Learning for Decision-Making under Uncertainty

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It is hard to make decisions under uncertainty

Transportation

Finance

Energy

But we have data!

Problem setup with uncertain constraints

optimization variable

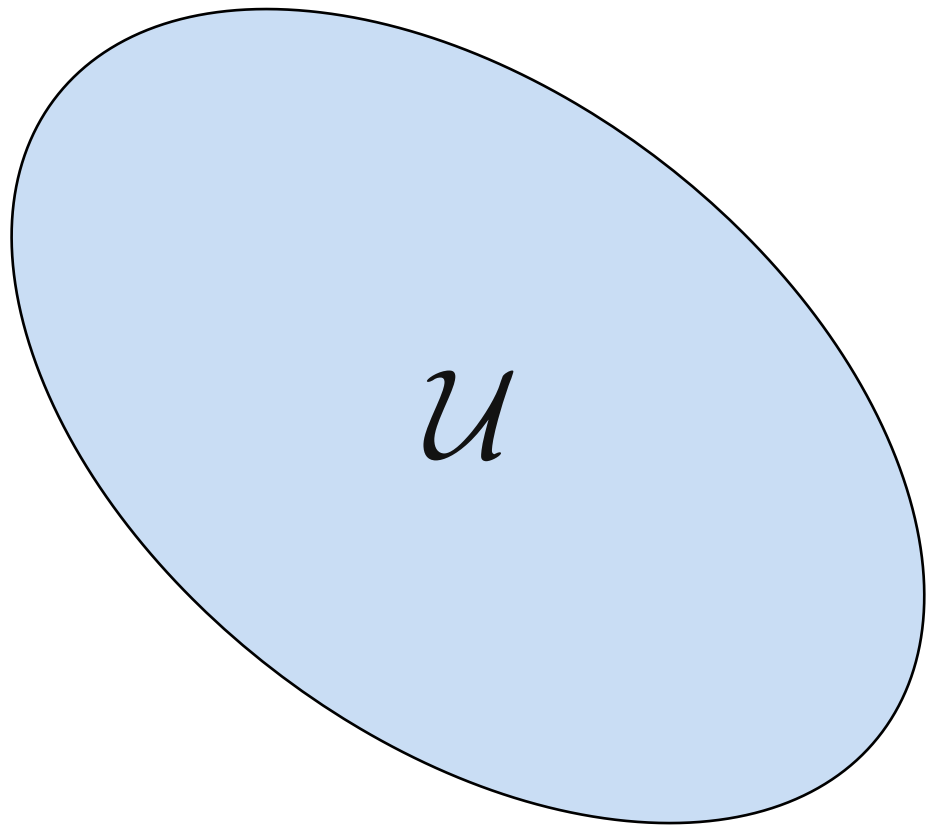
$$
\begin{array}{ll}
\text{minimize} & f(x) \\
\text{subject to} & g(u, x) \leq 0
\end{array}
$$

uncertain parameter

We want to guarantee constraint satisfaction

Robust optimization recipe

1. Pick uncertainty set $\mathcal{U}$
2. Ensure constraint satisfaction $\forall u \in \mathcal{U}$

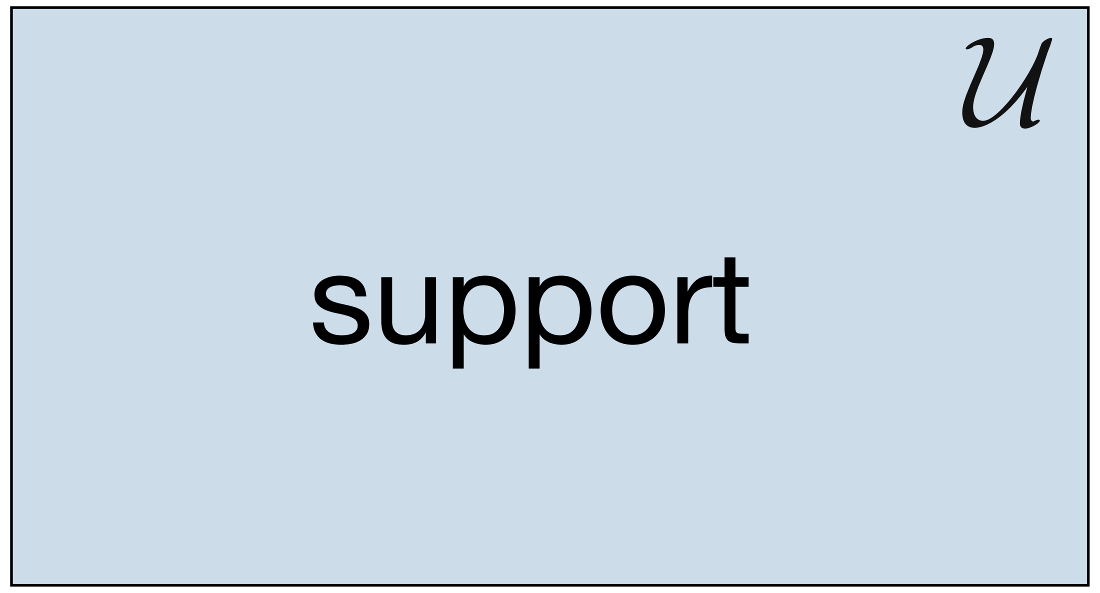


$$
\begin{array}{ll}
\text{minimize} & f(x) \\
\text{subject to} & g(u, x) \leq 0, \quad \forall u \in \mathcal{U}
\end{array}
$$

How do we pick the uncertainty set?

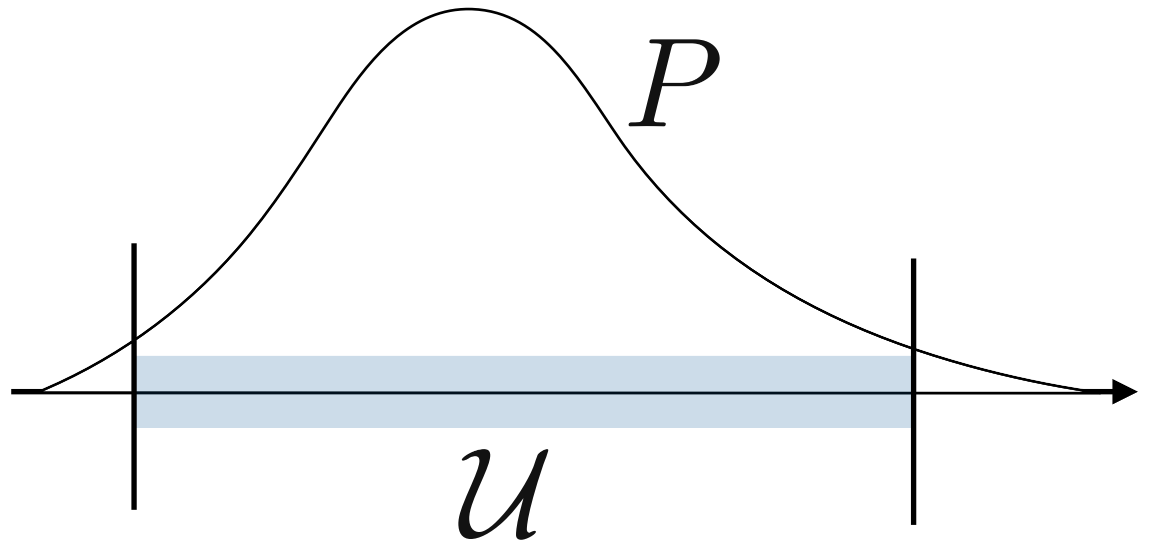
Picking the uncertainty set is difficult

Worst-case approach



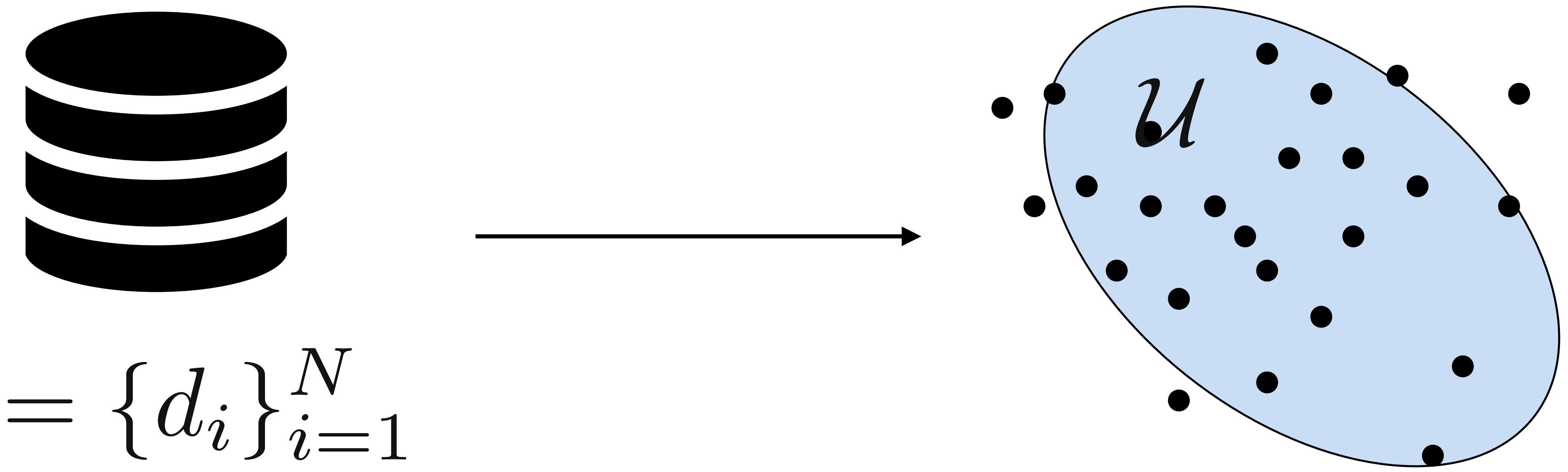
❌ Very conservative

Probabilistic approach



❌ nobody knows P

Can we use data?



