Mean Robust Optimization

Joint work with Irina Wang, Cole Becker, and Bart Van Parys

PRINCETON UNIVERSITY ORFE

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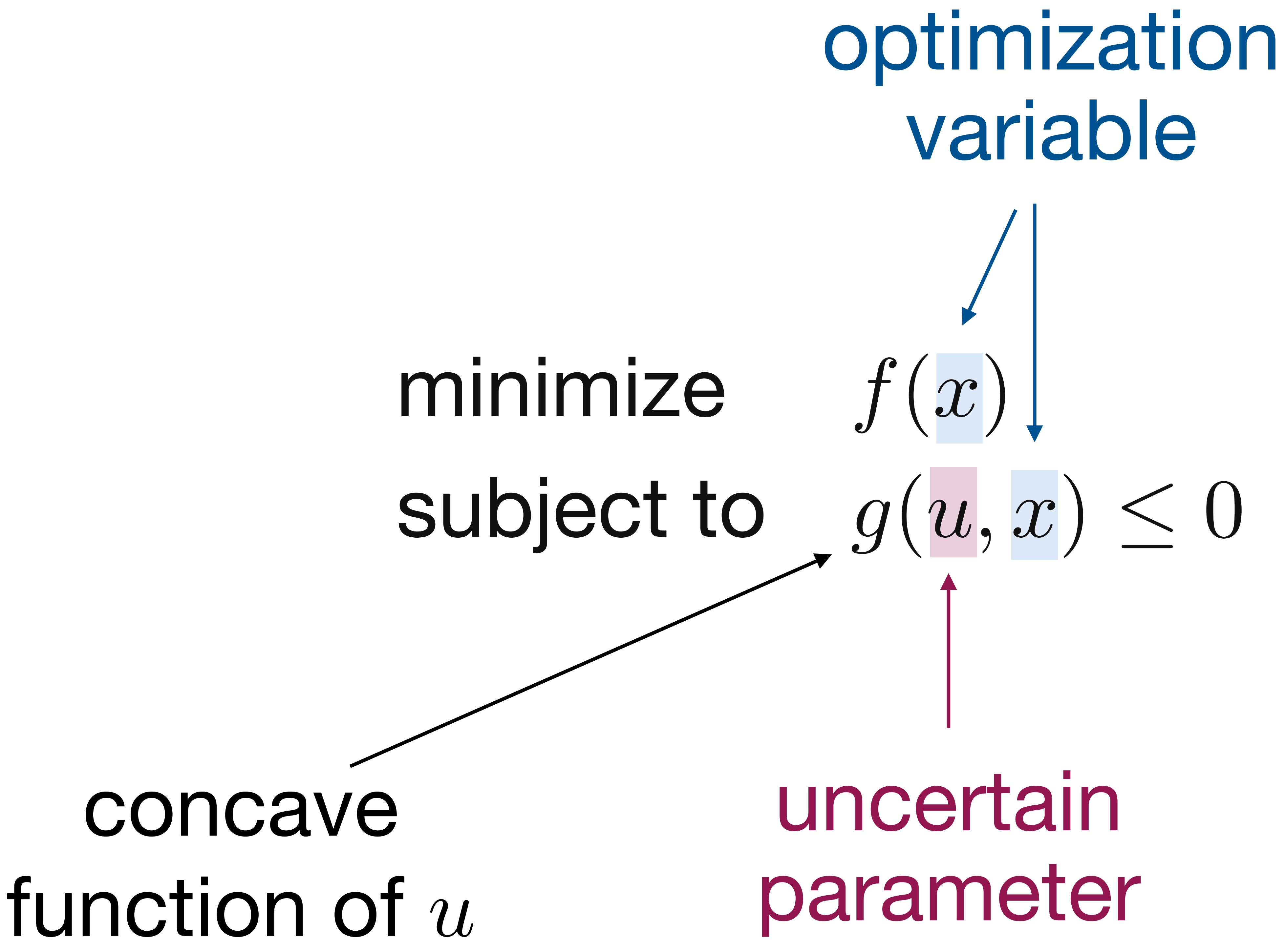
Decision-making with uncertainty is hard


COVID-19


Finance


Energy


optimization variable

$$
\begin{array}{ll}
\text{minimize} & f(x) \\
\text{subject to} & g(u, x) \le 0
\end{array}
$$

