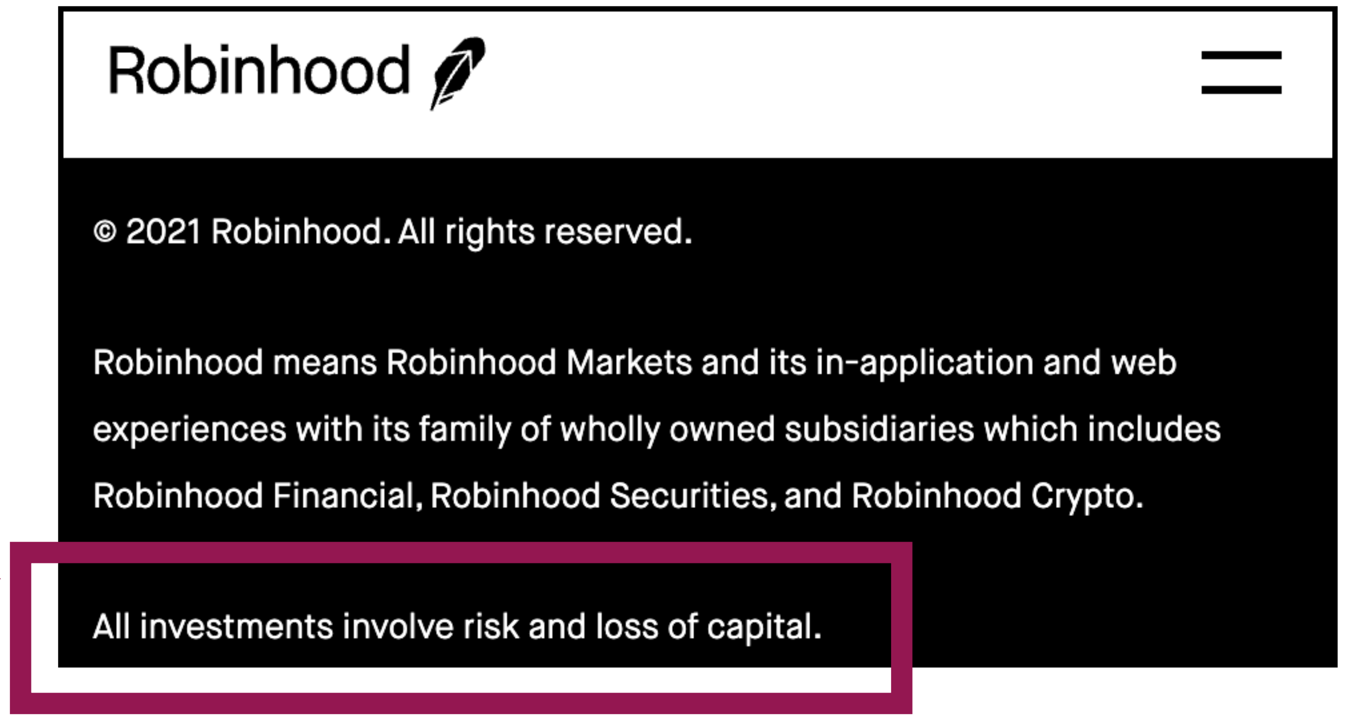
- Optimal portfolios on a **straight line**
- Line starts at risk-free portfolio ($\rho = 0$)
- $1/n$ much better than single portfolios



The big assumption

Future returns will look like past ones

- You are warned this is false, every time you invest
- It is often reasonable
- During crisis, market shifts, other big events not true