$\mathcal{D} = \{d_i\}_{i=1}^N$

"Probabilistic Guarantees in Robust Optimization", D. Bertsimas, D. den Hertog, and J. Pauphilet (2019)
"Data-driven robust optimization", D. Bertsimas, V. Gupta, and N. Kallus (2014)
"Learning-Based Robust Optimization: Procedures and Statistical Guarantees", L. Jeff Hong, Z. Huang, and H. Lam (2021)
"Data-driven Robust Optimization using Unsupervised Deep Learning", M. Goerigk, and J. Kurtz (2023)

Estimating true distribution from data

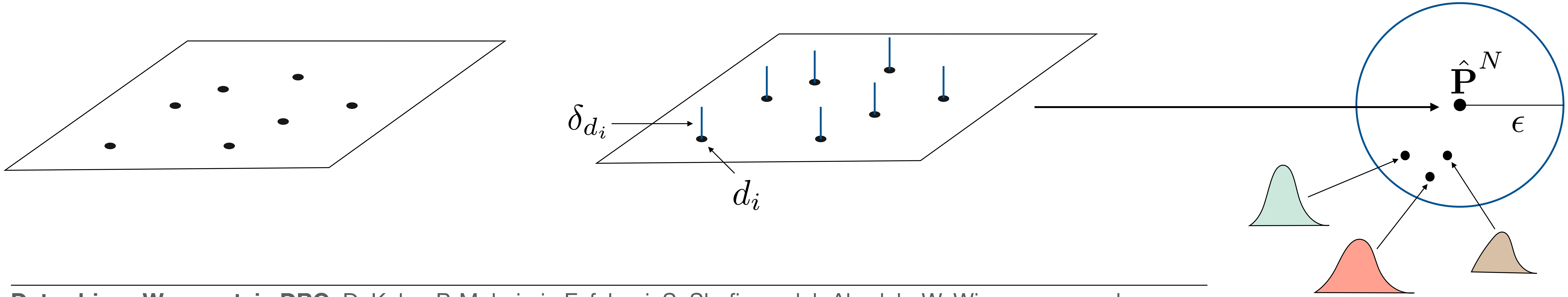
Data-Driven Distributionally Robust Optimization

Empirical Distribution | **Ambiguity Set**

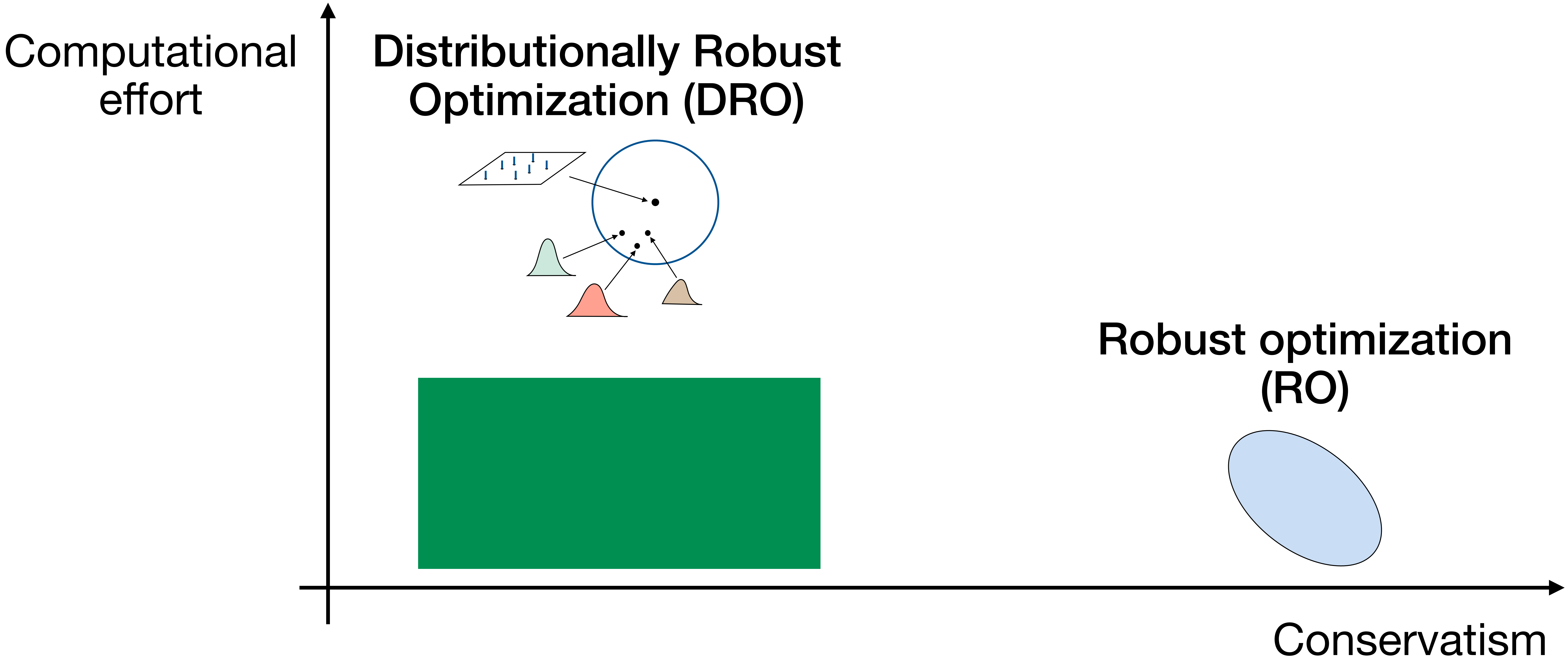
Data

$\mathcal{D} = \{d_i\}_{i=1}^N \longrightarrow \hat{\mathbf{P}}^N = \frac{1}{N}\sum_{i=1}^N \delta_{d_i} \longrightarrow \mathcal{P}_N = \left\{W_p(\hat{\mathbf{P}}^N, \mathbf{P}) \leq \epsilon\right\}$

Wasserstein distance



Robust vs Distributionally Robust Optimization

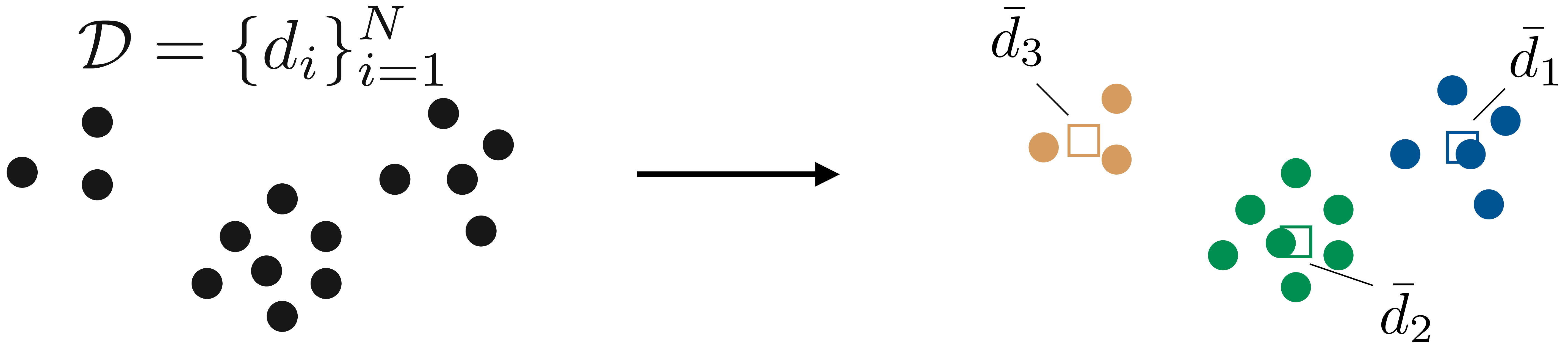


Can we get the best of both worlds?

Clustering reduces dimensionality

Data

$$\mathcal{D} = \{d_i\}_{i=1}^{N}$$



$$\text{minimize} \quad \sum_{k=1}^{K} \sum_{i \in C_k} \|d_i - \bar{d}_k\|^2$$

cluster centers

Main idea

Use cluster centers instead of original data

Probabilistic guarantees

$$\mathbf{E}(g(u,x)) \leq 0$$

concave function

$u \sim P$
(but we never know P!)