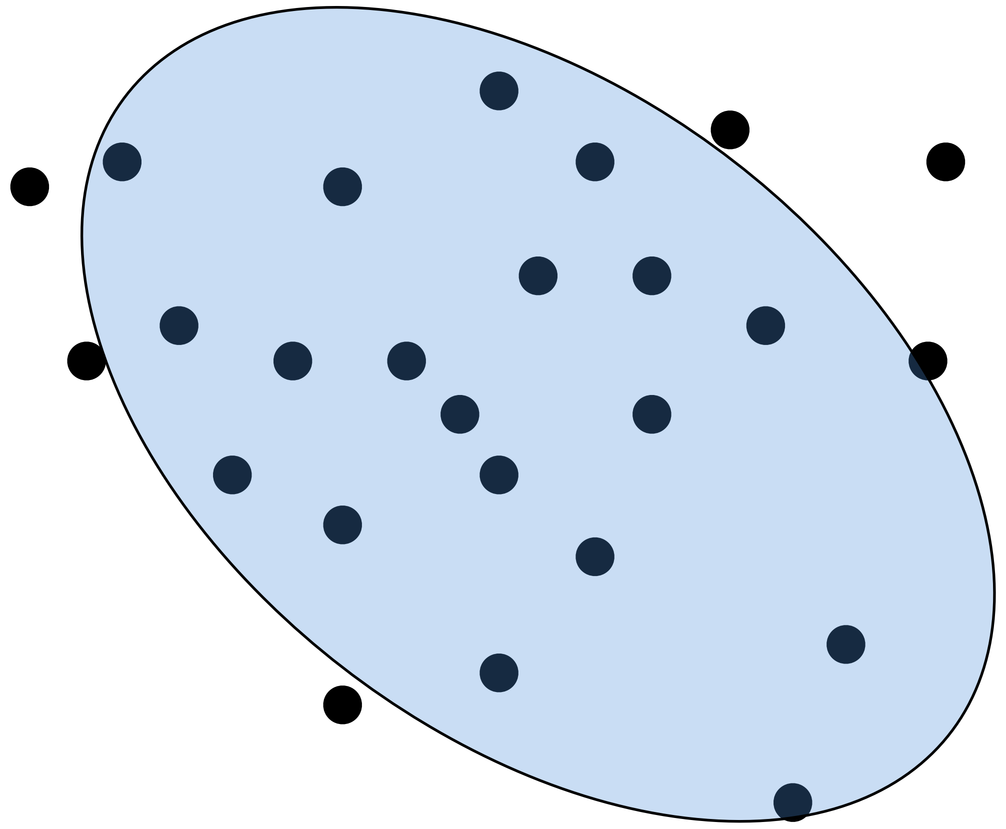
concave function of u

uncertain parameter

We want to guarantee constraint satisfaction

How do we solve it?

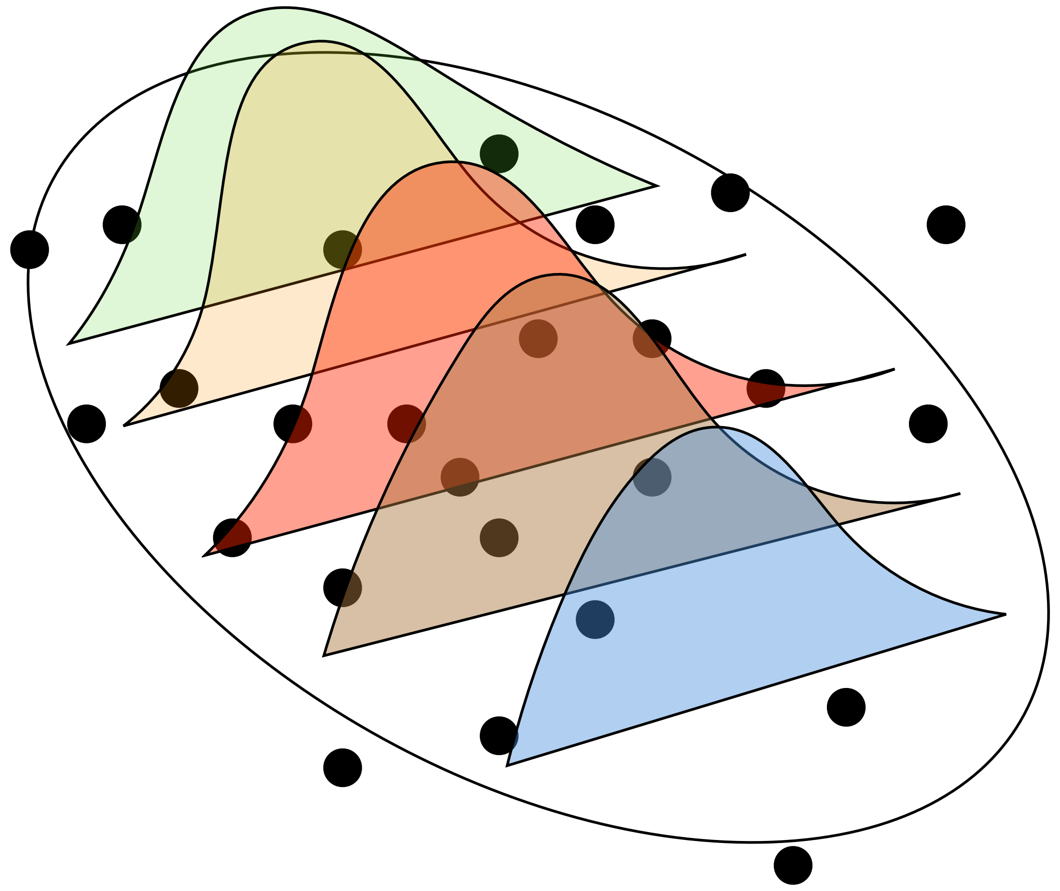
Robust optimization
(RO)



✓ Tractable, expressive

✗ Can be conservative

Distributionally Robust Optimization
(DRO)



✓ Can be less conservative

✗ Computationally expensive

Can we get the best of both worlds?