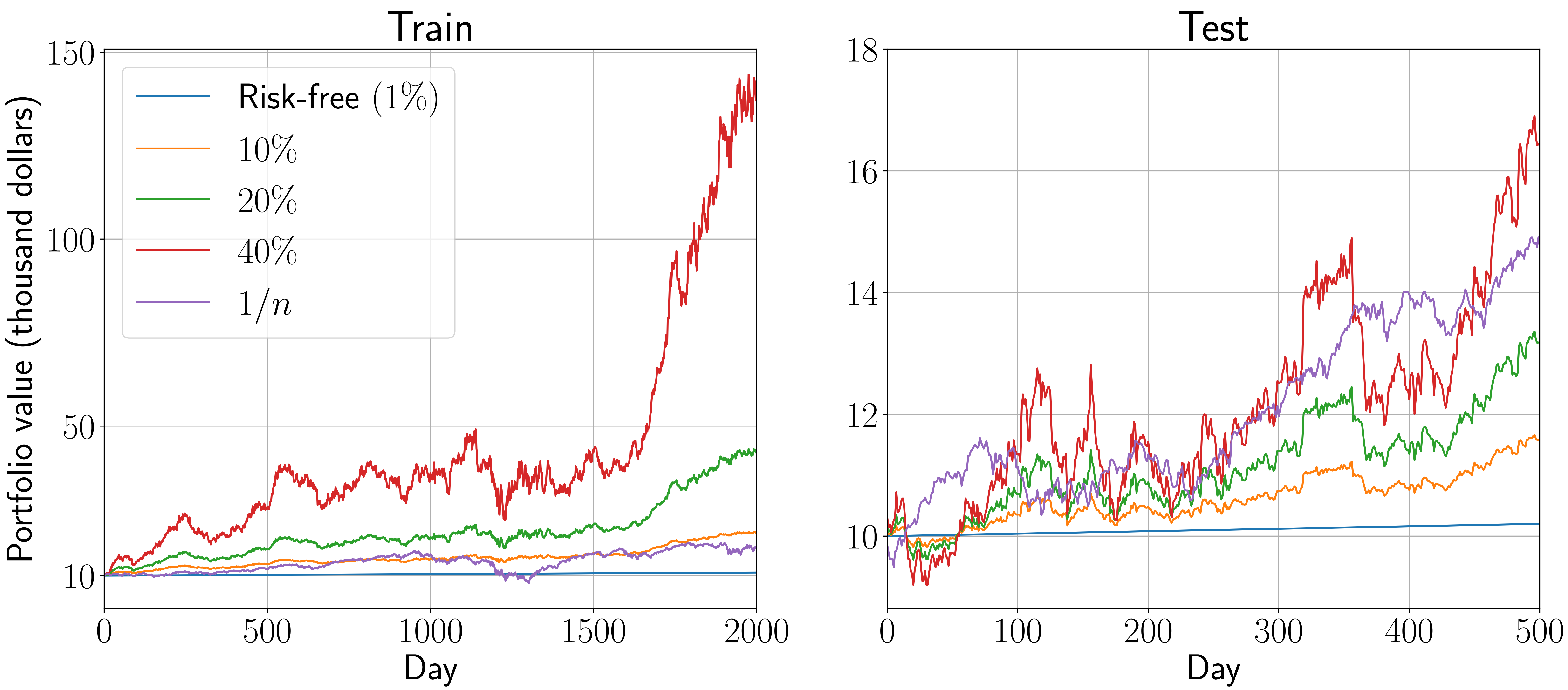
If assumption holds (even approximately), a good w on past returns leads to good future (unknown) returns

Example

- Pick w based on last 2 years of returns
- Use w during next 6 months

Total portfolio value

	Return		Risk		
	Train	Test	Train	Test	Leverage
Risk-free (1%)	0.01	0.01	0.00	0.00	1.00
10%	0.10	0.08	0.09	0.07	1.96
20%	0.20	0.15	0.18	0.15	3.03
40%	0.40	0.30	0.37	0.31	5.48
$1/n$	0.10	0.21	0.23	0.13	1.00



Build your quantitative hedge fund

Rolling portfolio optimization

For each period t, find weight w_t using L past returns

$$r_{t-1}, \ldots, r_{t-L}$$

Variations

- Update w every K periods (monthly, quarterly, ...)
- Add secondary objective $\lambda \|w_t - w_{t-1}\|^2$ to discourage turnover, reduce transaction cost
- Add logic to detect when the future is likely to not look like the past
- Add “signals” that predict future return of assets (Twitter sentiment analysis)

Constrained least squares

Today, we learned to:

- **Formulate** (linearly) and **solve** constrained least squares problems
- **Solve** portfolio allocations problems
- **Understand** the difference between past and future returns (be careful!)

References

- S. Boyd, L. Vandenberghe: Introduction to Applied Linear Algebra – Vectors, Matrices, and Least Squares
  - Chapter 16 and 17: constrained least squares

Next lecture

- Linear optimization