Data
$\mathcal{D} = \{d_i\}_{i=1}^N$

Data-driven probabilistic guarantees

$$\mathbf{P}^N\left(\mathbf{E}(g(u,\hat{x}_N)) \leq 0\right) \geq 1-\beta$$

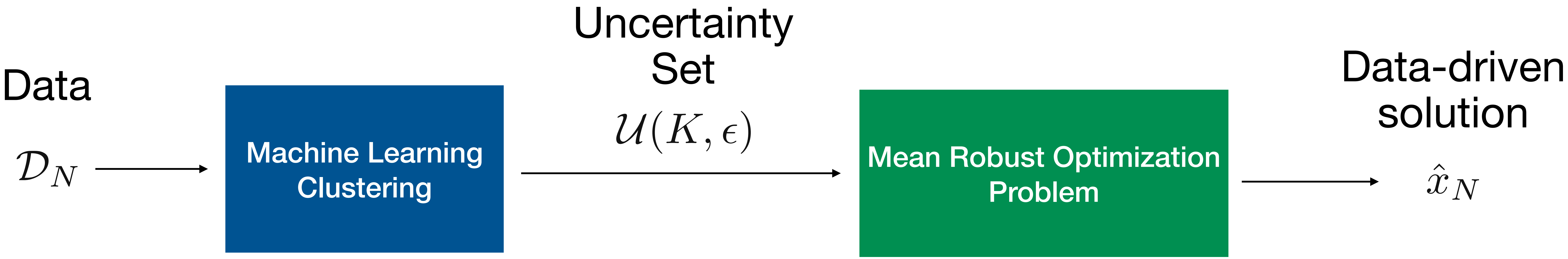
Product Distribution

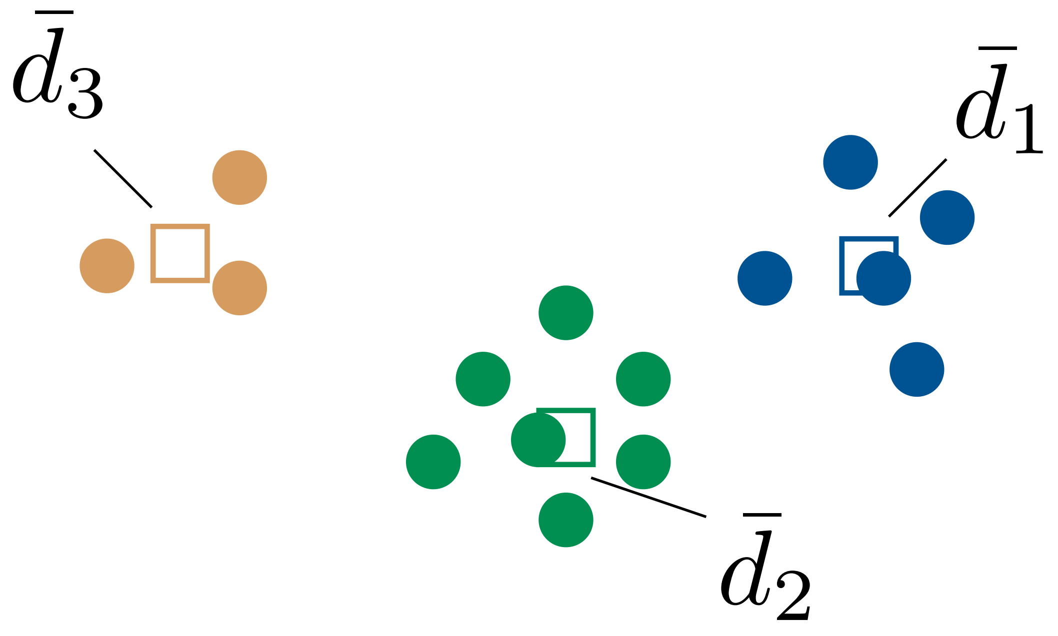
data-driven solution

probability of constraint satisfaction

Mean Robust Optimization

Mean Robust Optimization (MRO)



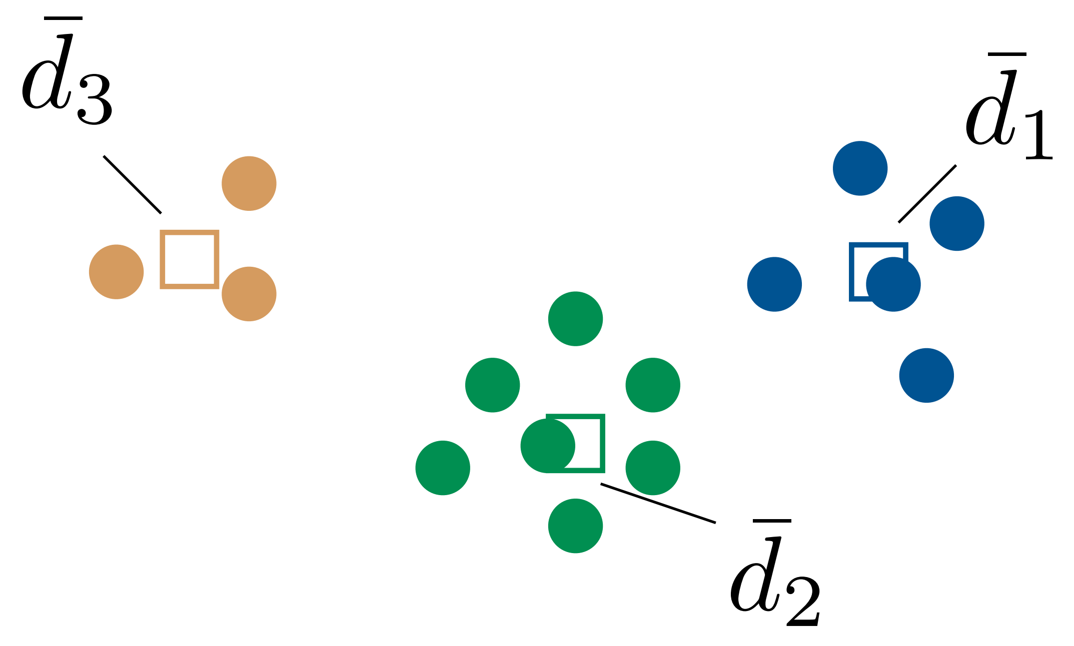


$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \bar{g}(u, x) \leq 0 \quad \forall u \in \mathcal{U}(K, \epsilon) \end{array}$$

constraint function ($\bar{g}(u, x)$)

uncertainty set ($\mathcal{U}(K, \epsilon)$)

Uncertainty set



cluster weights

order

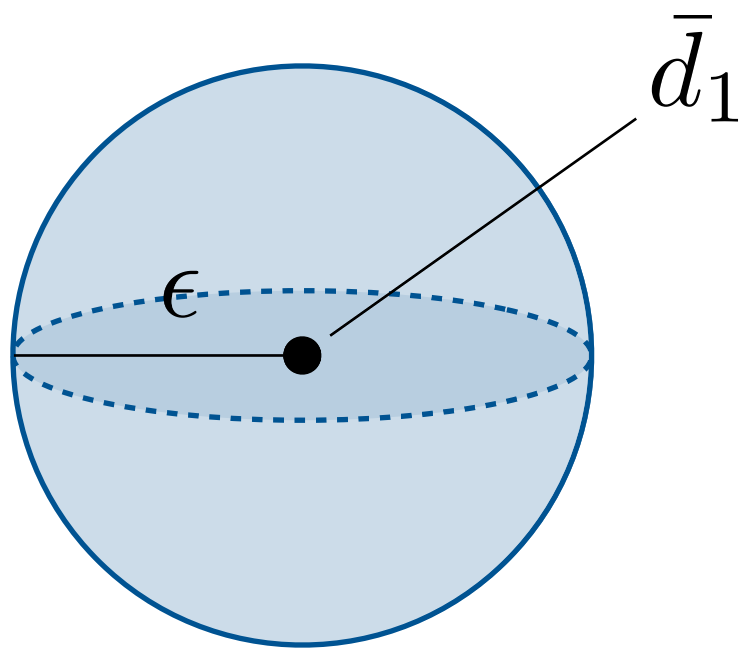
$$\mathcal{U}(K,\epsilon) = \left\{ u = (v_1,\dots,v_K) \;\middle|\; \sum_{k=1}^{K} w_k \|v_k - \bar{d}_k\|^p \le \epsilon^p \right\}$$