Probabilistic guarantees

$$\mathbf{E}(g(u,x)) \leq 0$$

$u \sim P$

(but we never know P!)

Data

$\mathcal{D} = \{d_i\}_{i=1}^N$

Data-driven probabilistic guarantees

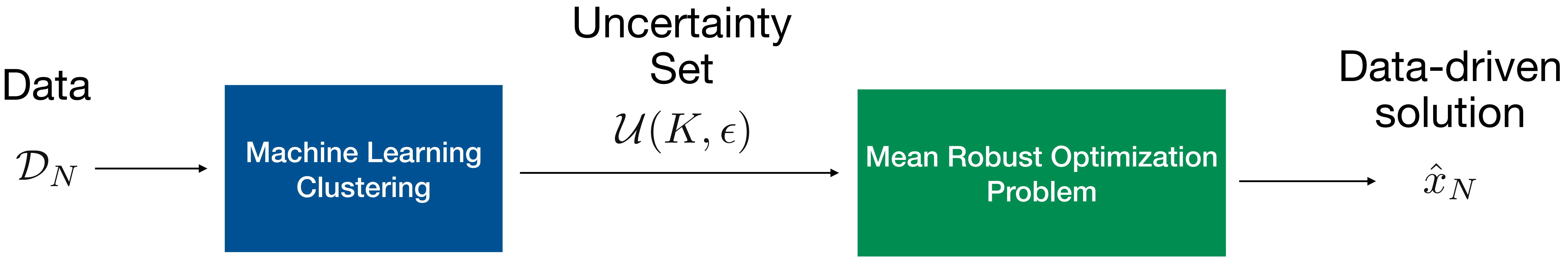
Product Distribution →

$$\mathbf{P}^N\left(\mathbf{E}(g(u,\hat{x}_N)) \leq 0\right) \geq 1-\beta$$

← probability of constraint satisfaction

↑ data-driven solution

Mean Robust Optimization (MRO)



Data

$\mathcal{D}_N$

Machine Learning Clustering

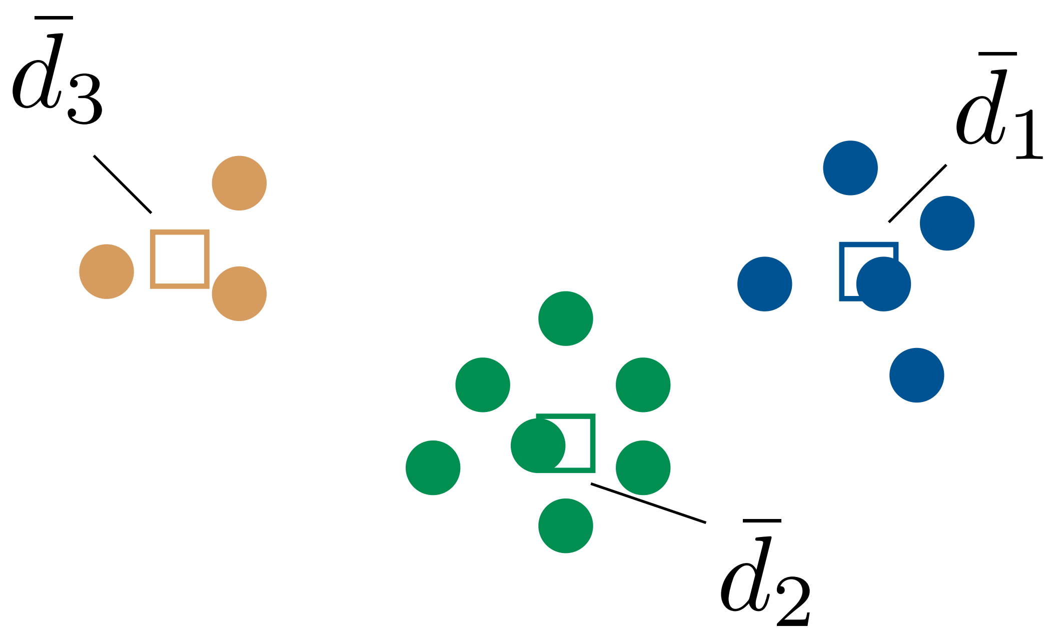
Uncertainty Set

$\mathcal{U}(K, \epsilon)$

Mean Robust Optimization Problem

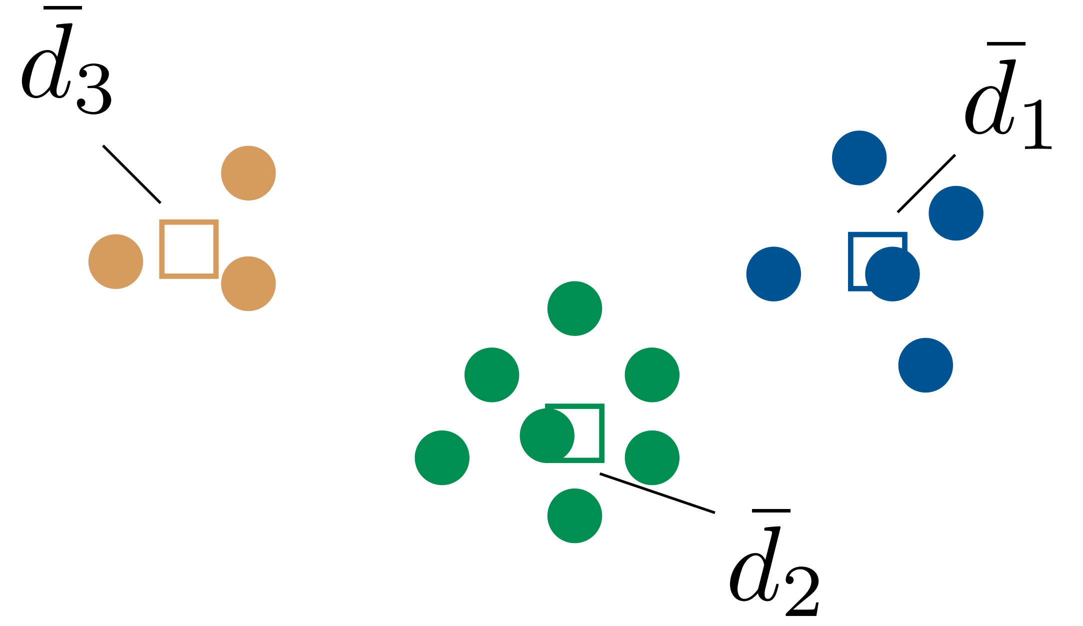
Data-driven solution

$\hat{x}_N$



$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \underbrace{\bar{g}(u,x)}_{\text{constraint function}} \leq 0 \quad \forall u \in \underbrace{\mathcal{U}(K,\epsilon)}_{\text{uncertainty set}} \end{array}$$

Uncertainty set



cluster weights

order

$$\mathcal{U}(K,\epsilon) = \left\{ u = (v_1,\dots,v_K) \;\middle|\; \sum_{k=1}^{K} w_k \|v_k - \bar{d}_k\|^p \leq \epsilon^p \right\}$$

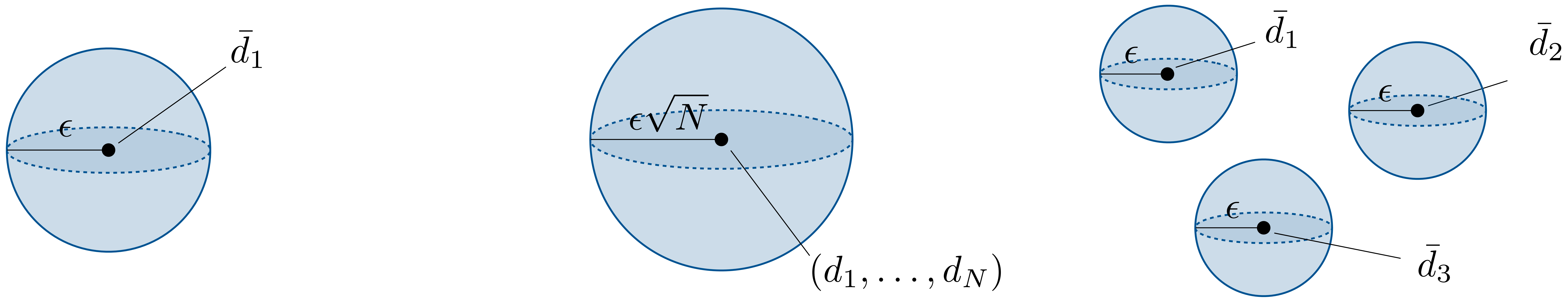
cluster centers

Examples

$K = 1$

$K = N, p = 2$

$K = 3, p = \infty$



Mean Robust Optimization Problem

uncertainty set

$$\left\{ \sum_{k=1}^{K} w_k \|v_k - \bar{d}_k\|^p \leq \epsilon^p \right\}$$

Uncertain variable lifting

$$u = (v_1, \dots, v_K)$$

→

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \bar{g}(u, x) \leq 0 \quad \forall u \in \mathcal{U}(K, \epsilon) \end{array}$$

constraint function

$$\sum_{k=1}^{K} w_k g(v_k, x)$$

Solving the MRO problem

Dualize constraint $\bar{g}(u, x) \leq 0, \; \forall u \in \mathcal{U}(K, \epsilon)$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \sum_{k=1}^{K} w_k s_k \leq 0 \\ & [-g]^*(z_k - y_k, x) - z_k^T \bar{d}_k + \phi(p)\lambda \|z_k/\lambda\|_*^{p/(p-1)} + \lambda\epsilon^p \leq s_k, \quad k = 1, \dots, K \\ & \lambda \geq 0 \end{array}$$