cluster centers

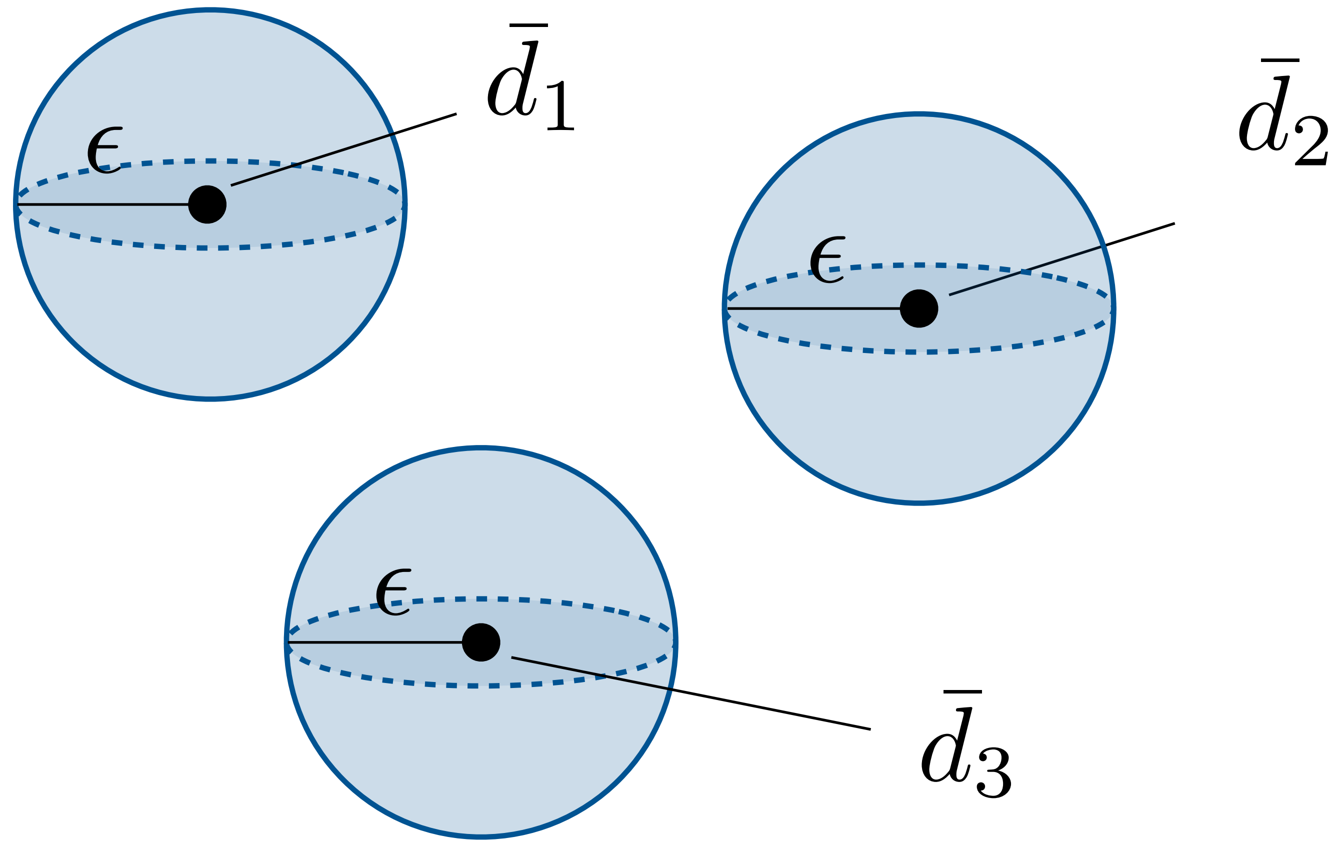
Examples

K

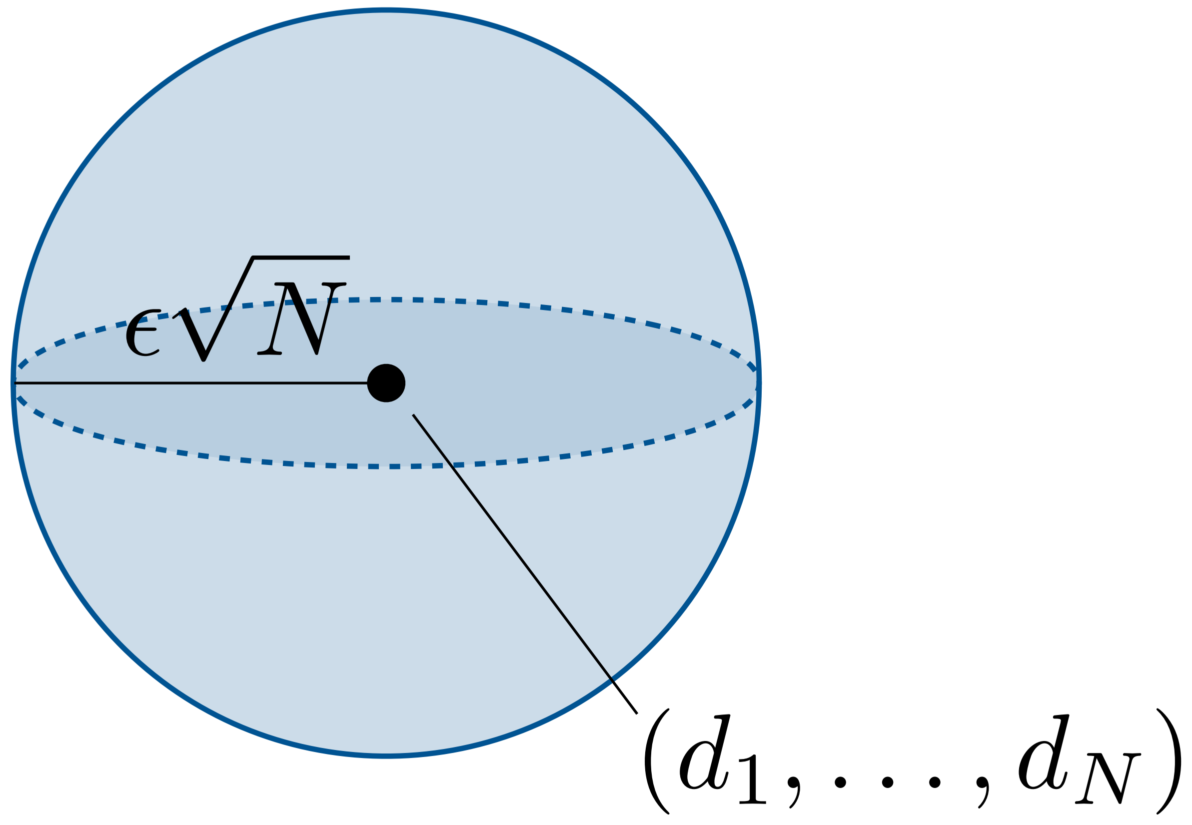
$K = 1$



$K = 3, p = \infty$



$K = N, p = 2$



Mean Robust Optimization Problem

uncertainty set

$$\left\{ \sum_{k=1}^{K} w_k \|v_k - \bar{d}_k\|^p \leq \epsilon^p \right\}$$

Uncertain variable lifting

$u = (v_1, \dots, v_K)$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \bar{g}(u, x) \leq 0 \quad \forall u \in \mathcal{U}(K, \epsilon) \end{array}$$

constraint function

$$\sum_{k=1}^{K} w_k g(v_k, x)$$

Solving the MRO problem

Dualize constraint $\bar{g}(u,x) \leq 0, \ \forall u \in \mathcal{U}(K,\epsilon)$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \sum_{k=1}^K w_k s_k \leq 0 \\ & [-g]^*(z_k, x) - z_k^T \bar{d}_k + \phi(p)\lambda \|z_k/\lambda\|_*^{p/(p-1)} + \lambda\epsilon^p \leq s_k, \quad k = 1,\dots,K \\ & \lambda \geq 0 \end{array}$$

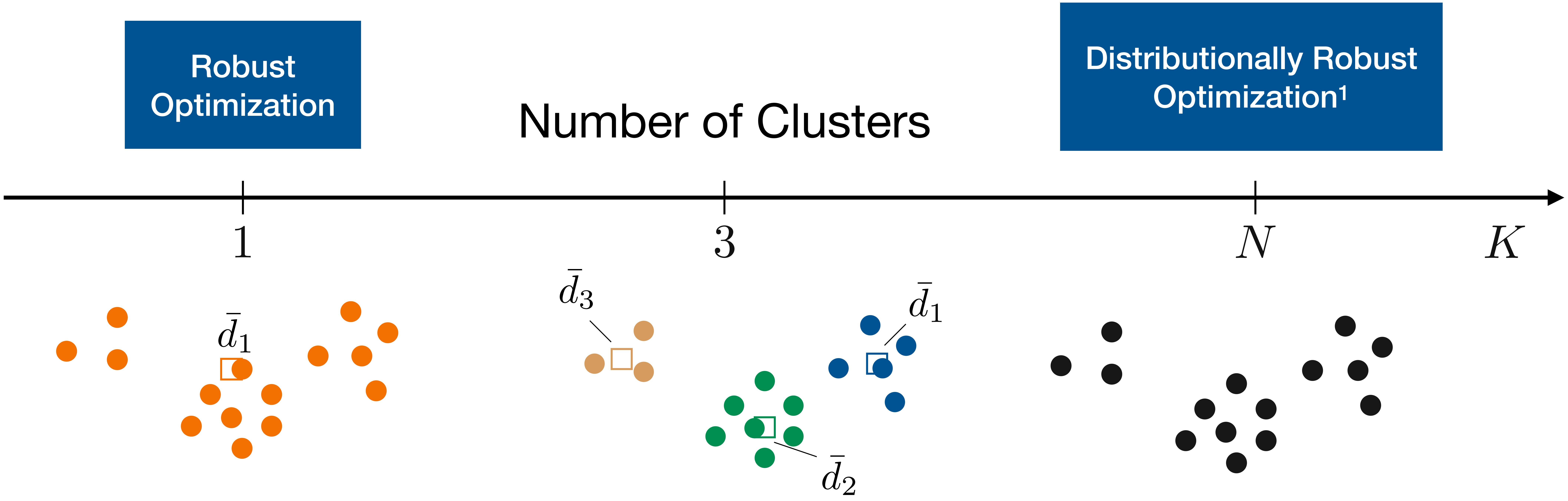
conjugate function

cluster centers

function of $p \geq 1$
$\phi(p) \to 1$ as $p \to \infty$
$\phi(1) = 0$

It can be very expensive when K is large (e.g., $K = N$)