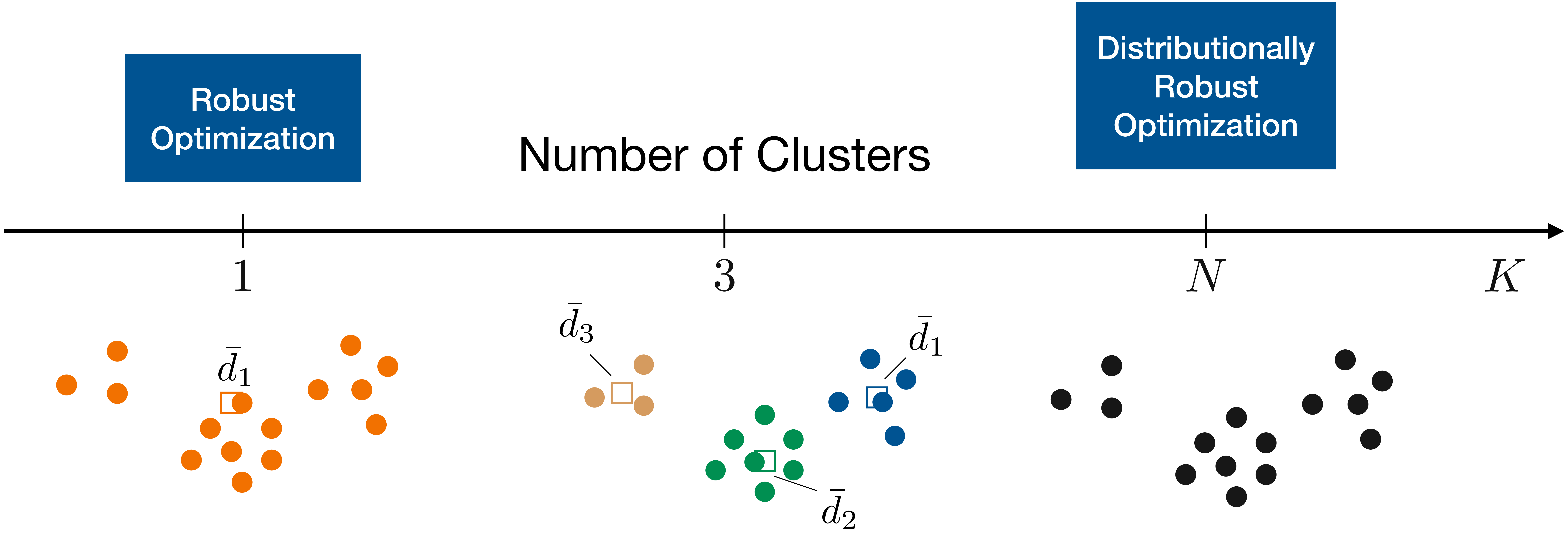
conjugate function

cluster centers

function of $p \geq 1$
$\phi(p) \to 1$ as $p \to \infty$
$\phi(1) = 0$

It can be very expensive when K is large (e.g., $K = N$)

MRO bridges between RO and DRO



Robust Optimization

Number of Clusters

Distributionally Robust Optimization

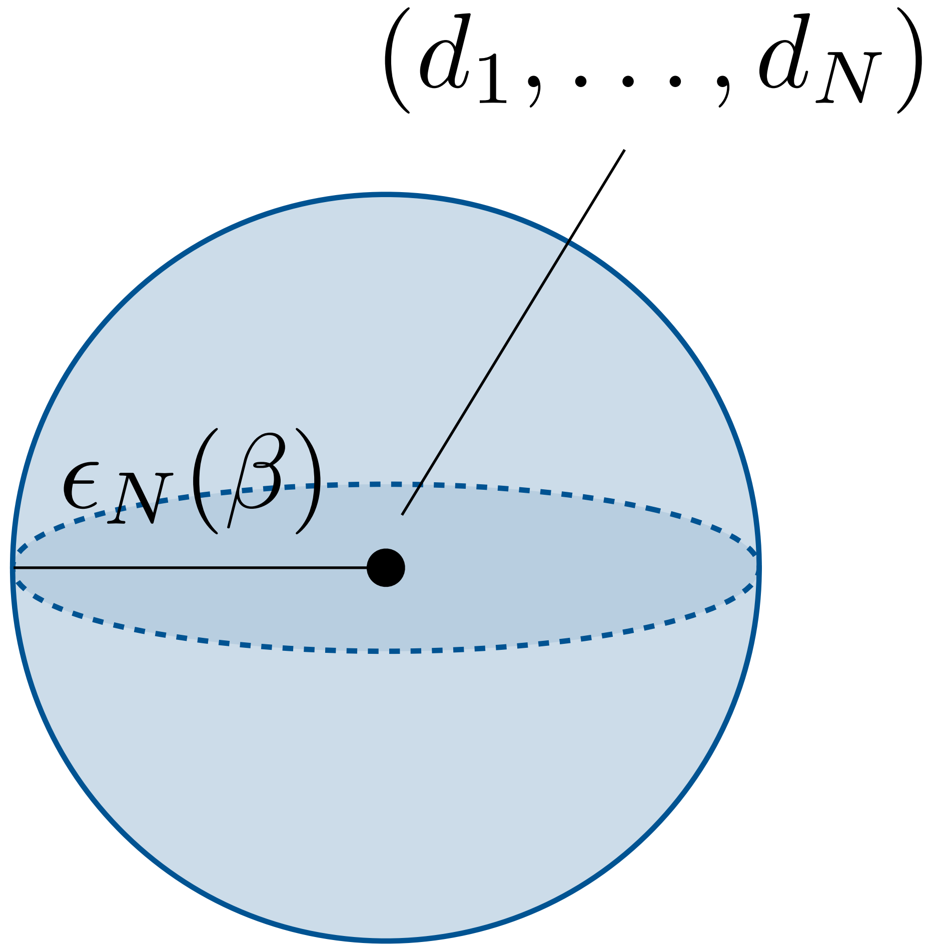
[KENSA19] D Kuhn, P M Esfahani, V A Nguyen, and S Shafieezadeh-Abadeh, “Wasserstein Distributionally Robust Optimization: Theory and Applications in Machine Learning