MRO bridges RO and DRO



1. D Kuhn, P M Esfahani, V A Nguyen, and S Shafieezadeh-Abadeh, “Wasserstein Distributionally Robust Optimization: Theory and Applications in Machine Learning”

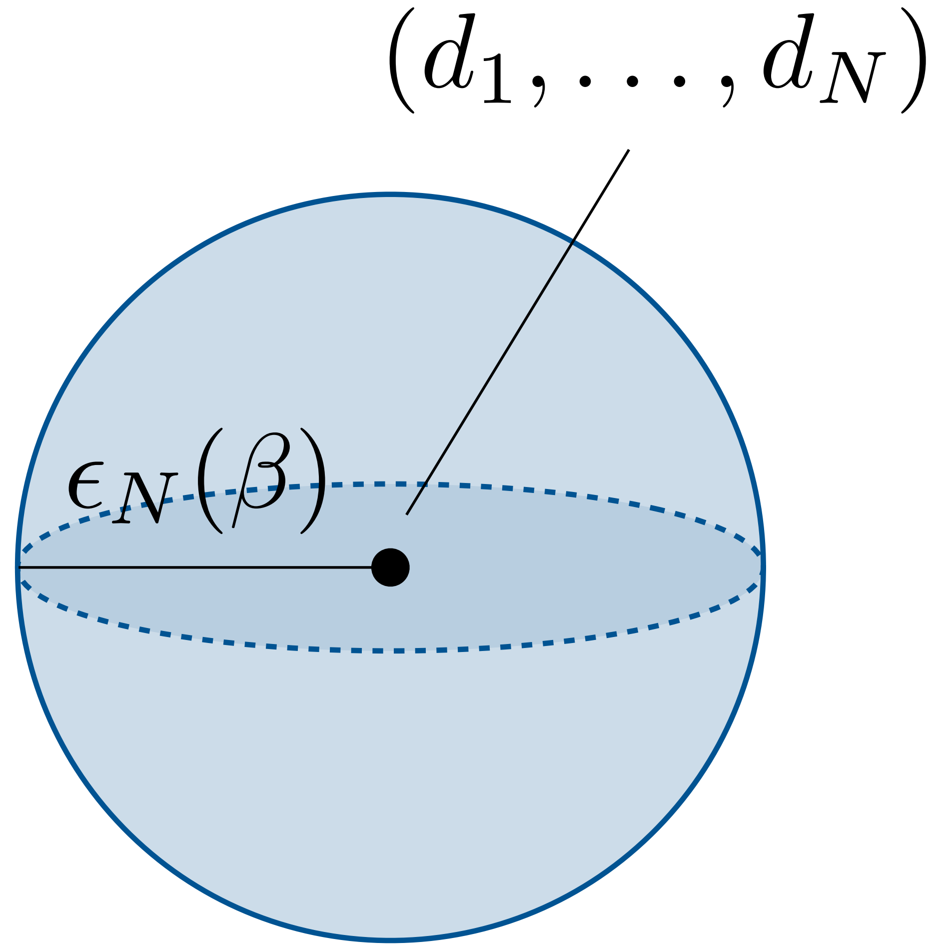
Guarantees

Satisfying the probabilistic guarantees

probability of constraint satisfaction

uncertainty set radius

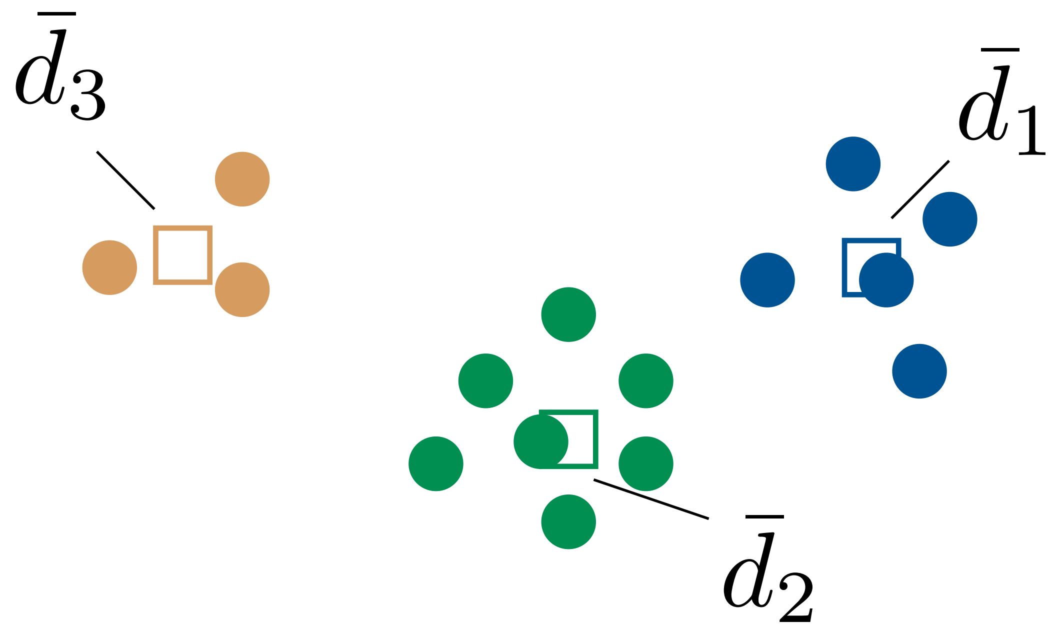
$$\mathbf{P}^N\left(\mathbf{E}(g(u,\hat{x}_N)) \le 0\right) \ge 1-\beta \xrightarrow{\text{light-tailed}} \mathcal{U}(N, \epsilon_N(\beta))$$



MRO clustering

$$\mathcal{U}(K, \epsilon_N(\beta) + \eta_N(K))$$

$$\frac{1}{N}\sum_{k=1}^{K}\sum_{d_i \in C_k} \|d_i - \bar{d}_k\|^p$$



Quite conservative bounds… can we do better?

Bounding the conservatism

Worst-case values

$$\bar{g}^{N}(x) = \underset{u \in \mathcal{U}(N,\epsilon)}{\text{maximize}} \quad \bar{g}(u,x)$$

$$\bar{g}^{K}(x) = \underset{u \in \mathcal{U}(K,\epsilon)}{\text{maximize}} \quad \bar{g}(u,x)$$

MRO constraint

$$\bar{g}(u,x) \leq 0 \quad \forall u \in \mathcal{U}(K,\epsilon)$$

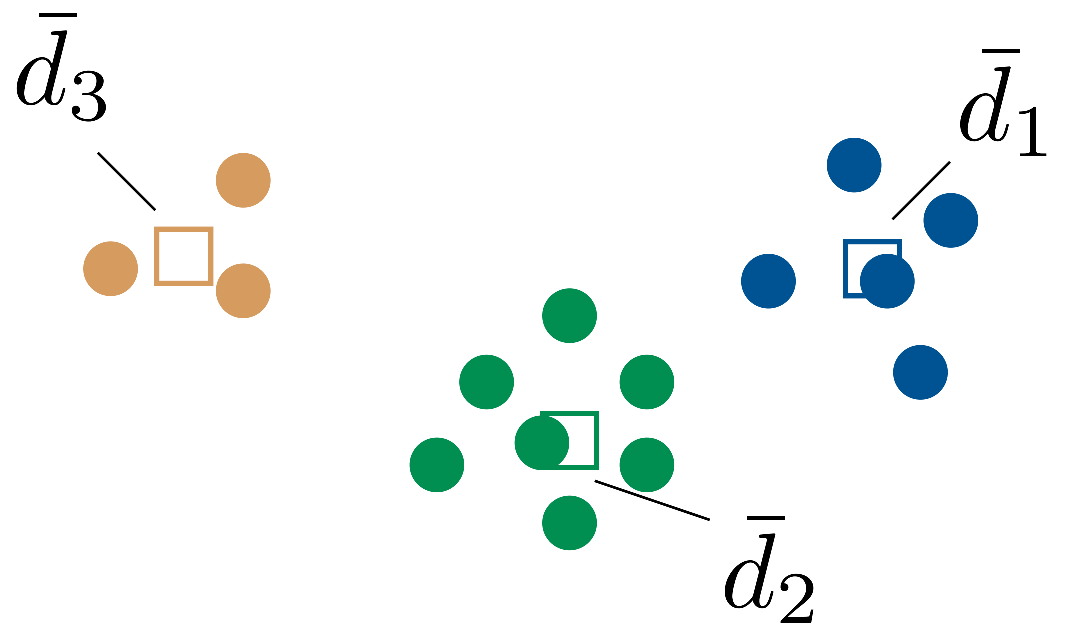
Theorem

If $-g$ is L-smooth in u, we have

$$\bar{g}^{N}(x) \leq \bar{g}^{K}(x) \leq \bar{g}^{N}(x) + \frac{L}{2} D(K)$$

clustering objective

$$D(K) \longleftarrow \min \ \frac{1}{N} \sum_{k=1}^{K} \sum_{d_i \in C_k} \|d_i - \bar{d}_k\|^2$$