Satisfying the probabilistic guarantees

probability of constraint satisfaction

uncertainty set radius

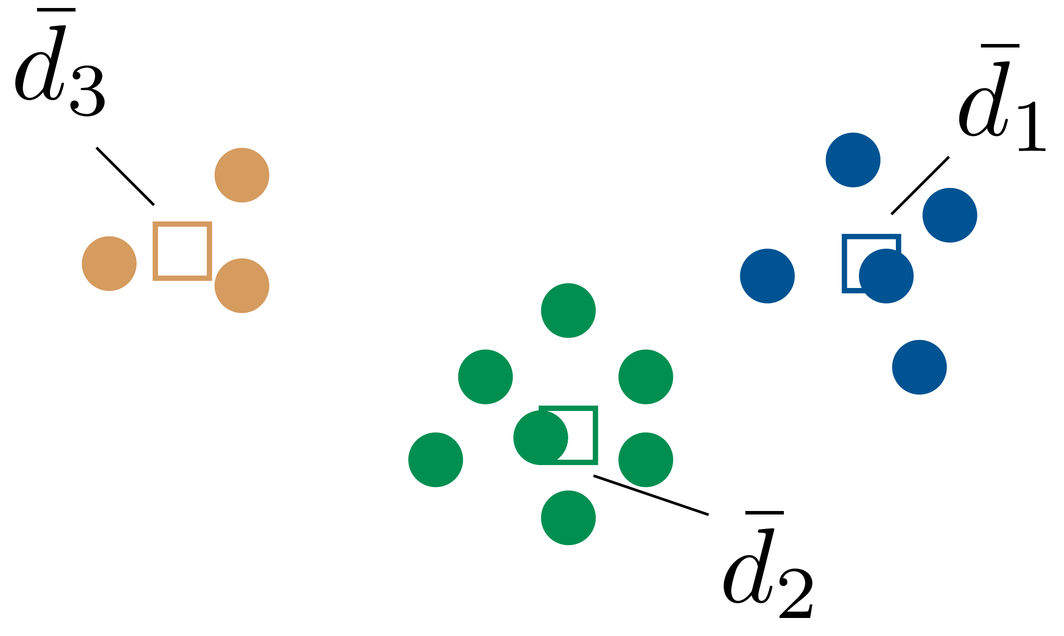
$$\mathbf{P}^N\left(\mathbf{E}(g(u, \hat{x}_N)) \leq 0\right) \geq 1 - \beta \xrightarrow{\text{light-tailed}} \mathcal{U}(N, \epsilon_N(\beta))$$



MRO clustering

$$\mathcal{U}(K, \epsilon_N(\beta) + \eta_N(K))$$

$$\max_{i \in C_k} \|d_i - \bar{d}_k\|$$



Quite conservative bounds… can we do better?

Bounding the conservatism

MRO constraint

$$\bar{g}(u,x) \leq 0 \quad \forall u \in \mathcal{U}(K,\epsilon)$$

Worst-case values

$$\bar{g}^{N}(x) = \underset{u \in \mathcal{U}(N,\epsilon)}{\text{maximize}} \quad \bar{g}(u,x)$$

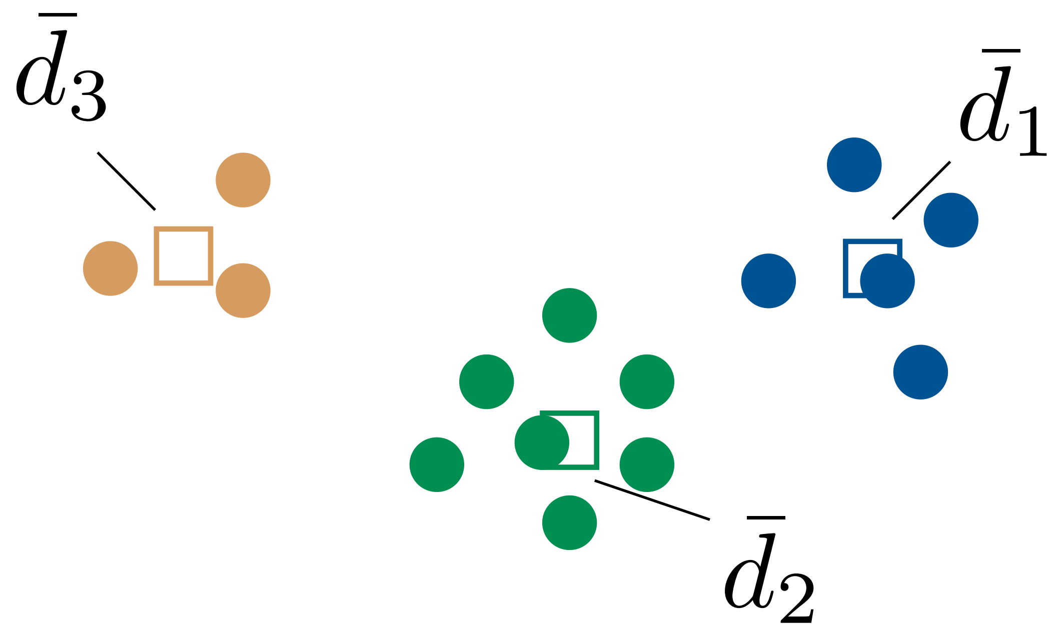
$$\bar{g}^{K}(x) = \underset{u \in \mathcal{U}(K,\epsilon)}{\text{maximize}} \quad \bar{g}(u,x)$$

Theorem

If $-g$ is L-smooth in u, we have

$$\bar{g}^{N}(x) \leq \bar{g}^{K}(x) \leq \bar{g}^{N}(x) + \frac{L}{2} D(K) \longleftarrow \min \frac{1}{N} \sum_{k=1}^{K} \sum_{d_i \in C_k} \|d_i - \bar{d}_k\|^2$$

clustering objective



When g is affine in u ($L=0$), clustering makes no difference to the optimal value or optimal solution

Capital budgeting example

Problem

total
net present value (NPV)

$$\begin{array}{ll} \text{maximize} & \eta(u)^T x \\ \text{subject to} & a^T x \le b \\ & x \in \{0,1\}^n \end{array}$$

budget
constraint

$$g(u, x, \tau) = -\eta(u)^T x - \tau$$

MRO formulation

$$\begin{array}{ll} \text{minimize} & \tau \\ \text{subject to} & \bar{g}(u, x, \tau) \le 0, \quad \forall u \in \mathcal{U}(K, \epsilon) \\ & a^T x \le b \\ & x \in \{0,1\}^n \end{array}$$

NPV of project j

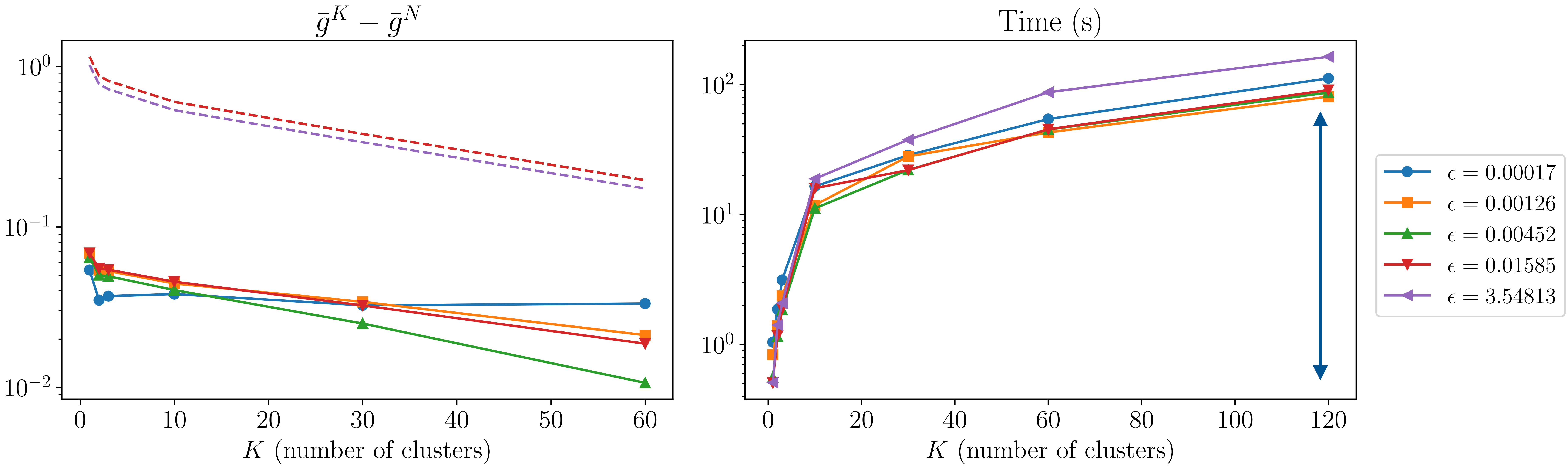
$$\eta_j(u) = \sum_{t=1}^{T} \frac{F_{jt}}{(1 + u_j)^t}$$