When g is affine in u ($L = 0$), clustering makes no difference to the optimal value or optimal solution

Example MRO with linear constraints

$p = \infty$

$$(a + Pu)^T x \leq b \longrightarrow g(u,x) = (a+Pu)^T x - b$$

$$[-g]^*(z_k, x) = \sup_u z_k^T u - (a+Pu)^T x + b = \begin{cases} a^T x - b & \text{if } z_k + P^T x = 0 \\ \infty & \text{otherwise} \end{cases}$$

$$z_1 = z_2 = \cdots = z_k = -P^T x$$

Convex reformulation

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & a^T x - b + (P^T x)^T \sum_{k=1}^K w_k \bar{d}_k + \epsilon \|P^T x\|_* \leq 0 \end{array}$$

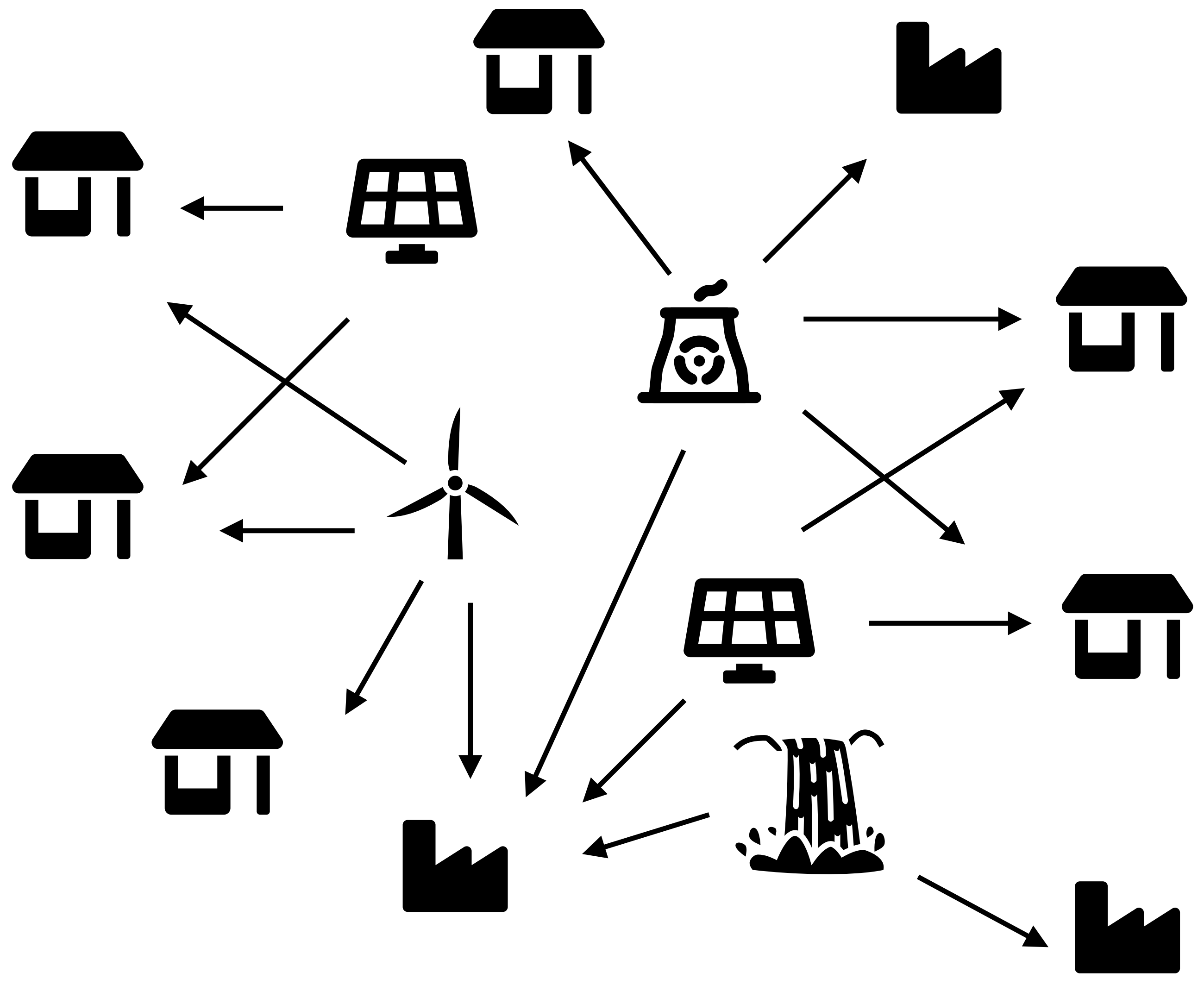
(clustering makes no difference)

average $\bar{d}$

Classical Robust Optimization reformulation

Numerical examples

Facility location example



cost of opening facilities

cost of energy distribution

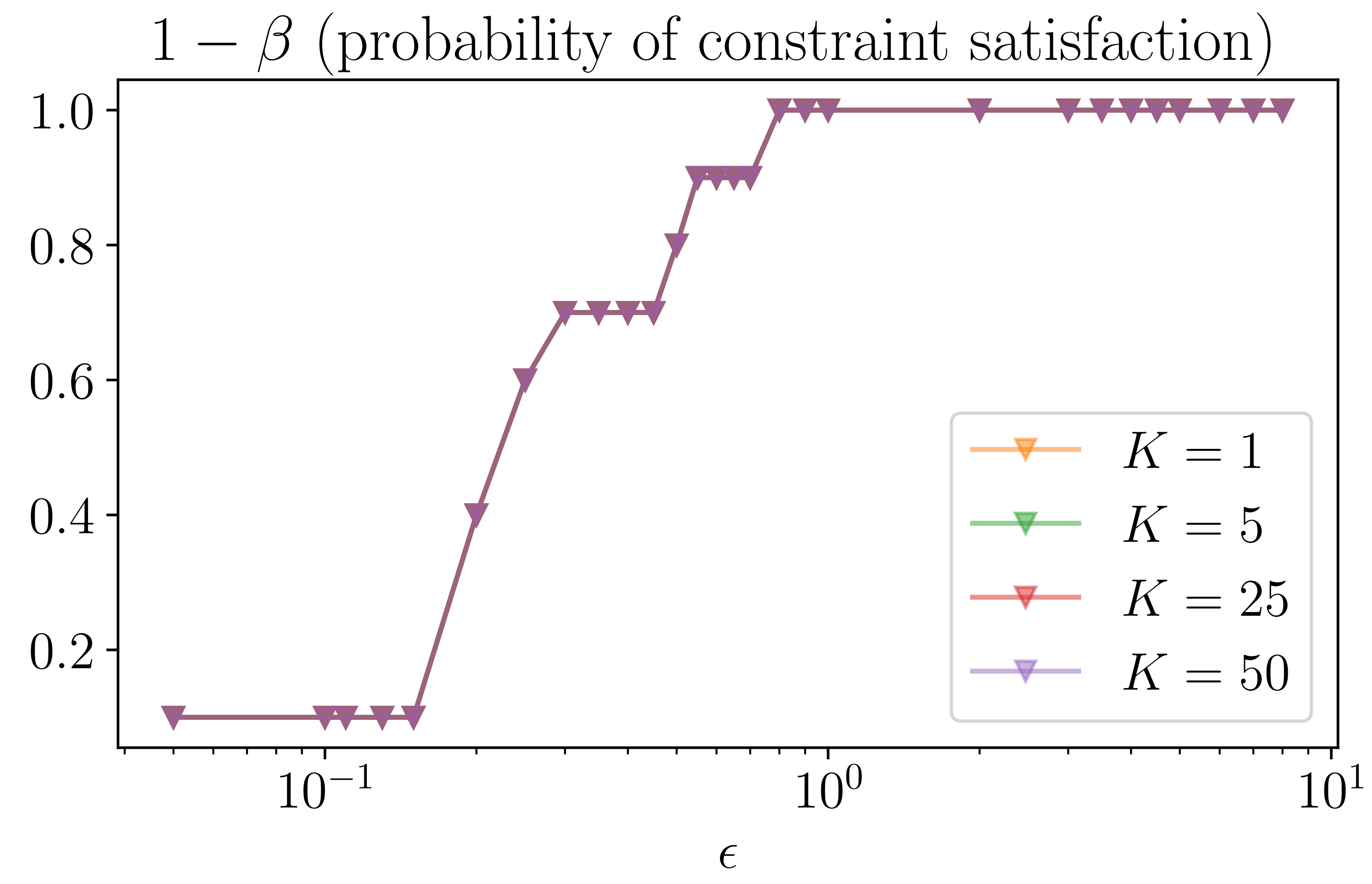
$$
\begin{array}{ll}
\text{minimize} & c^T x + \mathbf{tr}(C^T X) \\
\text{subject to} & \mathbf{1}^T X_j = 1, \quad j = 1, \dots, m \\
 & (X^T)_i u \le r_i x_i, \quad i = 1, \dots, n \\
 & x \in \{0,1\}^n, \quad X \in \mathbf{R}^{n \times m}
\end{array}
$$