cash
flow

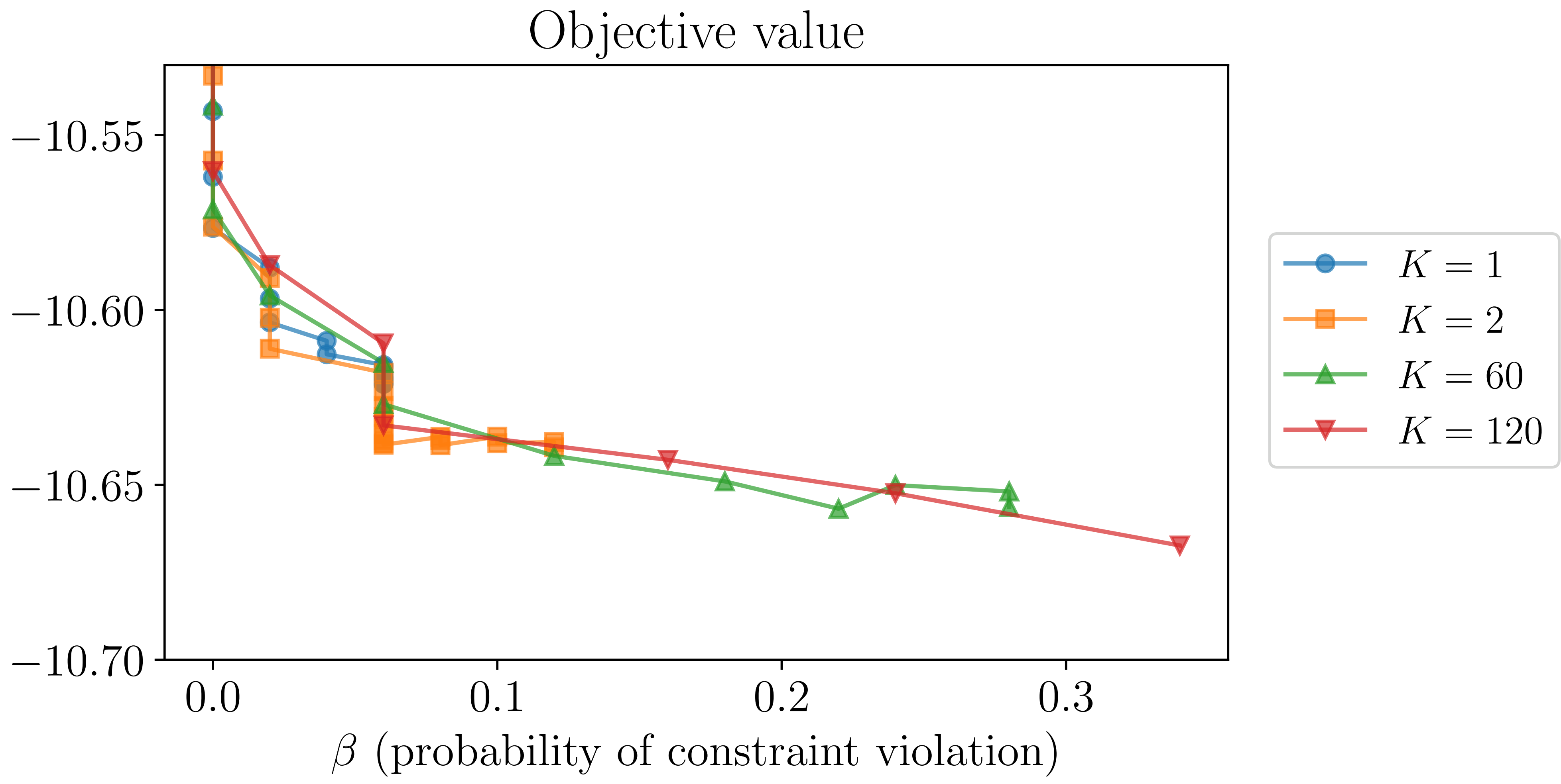
discount
rate

Capital budgeting results

$n = 20, N = 120, T = 5$



100x speedups!



2 clusters give near-optimal performance

Mean Robust Optimization

- **Bridge** between RO and DRO

- Clustering effect
  - g affine in u → **zero clustering effect!**
  - g concave in u → **performance bound**

- Multiple **orders of magnitude** speedups

https://github.com/stellatogrp/mro_experiments

https://arxiv.org/abs/2207.10820 ←

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stellato.io bstellato@princeton.edu @b_stellato