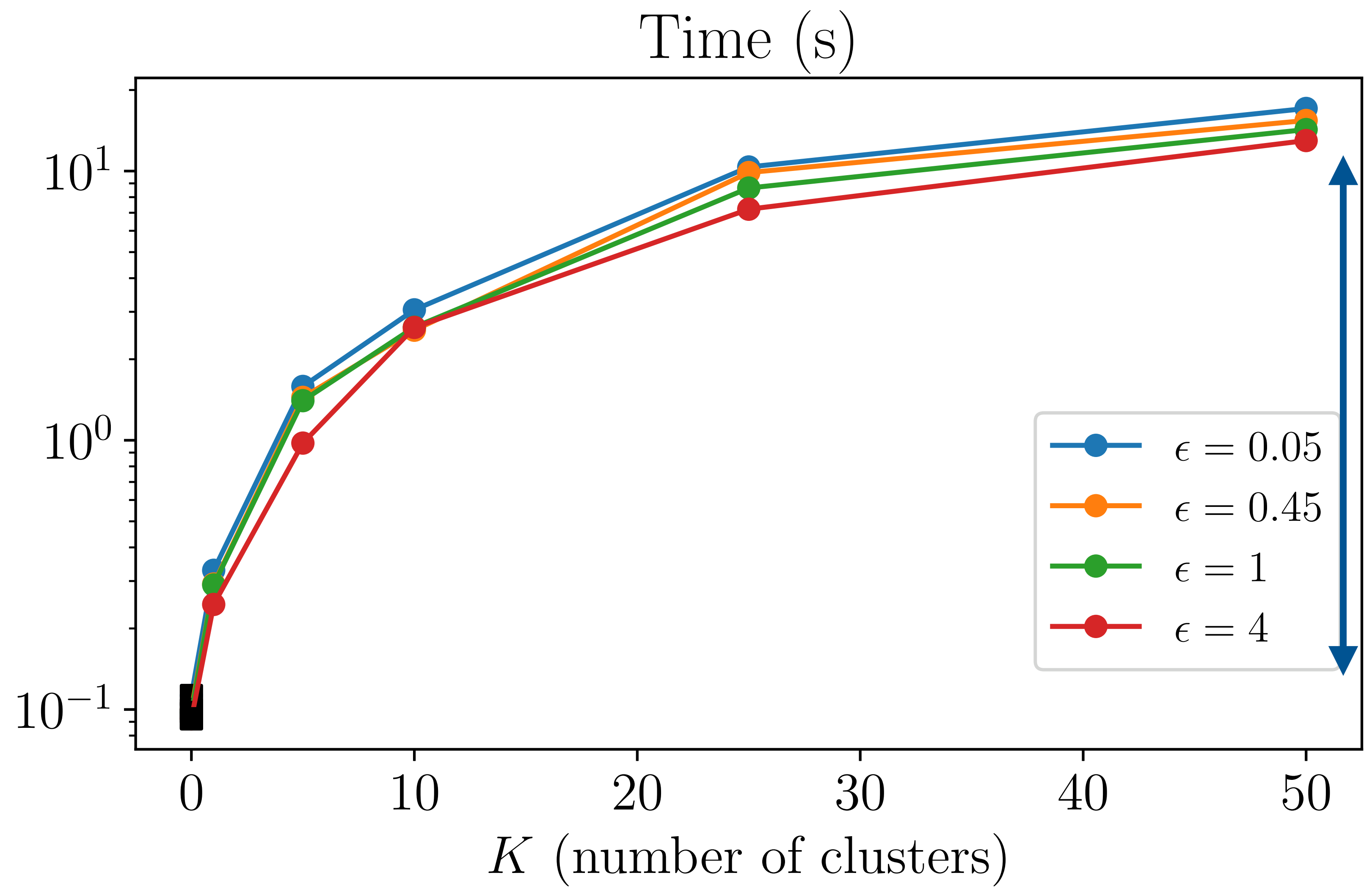
capacity constraints

$g(u, x, X) \le 0$

vector of uncertain energy demands

clustering does not affect constraint satisfaction





100x speedups!

Capital budgeting example

Problem

total net present value (NPV)

$$\begin{aligned} \text{maximize} \quad & \eta(u)^T x \\ \text{subject to} \quad & a^T x \leq b \\ & x \in \{0,1\}^n \end{aligned}$$

budget constraint

NPV of project j

$$\eta_j(u) = \sum_{t=1}^{T} \frac{F_{jt}}{(1+u_j)^t}$$

cash flow

discount rate

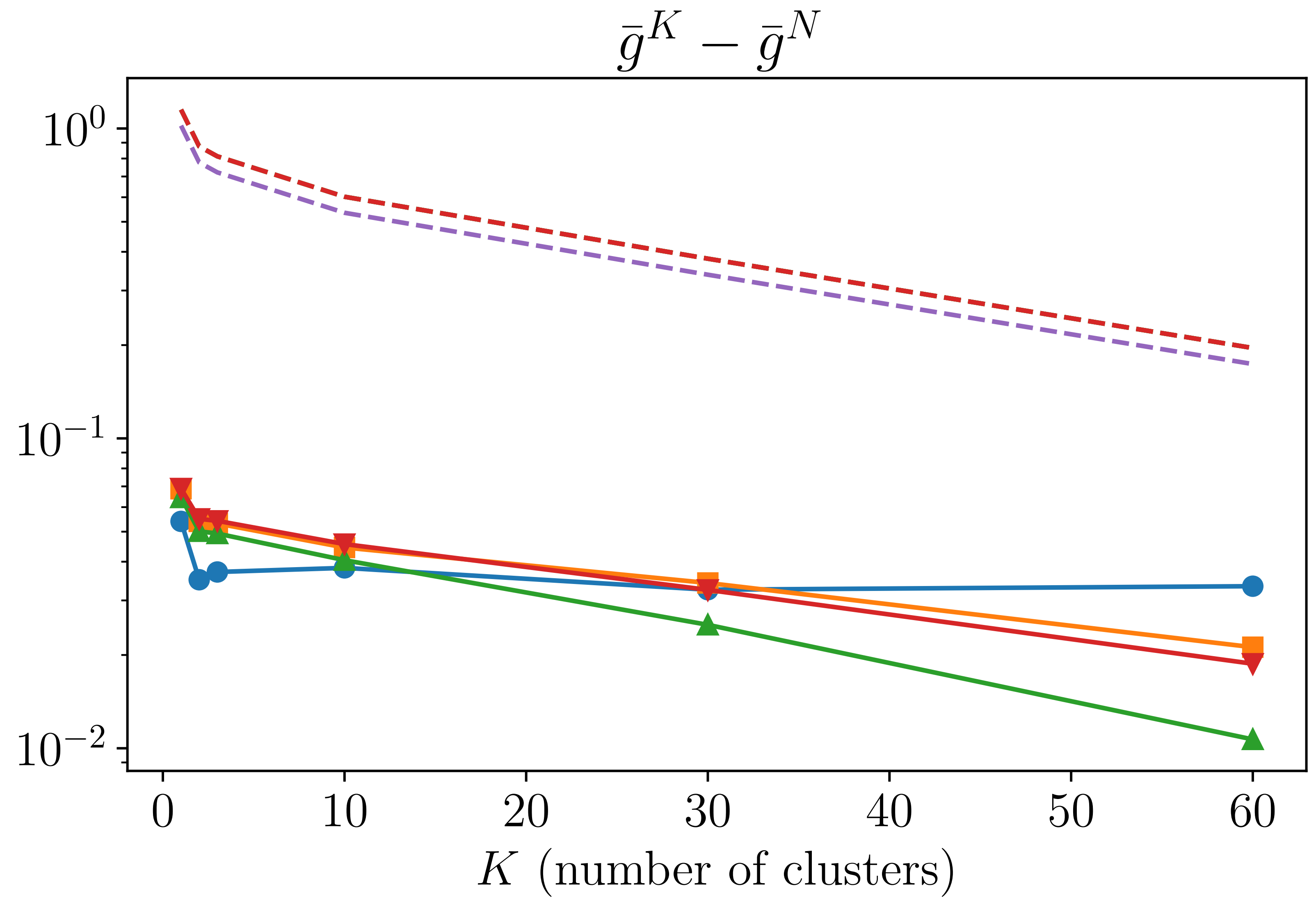
$$g(u, x, \tau) = -\eta(u)^T x - \tau$$

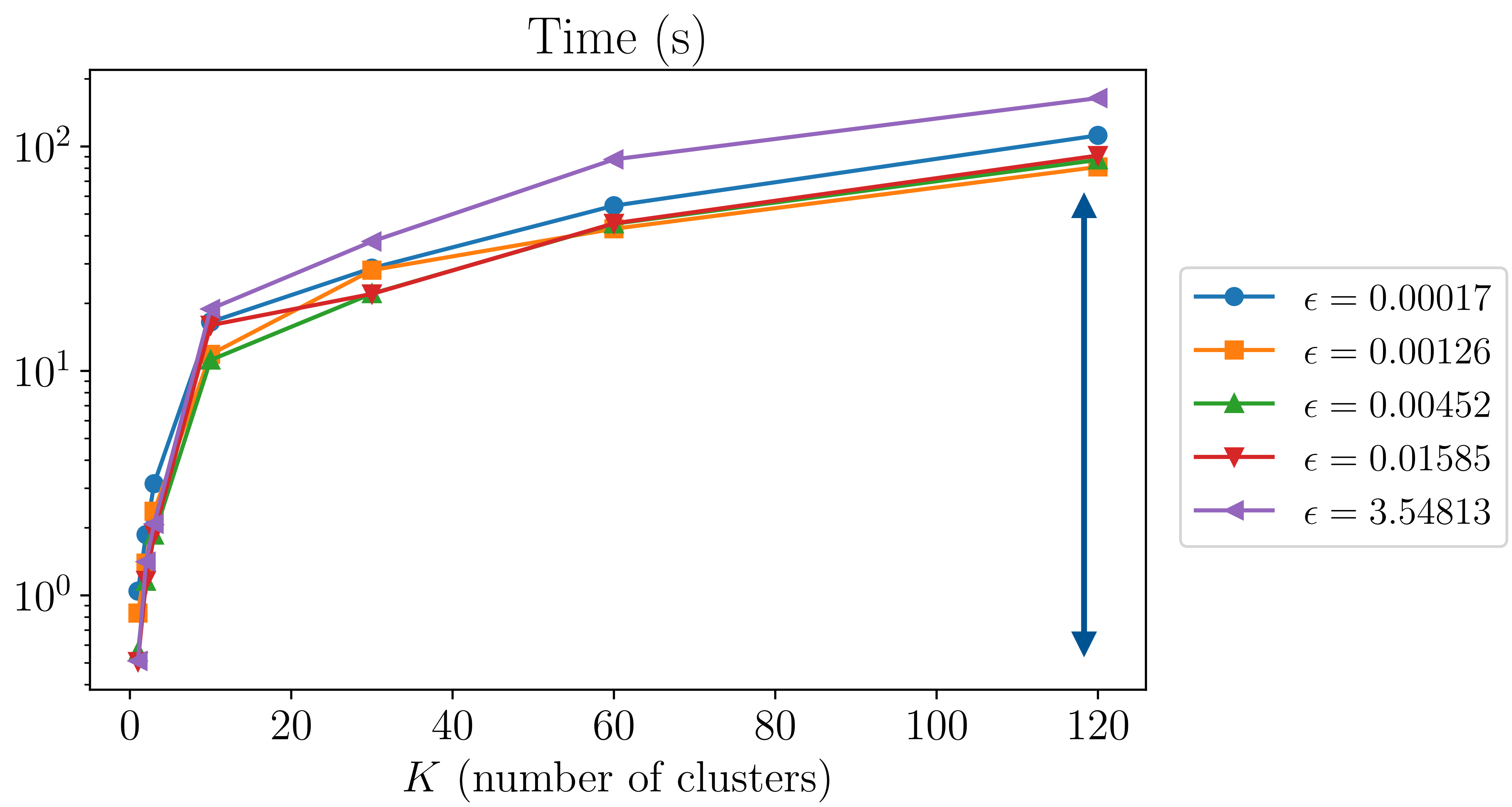
MRO formulation

$$\begin{aligned} \text{minimize} \quad & \tau \\ \text{subject to} \quad & \bar{g}(u, x, \tau) \leq 0, \quad \forall u \in \mathcal{U}(K, \epsilon) \\ & a^T x \leq b \\ & x \in \{0,1\}^n \end{aligned}$$

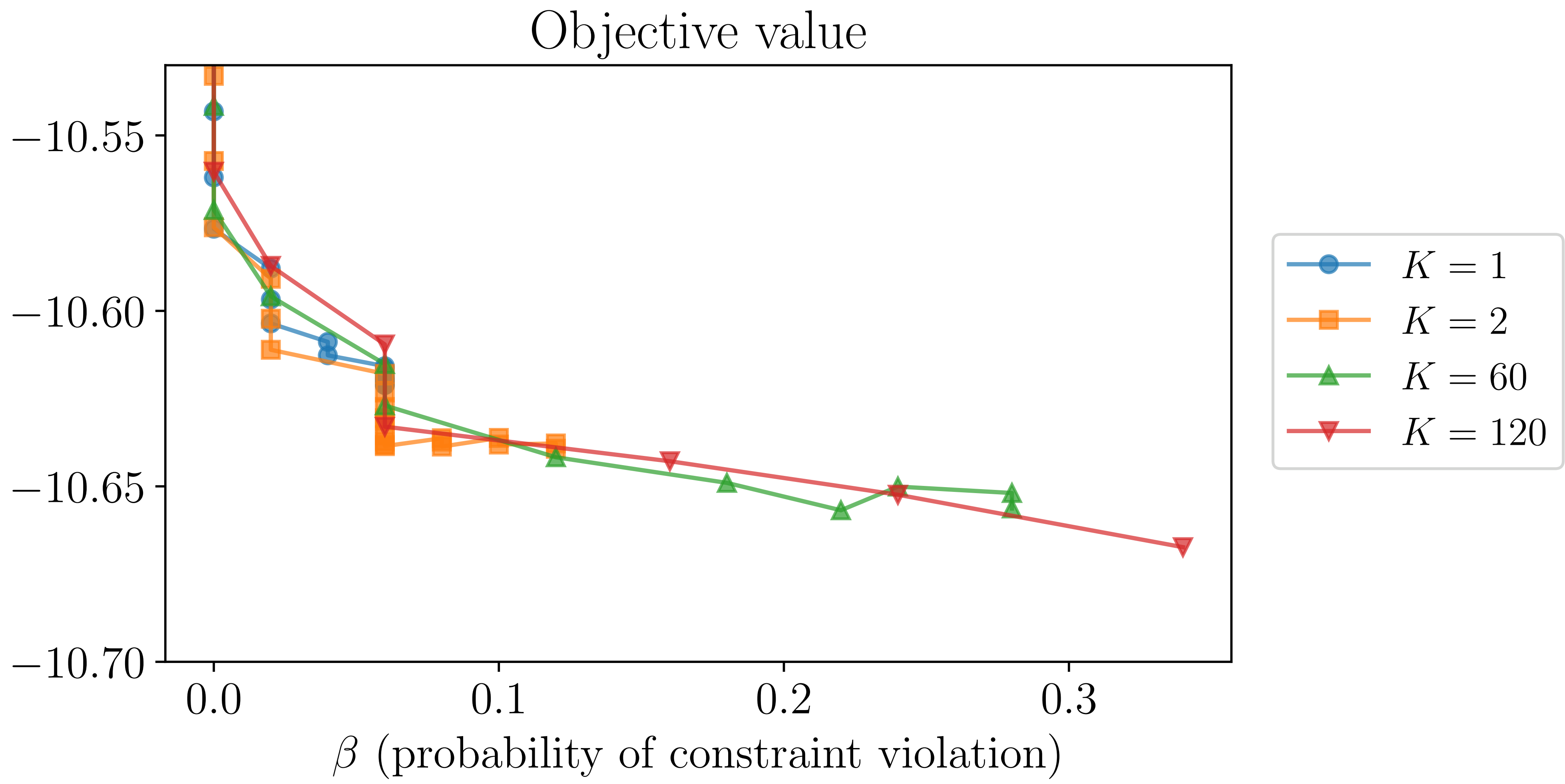
Capital budgeting results

$n = 20, N = 120, T = 5$





100x speedups!



2 clusters give near-optimal performance

Conclusions

Mean Robust Optimization

- **Bridge** RO and DRO

- Clustering effect
  - g affine in u → **zero clustering effect!**
  - g concave in u → **performance bound**

- Multiple **orders of magnitude** speedups

https://github.com/stellatogrp/mro_experiments

Mean Robust Optimization
Mathematical Programming (major revision)
I. Wang, C. Becker, B. Van Parys, and B. Stellato
arXiv e-prints:2207.10820, 2022

INFORMS Computing Society Student Paper Award Winner